§1. Hopfological algebra

In 1994, Crane and Frenkel published their seminal paper "Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases". There they proposed a program called "categorification", hoping to lift the combinatorially defined 3-manifold invariant constructed by Kuperberg to a 4d TQFT

$$\begin{tikzcd}
\text{Donaldson-Floer} \arrow[r, Leftrightarrow, no head] \arrow[d, Rightarrow] & \boxed{\text{A 4D TQFT ?}} \\
\begin{array}{l}\text{Kuperberg: } u_q^+(sl_2)\text{-invariant of} \\ \text{3-manifolds}\end{array} \arrow[r, phantom, "\sim"] & \begin{array}{c}\text{WRT- 3D TQFT} \\ \text{associated with } u_q(sl_2)\end{array} \arrow[u, Rightarrow, "\text{Categorification}"']
\end{tikzcd}$$

- q: a primitive n-th root of unity

Digression: Homological algebra

Assume, for simplicity, that we work over a ground field $\Bbbk$. Homological algebra has the following features.

(0). Chain complexes and their cohomology groups

$$(K^\bullet, d): \quad d: K^\bullet \longrightarrow K^{\bullet+1}, \quad d^2 = 0 \ ;$$

(1). Direct sums of chain complexes ;

(2). Tensor products of chain complexes : $K^\bullet \otimes L^\bullet$

(3). Inner homs between chain complexes: $\mathrm{HOM}^\bullet(K^\bullet, L^\bullet)$

$$\mathrm{HOM}^i(K^\bullet, L^\bullet) := \{ f : K^\bullet \longrightarrow L^\bullet \mid f(K^k) \subseteq L^{k+i} \}$$

$$d(f) = d \circ f - (-1)^{|f|} f \circ d.$$

(4). Triangular structures.

Homological shifts / cone constructions / s.e.s. leading to d.t.
TR1 — TR4 etc.

Homological algebra plays a fundamental role in categorification since it gives a systematic lifting of abelian structures

$$\begin{array}{ccc} D^b(\Bbbk) & \xrightarrow{\chi\,(K_0)} & \mathbb{Z} \\ K^\bullet & \mapsto & \sum(-1)^i \dim_i K^i \\ \oplus & \mapsto & \text{addition} \\ \otimes & \mapsto & \text{multiplication} \\ \text{tensor unit} & \mapsto & 1 \\ [1] & \mapsto & \text{multiplication by } (-1). \end{array}$$

Rmk: If we replace vector spaces by graded vector spaces, we get a systematic lifting of "quantum" abelian structures:

$$K_0(\Bbbk\text{-gvect}) \cong \mathbb{Z}[q,q^{-1}]$$

The grading shift $\{1\}$ decategorifies to multiplication by q.

- Observation: Feature (1) – (3) are reminiscent of some basic constructions in representation theory: If G is some group, $H=\Bbbk G$ is a Hopf algebra so that its category of modules H-mod has:

(1)'. $K \oplus L \in H\text{-mod}$
(2)'. $K \otimes L \in H\text{-mod}$ $\quad h(k\otimes \ell) := \sum (h_{(1)}k)\otimes(h_{(2)}\ell)$.
(3)'. $\mathrm{HOM}(K,L) \in H\text{-mod}$ $\quad (h\cdot f)(k) := \sum h_{(2)} f(S^{-1}(h_{(1)}(v)))$.

Thus (1) – (3) above can be viewed as a special case of (1)' – (3)' for the Hopf superalgebra of dual numbers $H=\Bbbk[d]/(d^2)$.

- Question: Are there analogues of the other features of homological algebra for H-mod ? For instance, what is "cohomology" ?

Any chain complex / $\Bbbk$ decomposes uniquely into direct sums :

$$(\oplus\ 0 \longrightarrow \Bbbk \longrightarrow 0) \oplus (\oplus\ 0 \longrightarrow \Bbbk \longrightarrow \Bbbk \longrightarrow 0)$$

Taking cohomology does nothing but killing the second factor, which is a direct sum of free $\Bbbk[d]/(d^2)$ - modules.

Less obvious is the fact that $(0 \longrightarrow \Bbbk \longrightarrow \Bbbk \longrightarrow 0)$ is also injective. In fact, $\Bbbk[d]/(d^2)$ is a Frobenius superalgebra.

Thm. (Sweedler) A Hopf algebra H is Frobenius iff it is finite-dim'l.

Our question reduces to asking how one can systematically kill projective-and-injective modules.

The stable category

Intuitively, the stable category $H\text{-}\underline{mod}$ is the categorical quotient of H-mod by the class of projective / injective objects.

Def. The category $H\text{-}\underline{mod}$ consists of the same objects as H-mod, while the morphism spaces between any objects K, L are given by

$$\mathrm{Hom}_{H\text{-}\underline{mod}}(K,L) := \mathrm{Hom}_{H\text{-}mod} \Big/ \left(\begin{array}{l} \text{morphisms that factor} \\ \text{through pro/injectives} \end{array} \right)$$

Rmk: The notion of stable categories makes sense for any self-injective algebra, not necessarily those coming from finite-dimensional Hopf algebras.

Thm. (Heller) If H is self-injective, then H-$\underline{\mathrm{mod}}$ is triangulated.

In general, the morphism spaces between objects in some stable category are hard to compute. But for a stable category arising from a finite dimensional Hopf algebra, there is a conceptually easy way to compute them. To do this we need the notion of integrals for Hopf algebras.

Def. Let H be a Hopf algebra $/\Bbbk$. An element $\Lambda \in H$ is called a left integral in H if $\forall h \in H$,

$$h \cdot \Lambda = \epsilon(h) \Lambda.$$

Thm. (Sweedler) Any finite dimensional H has a non-zero integral, unique up to a non-zero constant.

Examples (1). $H = \Bbbk G$ (G: finite group). $\Lambda = \sum_{g \in G} g$.

(2). $H = \Bbbk[d]/(d^2)$, $\Lambda = d$.

(3). $H = \Bbbk[\partial]/(\partial^p)$, $(\mathrm{char}\,\Bbbk = p > 0)$, $\Lambda = \partial^{p-1}$.

Prop. Let H be a finite-dim'l Hopf algebra, and K, L be H-modules. Then

$$\begin{aligned} \mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(K, L) &\cong \mathrm{Hom}_H(K, L) / \Lambda \cdot \mathrm{HOM}(K, L) \\ &\cong \mathrm{HOM}(K, L)^H / \Lambda \cdot \mathrm{HOM}(K, L) \end{aligned}$$

We will prove the prop shortly. Before that, we look at a couple of examples.

Examples. (1) $H = \Bbbk G$, a finite group, $\Lambda = \sum_{g \in G} g$. Recall that H is semisimple iff $\Bbbk$ is a projective module. This is equivalent to requiring that $\mathrm{Hom}_{\underline{H\text{-mod}}}(\Bbbk,\Bbbk)=0$. But $\Lambda\cdot \mathrm{HOM}(\Bbbk,\Bbbk)=\epsilon(\Lambda)\Bbbk=|G|\cdot\Bbbk$. Thus $\Bbbk G$ is semisimple iff $|G|\in\Bbbk^*$.

(2) $H=\Bbbk[d]/(d^2)$: $\Lambda\cdot f=d\cdot f=d\circ f-(-1)^{|f|}f\circ d$

(3) $H=\Bbbk[\partial]/(\partial^{p-1})$ $(\mathrm{char}\,\Bbbk=p>0)$

$$\Lambda\cdot f=\partial^{p-1}(f)=\textstyle\sum_{i=0}^{p-1}\partial^i\circ f\circ\partial^{p-1-i}.$$

Lemma. An H-module map $K\longrightarrow L$ factors through an injective H-module iff there exists a factorization

$$\begin{tikzcd}
K \arrow[rr] \arrow[dr, "\mathrm{Id}\otimes\Lambda"'] & & L \\
& K\otimes H \arrow[ur] &
\end{tikzcd}$$

Proof. It suffices to show this when L is injective. Consider the following commutative diagram

$$\begin{tikzcd}
K \arrow[r, "\varphi"] \arrow[d, "\mathrm{Id}_K\otimes\Lambda"'] & L \arrow[d, "\mathrm{Id}_L\otimes\Lambda"'] \\
K\otimes H \arrow[r, "\varphi\otimes\mathrm{Id}"] & L\otimes H \arrow[u, bend right, "g"']
\end{tikzcd},$$

since L is injective, the injection $\mathrm{Id}_L\otimes\Lambda: L\longrightarrow L\otimes H$ must split. Choose a splitting g. Then φ factors as $g\circ(\varphi\otimes\mathrm{Id})\circ(\mathrm{Id}_K\otimes\Lambda)$. □

Lemma. An H-module map $\varphi: K\longrightarrow L$ factors through $\mathrm{Id}_K\otimes\Lambda: K\longrightarrow K\otimes H$ iff there is a $\Bbbk$-linear map ψ s.t. $\varphi=\Lambda\cdot\psi$.

Proof. If $\varphi=\Lambda\cdot\psi$ for some $\psi\in\mathrm{Hom}_{\Bbbk}(K,L)$, we will extend ψ to $\tilde{\Psi}: K\otimes H\longrightarrow L$ by

$$\tilde{\Psi}(k\otimes h):=(h\cdot\psi)(k)=h_{(2)}\psi(S^{-1}h_{(1)}k)$$

Then $\tilde{\Psi}$ is H-linear: $\forall x,h\in H$, $k\in K$, we have

$$\tilde{\Psi}(x\cdot(k\otimes h))=\tilde{\Psi}(x_{(1)}k\otimes x_{(2)}h)$$

$$
\begin{aligned}
&= (x_{(2)}h)_{(2)}\tilde{\psi}(S^{-1}((x_{(2)}h)_{(1)})\,x_{(1)}k) \\
&= x_{(3)}h_{(2)}\tilde{\psi}(S^{-1}(h_{(1)})\,S^{-1}(x_{(2)})\,x_{(1)}k) \\
&= x_{(2)}h_{(2)}\tilde{\psi}(S^{-1}(h_{(1)})\,\epsilon(x_{(1)})k) \\
&= x\,h_{(2)}\tilde{\psi}(S^{-1}(h_{(1)})k) \\
&= x\,(h\cdot\tilde{\psi})(k) \\
&= x\,\tilde{\psi}(k\otimes h)
\end{aligned}
$$

Then, φ factors through $\varphi: K \xrightarrow{\mathrm{Id}\otimes\Lambda} K\otimes H \xrightarrow{\tilde{\psi}} L$.

Conversely, given such a factorization of H-module maps

$$\varphi: K \xrightarrow{\mathrm{Id}\otimes\Lambda} K\otimes H \xrightarrow{\tilde{\psi}} L.$$

Let ψ be the $\Bbbk$-linear composition map $K \xrightarrow{\cong} K\otimes 1 \hookrightarrow K\otimes H \xrightarrow{\tilde{\psi}} L$. Then $\varphi = \Lambda\cdot\psi$. Indeed, $\forall k\in K$,

$$
\begin{aligned}
(\Lambda\cdot\psi)(k) &= \Lambda_{(2)}\psi(S^{-1}(\Lambda_{(1)})k) \\
&= \Lambda_{(2)}\tilde{\psi}(S^{-1}(\Lambda_{(1)})k\otimes 1) \\
&= \tilde{\psi}(\Lambda_{(2)}\cdot(S^{-1}(\Lambda_{(1)})k\otimes 1)) \\
&= \tilde{\psi}(\Lambda_{(2)}S^{-1}(\Lambda_{(1)})k\otimes\Lambda_{(3)}) \\
&= \tilde{\psi}(\epsilon(\Lambda_{(1)})k\otimes\Lambda_{(2)}) \\
&= \tilde{\psi}(k\otimes\Lambda) = \varphi(k).
\end{aligned}
$$

□

Relation to categorification

Def. Let H be the graded Hopf algebra $\Bbbk[\partial]/(\partial^p)$, $\deg\partial := 1$. We call H-gmod the category of p-complexes, while H-$\underline{\mathrm{gmod}}$ the homotopy category of p-complexes.

Historically, the first consideration of p-complexes and their homotopy category is due to Mayer (1942). In the definition of simplicial homology theory, the differential $d = \sum_i (-1)^i d_i$ satisfies $d^2 \equiv 0$. Mayer noticed that, if we work over a field of char $p>0$, and set $\partial := \sum_i d_i$. Then $\partial^p \equiv 0$, and there are the corresponding notions of (Mayer) homology. However, Spanier soon found out that Mayer's homology can be recovered from the usual homology groups ($d^2=0$), and thus are less interesting.

Then, why do we care about p-complexes? This is due to the following simple observation.

Lemma (Bernstein-Khovanov). If $H=\Bbbk[\partial]/(\partial^p)$, $\deg(\partial)=1$, then

$$K_0(H\text{-gmod}) \cong \mathbb{Z}[q,q^{-1}]$$

$$K_0(H\text{-}\underline{\text{gmod}}) \cong \mathbb{Z}[q,q^{-1}]/(1+q+\cdots+q^{p-1}) := \mathcal{O}_p.$$

Indeed, K_0 of the homotopy category is generated by the symbol $[\Bbbk]$, subject to the only relation

$$0=[H]=[\Bbbk]+[\Bbbk\{1\}]+\cdots+[\Bbbk\{p-1\}]=(1+q+\cdots+q^{p-1})[\Bbbk].$$

In other words, H-$\underline{\text{gmod}}$ is a categorical interpretation of the ring of the pth cyclotomic integers.

$$\begin{tikzcd}
H\text{-}\underline{\text{gmod}} \arrow[r, "K_0"] & \mathbb{Z}[q,q^{-1}]/(1+\cdots+q^{p-1}) \\
\oplus \quad \otimes \arrow[r, mapsto] & +,\ - \\
{[1]} \arrow[r, mapsto] & -1 \\
\{1\} \arrow[r, mapsto] & q
\end{tikzcd}$$

Here, the homological shift is defined as follows: $M \in H\text{-mod}$, then we have the canonical

$$\varphi_M : M \xrightarrow{\mathrm{Id}\otimes\Lambda} M\otimes H\{-\deg\Lambda\}$$

Then

$$M[1] := \mathrm{coker}(\varphi_M).$$

To utilize this categorical $\mathcal{O}_p$, we need to find interesting "algebras" in $H\text{-}g\underline{\mathrm{mod}}$. Then the Grothendieck groups of these "algebras" will be $\mathcal{O}_p$-modules. As a motivation, note that many interesting algebra objects in the usual homotopy category of chain complexes ($H = \Bbbk[d]/(d^2)$) arise as differential graded algebras (DG algebras).

Def. A p-DG algebra A over a field of $\mathrm{char}\Bbbk = p > 0$ is a graded algebra together with a differential ∂ s.t. $\forall\, a, b \in A$.

$$\partial(ab) = \partial(a)b + a\partial(b),$$
$$\partial^p(a) = 0.$$

More generally, one has the notion of an H-module algebra, which in turn gives rise to an algebra object in $H\text{-}\underline{\mathrm{mod}}$. We refer to the study of homological properties of algebra objects in $H\text{-}\underline{\mathrm{mod}}$ as "hopfological algebra."

In analogy with the usual DG-algebras, we have

$$\begin{tikzcd}
\boxed{(A,\partial)\text{-mod}} \arrow[r, "\text{mod out}", "\text{nullhomotopy}"'] & \boxed{C(A,\partial)} \arrow[r, "\text{inverting}", "\text{qis}"'] & \boxed{D(A,\partial)}
\end{tikzcd}$$

Much of my thesis is about developing some necessary tools in establishing the following result.

Thm. (Khovanov, Qi) The homotopy and derived categories of a p-DG algebra are module-categories over $H\text{-}g\underline{\mathrm{mod}}$. Under taking Grothendieck groups (in some appropriate sense), $K_0(D(A,\partial))$ has the structure of an $\mathcal{O}_p$-module.

In other words, we have the following diagram:

$$\begin{tikzcd}
H\text{-}\underline{\text{gmod}} \times \mathcal{D}(A,\partial) \arrow[r, "\otimes"] \arrow[d, Rightarrow, "K_0"] & \mathcal{D}(A,\partial) \arrow[d, Rightarrow, "K_0"] \\
\mathcal{O}_p \times K_0(A,\partial) \arrow[r, "\text{mult}"] & K_0(A,\partial)
\end{tikzcd}$$

Question: Are there other symmetric monoidal categories whose Grothendieck rings are isomorphic to rings of integers in number fields? Or $\mathbb{Q}$ / $\mathbb{R}$ / $\mathbb{C}$ etc.?

§2 Categorified Quantum $\mathfrak{sl}(2)$ at Prime Roots of Unity 🚩

- Why do we want to categorify $U_q(\mathfrak{sl}_2)$?

- Reshetikhin-Turaev-Witten :

$U_q(\mathfrak{sl}_2)$ is the quantized gauge group of 3d Chern-Simons theory.

- Crane-Frenkel :

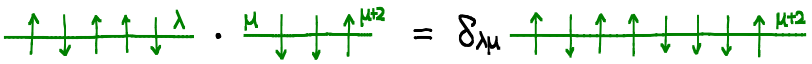
Categorify 3d Chern-Simons to a 4d-TQFT.
$\mathcal{U}_q(\mathfrak{sl}_2)$: quantized 2-gauge group ?

- Quantum $\mathfrak{sl}(2)$ at roots of unity.

We are interested in the idempotented version of $U_q(\mathfrak{sl}_2)$. It is generated over $\mathbb{Z}[q,q^{-1}]$ by pictures of the form

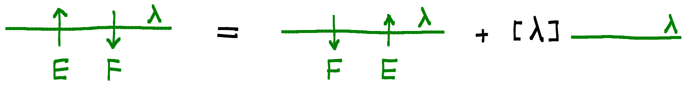
$(\lambda \in \mathbb{Z})$

with the algebra structure

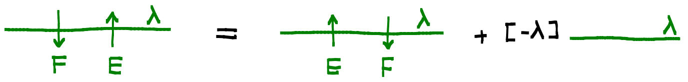


(etc)

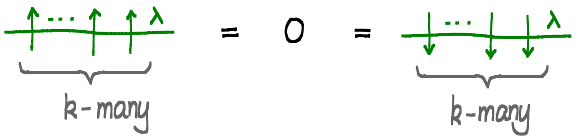
Modulo relations (at a 2k-th root of unity, k odd)



$(\lambda \geq 0)$



$(\lambda \leq 0)$



(Nilpotency relation)

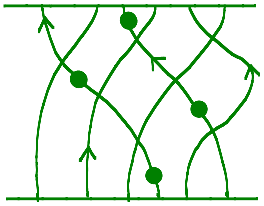
• Categorification of $U_q(\mathfrak{sl}_2)$

Below we present Lauda's diagrammatic calculus for $U_q(\mathfrak{sl}_2)$ at a generic q-value. The rough idea is that:

- Pictures = Isomorphism class / symbol of some modules
- Sum of pictures = symbol of direct sum of modules
- Equalities of pictures = isomorphisms of modules.

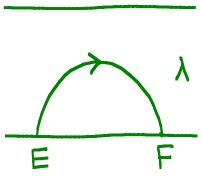
In general, isomorphisms are rare between modules. Instead, study homomorphisms between them. Intuitively, homomorphisms = evolution of pictures, which is not necessarily reversible.

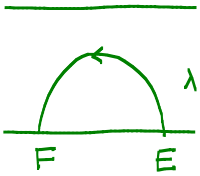
- Maps just among E's (or F's) (Khovanov - Lauda - Rouquier)

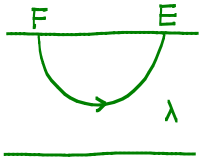


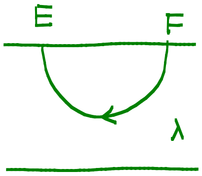
(Nil-Hecke algebra)

- To categorically "Drinfeld-double" E's, Lauda introduces cups and caps



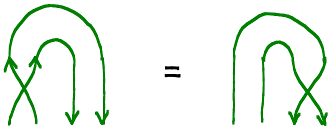


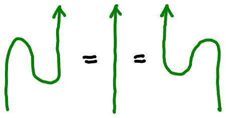




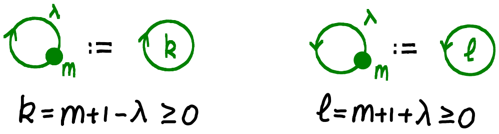
Together with the nilHecke algebra generators, cups and caps satisfy certain relations

(i) Biadjointness E.g.



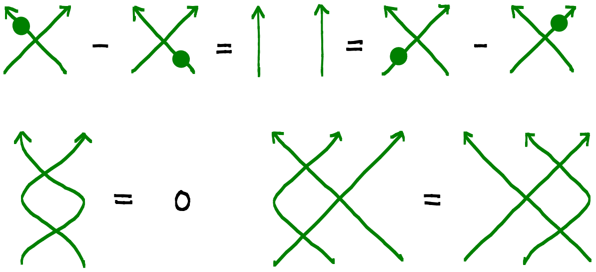


(ii) Bubble positivity (degrees of

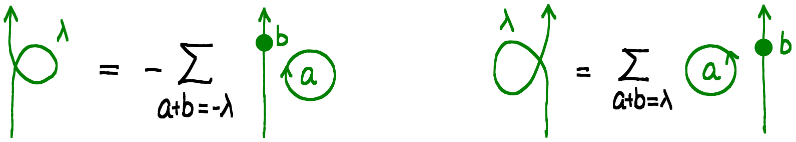


must be ≥ 0)

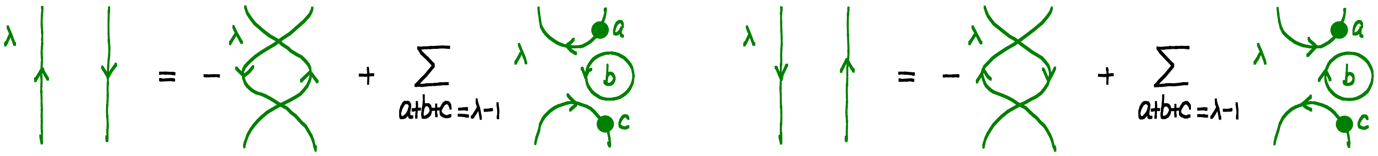
(iii). NilHecke relations



(iv) Reduction to bubbles



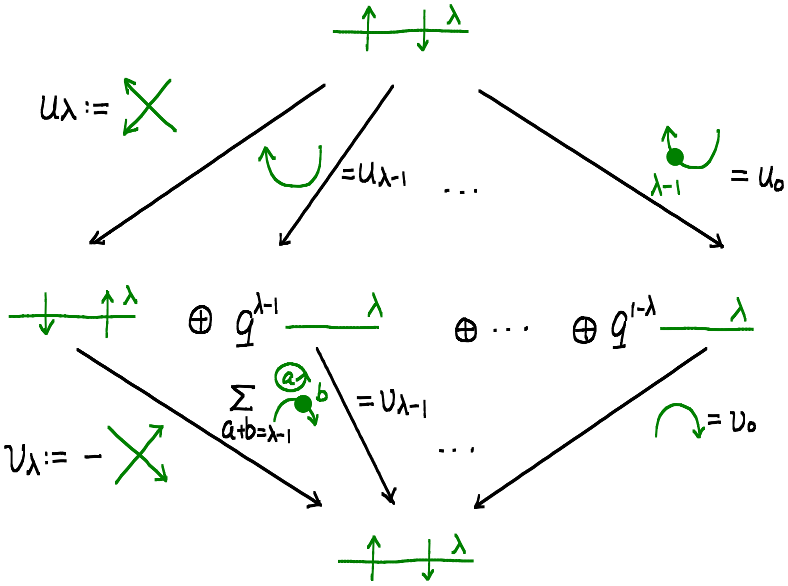
(v). Identity decomposition



Thm. (Lauda) This graphical calculus is non-degenerate and categorifies $\dot{U}_q(\mathfrak{sl}_2)$ at a generic q-value.

Rmk: Lauda's calculus is a 2-dim'l idempotented algebra, i.e. it has two compatible multiplication structures (vertical and horizontal). Such idempotented algebras are also known as a 2-category)

To see the plausibility of this categorification, we consider how $EF1_\lambda$ can "evolve" into $FE1_\lambda \oplus 1_\lambda^{\oplus[\lambda]}$



These elements $\{u_\lambda\}$, $\{v_\lambda\}$ satisfy

$$\begin{cases} u_i v_i u_i = u_i \\ v_i u_i v_i = v_i \\ v_i u_j = 0 \quad (i \neq j) \end{cases},$$

which follows from the identity decomposation relation. Consequently $\{u_i v_i \mid i=0,\cdots,\lambda\}$ form an orthogonal set of idempotents in $\mathrm{End}_{\dot{\mathcal{U}}}(\mathcal{E}\mathcal{F}1_\lambda)$

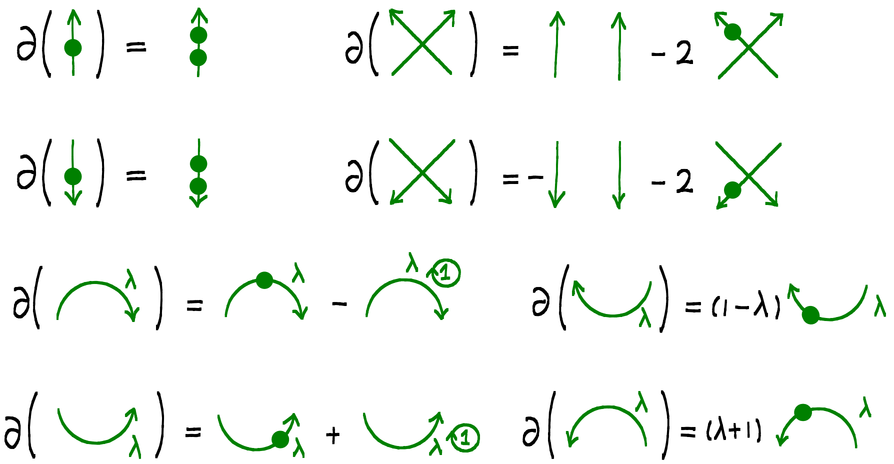
(Factorization of idempotents)

• Enhancing $\dot{\mathcal{U}}$ with a p-differential

As we have learnt from §1, if A is a p-DG algebra, then the derived category of p-DG modules over A is a module-category over the homotopy category of p-complexes.

$$\begin{tikzcd}
\Bbbk[\partial]/(\partial^p)\text{-}\underline{\mathrm{gmod}} \arrow[d, Rightarrow, "K_0"] & \times & \mathcal{D}(A,\partial) \arrow[r, "\otimes"] \arrow[d, Rightarrow, "K_0"] & \mathcal{D}(A,\partial) \arrow[d, Rightarrow, "K_0"] \\
\mathcal{O}_p & \times & K_0(A,\partial) \arrow[r, "\times"] & K_0(A,\partial)
\end{tikzcd}$$

Def. Let $(\mathcal{U},\partial)$ be Lauda's 2-dimensional algebra equipped with the differential ∂-action on generators given by



Lemma. The above ∂ preserves all relations of $\mathcal{U}$, and it is p-nilpotent over a field of characteristic $p>0$.

Thm. (Elias-Q.) The derived module category $\mathcal{D}^b(\mathcal{U},\partial)$ is Karoubian, and it categorifies $\dot{U}_q(\mathfrak{sl}_2)$ at a p-th primitive root of unity.

$$K_0(\mathcal{U},\partial) \cong \dot{U}_{\mathbb{q}}(\mathfrak{sl}_2)$$

- Decomposition v.s. filtration.

In Lauda's abelian categorification, the relations in $\dot{U}_q(\mathfrak{sl}_2)$ are usually realized as different ways of decomposing projective $\mathcal{U}$-modules.

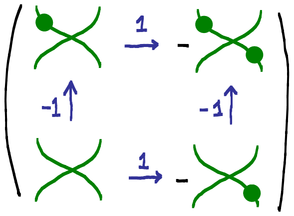
In the realm of triangulated categories, direct sum decompositions are very rare. Instead, a short exact sequence of p-DG $\mathcal{U}$-modules gives rise to a distinguished triangle in $\mathcal{D}(\mathcal{U},\partial)$.

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] & B \arrow[r] \arrow[d, squiggly] & C \arrow[r] & 0 & \text{in} \quad (\mathcal{U},\partial)\text{-mod} \\ A \arrow[r] & B \arrow[r] & C \arrow[r] & A[1] & \text{in} \quad \mathcal{D}(\mathcal{U},\partial) \Longrightarrow [B]=[A]+[C] \in K_0(\mathcal{U},\partial) \end{tikzcd}$$

More generally, a filtered p-DG module $(M, F^\bullet)$ presents M as a convolution (Postnikov tower) of $\mathrm{gr}F^\bullet$.

Example In the nilHecke algebra NH_2:

$NH_2 \cong Sym_2 \cdot$



$\Rightarrow$ $0 \longrightarrow P_2\{1\} \longrightarrow NH_2 \longrightarrow P_2\{1\} \longrightarrow 0$ is a s.e.s. of $(\mathcal{U}, \partial)$-modules.

$\Rightarrow$ In $K_0(\mathcal{U}, \partial)$, $E^2 = [(NH_2, \partial)] = q[P_2] + q^{-1}[P_2] = (q+q^{-1})E^{(2)}$

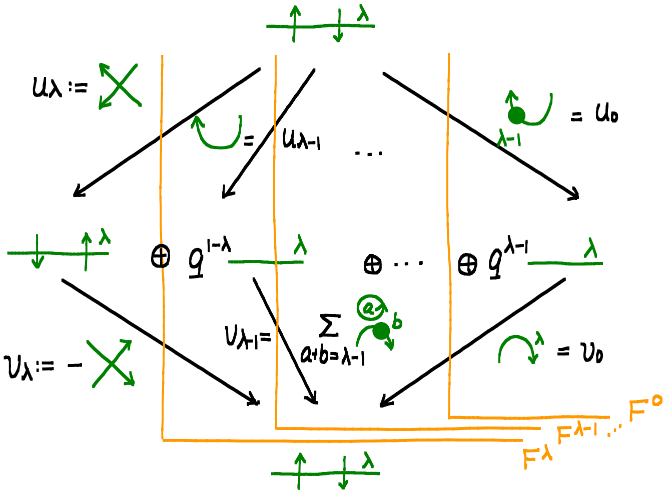
Prop. Let $\{(u_i, v_i) \mid i \in I\}$ be factorization of idempotents in a p-DG algebra R. If there is a total ordering on I such that

$$\begin{cases} v_i\, \partial(u_i) = 0 \\ u_i\, \partial(v_i) \equiv 0 \quad \text{(modulo lower order terms)} \end{cases}$$

Then if $\varepsilon = \sum_{i \in I} u_i v_i$, then the p-DG module $R\varepsilon$ admits a filtration $F^\bullet$ whose subquotients are isomorphic to Rv_iu_i's

Cor. (Fantastic!) In the situation of the Prop. $[R\varepsilon] = \sum_{i \in I} [Rv_iu_i]$.

Cor. Under the differential defined earlier on $\mathcal{U}$, there is a filtration on $\mathcal{E}\mathcal{F}1_\lambda$



• Uniqueness: a small surprise!

Lauda's factorization of idempotents, in general, is not unique.

However, in the presence of a diagrammatically local differential (not necessarily the differential we defined here, but any ∂ compatible with the local relations of $\mathcal{U}$), we have, up to conjugation by diagrammatic automorphisms

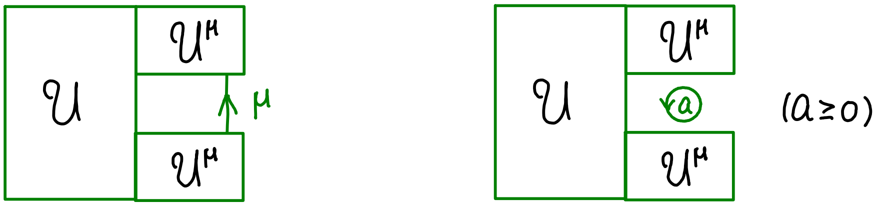
- The differential we defined here is the unique differential such that the modules $\mathcal{E}\mathcal{F}\mathbb{1}_\lambda$ $(\lambda \geq 0)$ admit filtrations whose subquotients are isomorphic to $\mathcal{F}\mathcal{E}\mathbb{1}_\lambda$, $\mathbb{1}_\lambda\{1-\lambda\}, \cdots \mathbb{1}_\lambda\{\lambda-1\}$.

- Lauda's factorization of idempotents is the unique choice that is compatible with the differential. (Fantastic Filtration)

• Application: categorification of simple $U_q(\mathfrak{sl}_2)$-modules

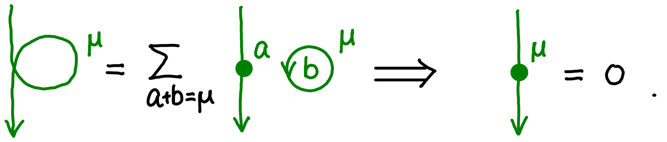
For any weight μ, the subcategory $\mathcal{U}_\mu$, in which pictures have the right most region labelled by μ, forms a natural left $\mathcal{U}$-module category by horizontal composition of pictures.

Def. The cyclotomic quotient category $\mathcal{U}^\mu$ is the quotient of $\mathcal{U}_\mu$ by the ideal



The ideal is clearly closed under ∂, so that $\mathcal{U}^\mu$ inherits the quotient differential. $(\mathcal{U}^\mu, \partial)$ is called the cyclotomic quotient p-DG category.

The name of the category arises from the relation that



The nilHecke algebra $NH_n/(x_n^{\mu})$ is called the level-μ cyclotomic quotient.

Thm (Elias - Q) If $0 \leq \mu \leq p-1$, then $\mathcal{D}(\mathcal{V}^{\mu}, \partial)$ categorifies the highest weight-μ simple $U_q(\mathfrak{sl}_2)$-module.

Sketch of proof. As a quotient category, $(\mathcal{V}^{\mu}, \partial)$ inherits the $\mathcal{E}\mathcal{F}$ and $\mathcal{F}\mathcal{E}$ fantanstic filtration. Thus 1-morphisms in this category can be reduced to those of the form $\mathcal{F}^b\mathcal{E}^a\mathbb{1}_{\mu}$ for $a, b \geq 0$. Since $\mathcal{E}^a\mathbb{1}_{\mu} = 0$ and $\mathcal{F}^b\mathcal{E}^a\mathbb{1}_{\mu}$ is acyclic if $b \geq p$, the category $(\mathcal{V}^{\mu}, \partial)$ is generated by $\{\mathcal{F}^b\mathbb{1}_{\mu},\ 0 \leq b \leq p-1\}$. The endomorphism p-DG algebra

$$\mathrm{End}_{(\mathcal{V}^{\mu},\partial)}(\mathcal{F}^b\mathbb{1}_{\mu}) \cong NH_b/(x_b^{\mu}) \cong \begin{cases} \mathrm{Mat}(b!, H^*(Gr(b,\mu))) & 0 \leq b \leq \mu \\ 0 & \text{otherwise} \end{cases}$$

The result follows from the fact that $\mathrm{Mat}(b!, H^*(Gr(b,\mu)))$ has rank-1 Grothendieck group. □