Algebraic Geometry

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A graduate course taught by
Professor Asie Johan de Jong
Columbia University, Spring 2009
Notes taken by Qi You

§1. Sheaves on spaces

$f: Y \longrightarrow X$, continuous map between topological spaces. $V \subseteq U \subseteq X$ open subsets

Let $\mathcal{F}(U) \triangleq \{s: U \longrightarrow Y$, continuous map s.t. $f \circ s = id_U\}$

Note that the restriction of s to V gives a map between sets:

$$\rho^U_V: \mathcal{F}(U) \longrightarrow \mathcal{F}(V), \quad s \mapsto \rho^U_V(s) \ (\triangleq s|_V \text{ notationally})$$

By definition, we have $(s|_V)|_W = s|_W$, whenever we have inclusion of open sets $W \subseteq V \subseteq U$, or equivalently:

$$\rho^U_U = id \ ; \quad \rho^U_W = \rho^V_W \circ \rho^U_V \qquad ①$$

Def. A presheaf of sets $\mathcal{F}$ assigns every open subset U of X a set $\mathcal{F}(U)$, and every inclusion of open subsets $V \subseteq U$ a map of sets

$$\rho^U_V: \mathcal{F}(U) \longrightarrow \mathcal{F}(V)$$

such that ① holds for any inclusion of open sets $W \subseteq V \subseteq U$.

A presheaf $\mathcal{F}$ is a sheaf if it satisfies the sheaf condition (of sets):

If (i) $U = \bigcup_{i \in I} U_i$ is an open cover of U.

(ii) $s_i \in \mathcal{F}(U_i)$

(iii) $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$

then $\exists!\ s \in \mathcal{F}(U)$ s.t. $s|_{U_i} = s_i$, $\forall i \in I$.

Equivalently,

$$\mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{ij} \mathcal{F}(U_i \cap U_j)$$

$$(s_i) \mapsto (s_i|_{U_i \cap U_j}), \quad (s_i) \mapsto (s_j|_{U_i \cap U_j})$$

is an equalizer diagram.

Remark: Note that by an open covering of U we mean: (i) I is any set, possibly empty; (ii) each U_i is open, possibly empty. Thus for a sheaf $\mathcal{F}$, since $\phi = \bigcup_{i \in \phi} U_i$, $\mathcal{F}(\phi) = \{*\}$, the final object in the category of sets! In particular, if $U = V \sqcup W$, the equalizer diagram $\Rightarrow$ $\mathcal{F}(U) = \mathcal{F}(V) \times_{\mathcal{F}(\phi)} \mathcal{F}(W) = \mathcal{F}(U) \times \mathcal{F}(W)$.

Examples

1). Constant sheaf.

X: a topological space, S: a fixed set.

Define $\mathcal{F}(U) = S$, $\forall U \subseteq X$ open. ($\mathcal{F}(\phi) \triangleq \{*\}$). $\forall V \subseteq U \subseteq X$, $\rho^U_V \triangleq id_S$

- Is $\mathcal{F}$ a sheaf on X?
- Answer: no!

But there is such a sheaf that $\mathcal{F}(U) = S$ whenever $\phi \neq U$ is connected, denoted $\underline{S}_X$: give S the discrete topology, then $\pi: X \times S \longrightarrow X$ is continuous. $\forall U \neq \phi$ $\underline{S}_X(U) \triangleq \{$locally constant sections of $\pi: X \times S \longrightarrow X\}$.

For instance, if X is a "reasonable" topological space, the cohomology of $\underline{\mathbb{Z}}_X$ is isomorphic to the singular cohomology of X with $\mathbb{Z}$ coefficients.

2). X: a topological space: $\mathcal{C}^0_X \triangleq$ the sheaf of real valued functions on X.

3). X: differentiable manifold. $\mathcal{C}^\infty_X \triangleq$ the sheaf of smooth functions on X.

Notation: X: a topological space.
$\mathcal{P}Sh(X)$: presheaf of sets on X;
$\mathcal{S}h(X)$: sheaf of sets on X;
$\mathcal{P}\mathcal{A}b(X)$: presheaf of abelian groups on X;
$\mathcal{A}b(X)$: sheaf of abelian groups on X;
$\mathcal{M}od(\mathcal{O}_X)$: sheaf of $\mathcal{O}_X$-modules on X.

Def. A ringed space is a pair $(X, \mathcal{O}_X)$, where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings.

Def: $(X, \mathcal{O}_X)$: ringed space. A sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ is given by a sheaf of abelian groups $\mathcal{F}$ endowed with a map of sheaves:

$$\mathcal{O}_X \times \mathcal{F} \longrightarrow \mathcal{F}$$

s.t. $\forall U \subseteq X$ open, $\mathcal{O}_X(U) \times \mathcal{F}(U) \longrightarrow \mathcal{F}(U)$ makes $\mathcal{F}(U)$ an $\mathcal{O}_X(U)$-module.
($\mathcal{O}_X \times \mathcal{F}$ is the sheaf of sets: $(\mathcal{O}_X \times \mathcal{F})(U) \triangleq \mathcal{O}_X(U) \times \mathcal{F}(U)$.)

Question: $\mathcal{F}, \mathcal{G}$: sheaf of $\mathcal{O}_X$-modules. How to define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$?

Answer: $U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U)$, works (only) in $\mathcal{P}Mod(\mathcal{O}_X)$.

- Adjoint functors

$\mathcal{C} \underset{v}{\overset{u}{\rightleftarrows}} \mathcal{D}$: functors between two categories.

Def. u, v are called adjoint if $Mor_{\mathcal{C}}(X, vY) \cong Mor_{\mathcal{D}}(uX, Y)$, $\forall X \in Ob(\mathcal{C})$, $Y \in Ob(\mathcal{D})$, and the isomorphism is bi-functorial in X and Y. If so, we say u is a left adjoint of v.

Examples:

(a). Consider the functor from the category of abelian groups to the category of sets $u: \mathcal{A}b \longrightarrow \mathcal{S}ets$, $M \mapsto M$ as a set (forgetful functor).

$F: \mathcal{S}ets \longrightarrow \mathcal{A}b$, $F(S) =$ the free abelian group on $S = \bigoplus_{s \in S} \mathbb{Z} \cdot [s]$.

Then $Mor_{\mathcal{S}ets}(S, uM) \cong Mor_{\mathcal{A}b}(F(S), M)$.

(b). $R \longrightarrow S$: a ring map.

Then $Hom_R(N, M_R) \cong Hom_S(S \otimes_R N, M)$, i.e. $Mod_R \underset{(\cdot)_R}{\overset{S\otimes_R}{\rightleftarrows}} Mod_S$ are adjoint functors.

(c). Sheafification: $PSh(X) \longrightarrow Sh(X)$, $\mathcal{F} \longmapsto \mathcal{F}^{\#}$, adjoint to the inclusion functor $PSh(X) \underset{i}{\overset{(\cdot)^{\#}}{\rightleftarrows}} Sh(X)$: $Mor_{PSh(X)}(\mathcal{F}, i\mathcal{G}) = Mor_{Sh(X)}(\mathcal{F}^{\#}, \mathcal{G})$. We shall describe $(\cdot)^{\#}$ in detail below.

The sheafification $\mathcal{F}^{\#}$ of $\mathcal{F}$ will come with a map of presheaves $f_c: \mathcal{F} \longrightarrow i\mathcal{F}^{\#}$ s.t. $\forall$ any morphism of presheaves $\alpha: \mathcal{F} \longrightarrow i\mathcal{G}$, $\exists!$ factorization

$$\begin{tikzcd} \mathcal{F} \arrow[r] \arrow[d, "f_c"'] & i\mathcal{G} \\ i\mathcal{F}^{\#} \arrow[ur, dashed, "\exists!\,\alpha^{\#}"'] & \end{tikzcd}$$

- Sheafification

Idea: Force the sheaf condition to hold.

$U \subseteq X$ an open set, $\mathcal{U}: \{U_i\}_{i \in I}$ an open covering of U.

Def. Let $\mathcal{F}$ be a presheaf of sets on X.

$$\check{H}^0(\mathcal{U},\mathcal{F}) \triangleq \ker\left(\prod_{i\in I}\mathcal{F}(U_i) \rightrightarrows \prod_{i,j\in I}\mathcal{F}(U_i\cap U_j)\right).$$
$$= \{(s_i)_{i\in I} \mid s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j}\}$$

Lemma. There is a natural map $\mathcal{F}(U) \longrightarrow \check{H}^0(\mathcal{U},\mathcal{F})$. $s \mapsto (s|_{U_i})$. If $\mathcal{F}$ is a sheaf, then it's an isomorphism. □

Def. Given a presheaf, we define $\mathcal{F}^+$ to be the presheaf with value:

$$\mathcal{F}^+(U) \triangleq \operatorname{colim}_{\mathcal{U}} \check{H}^0(\mathcal{U},\mathcal{F}).$$

The restriction maps are given below.

Def. A covering $\mathcal{U} = \bigcup_{i\in I} U_i$ is a refinement of $\mathcal{U}' = \bigcup_{i'\in I'} U'_{i'}$ iff $\exists\, \alpha: I \longrightarrow I'$ s.t. $U_i \subseteq U'_{\alpha(i)}$, $\forall\, i \in I$.

Given such an α, we may define $\check{H}^0(\mathcal{U}',\mathcal{F}) \longrightarrow \check{H}^0(\mathcal{U},\mathcal{F})$ by the rule:

$$(s_{i'})_{i'\in I'} \longmapsto (s_{\alpha(i)}|_{U_i})_{i\in I}.$$

Then this is a well-defined map and is independent of choices of α. (Indeed, $U_i \subseteq U_{\alpha(i)}$, $U_i \subseteq U_{\beta(i)} \Rightarrow U_i \subseteq U_{\alpha(i)} \cap U_{\beta(i)}$, thus $s_{\alpha(i)}|_{U_i} = (s_{\alpha(i)}|_{U_{\alpha(i)}\cap U_{\beta(i)}})|_{U_i} = (s_{\beta(i)})|_{U_i}$).

Observation: Any two open coverings $\mathcal{U}_1, \mathcal{U}_2$ of U have a common refinement. Then the partially ordered set (POSet) is also directed. Thus $\operatorname{colim}_{\mathcal{U}}$ is a directed colimit. Thus $\mathcal{F}^+(U) = \coprod \check{H}^0(\mathcal{U},\mathcal{F})/\sim$, where $s_i \in \check{H}^0(\mathcal{U}_i,\mathcal{F})$, $i=1,2$, are called equivalent if $\exists$ common refinement $\mathcal{U}$ s.t. the images of s_i in $\check{H}^0(\mathcal{U},\mathcal{F})$ are the same.

Restriction mappings.

If $V \subseteq U \subseteq X$ are open subsets. $\mathcal{U}: U = \bigcup_{i\in I} U_i$, $s = (s_i)_{i\in I} \in \check{H}^0(\mathcal{U},\mathcal{F})$, then:

$\mathcal{V}: V = \bigcup_{i\in I} U_i \cap V$, $s|_V \triangleq (s_i|_{U_i\cap V}) \in \check{H}(\mathcal{V},\mathcal{F})$.

Note that there is a canonical map of presheaves: $\theta: \mathcal{F} \longrightarrow \mathcal{F}^+$. By regarding U as an open cover of itself (only one open set, $\mathcal{U} = U$):

$$\begin{aligned} \mathcal{F}(U) &\longrightarrow \mathcal{F}^+(U) \\ s &\longmapsto [s] \in \check{H}^0(\mathfrak{U}, \mathcal{F})/\sim . \end{aligned}$$

Def. A presheaf $\mathcal{F}$ is called seperated if for all open covering $\mathfrak{U}$: $U = \bigcup_{i\in I} U_i$ the map $\mathcal{F}(U) \longrightarrow \prod_{i\in I} \mathcal{F}(U_i)$ is injective.

Example: Let $X = \{x, y\}$ with the discrete topology. Define a presheaf $\mathcal{F}$ as follows:

$\mathcal{F}(\phi) = \{0\}$ $\quad \mathcal{F}(\{x\}) = \mathbb{Z}/2$, $\mathcal{F}(\{y\}) = \mathbb{Z}/2$, $\mathcal{F}(X) = \mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$

$\mathcal{F}(X) \to \mathcal{F}(\{x\})$ is given by projection onto the first factor;

$\mathcal{F}(X) \to \mathcal{F}(\{y\})$ is given by projection onto the second factor.

Then $\mathcal{F}$ is not seperated as $\mathcal{F}(\{x,y\}) \to \mathcal{F}(\{x\}) \times \mathcal{F}(\{y\})$ is not injective.

Thm. Let $\mathcal{F}$ be a presheaf. Then:

(i). $\mathcal{F}^+$ is seperated

(ii). If $\mathcal{F}$ is seperated, then $\mathcal{F}^+$ is a sheaf

(iii). If $\mathcal{F}$ is a sheaf, then $\mathcal{F}^+ = \mathcal{F}$.

(iv). The construction $\mathcal{F} \rightsquigarrow (\mathcal{F} \xrightarrow{\theta} \mathcal{F}^+)$ is functorial in $\mathcal{F}$. Moreover, for any $\mathcal{G}$ a sheaf, and morphism of presheaves $\varphi: \mathcal{F} \longrightarrow \mathcal{G}$, $\exists!$ factorization:

$$\begin{tikzcd} \mathcal{F} \arrow[rr, "\varphi"] \arrow[dr, "\theta"'] & & \mathcal{G} \\ & \mathcal{F}^+ \arrow[ur, "\exists!\varphi^+"'] & \end{tikzcd}$$

Proof of Thm.

(ii). $\mathcal{F}$ is a seperated presheaf. Let:

(1). $U \subseteq X$ be open, $\mathfrak{U}$: $U = \bigcup_{i\in I} U_i$ be an open cover.

(2). $s_i \in \mathcal{F}^+(U_i)$, i.e. $\mathfrak{U}_i$: $U_i = \bigcup_{k\in K_i} U_{ik}$, and $s_{ik} \in \mathcal{F}(U_{ik})$, $s_{ik}|_{U_{ik}\cap U_{ik'}} = s_{ik'}|_{U_{ik}\cap U_{ik'}}$.

(3). $s_i|_{U_i\cap U_j} = s_j|_{U_i\cap U_j} \in \mathcal{F}^+(U_i \cap U_j)$

Now $s_i|_{U_i\cap U_j}$ is given by $(s_{ik}|_{U_{ik}\cap U_j})_{k\in K_i}$; $s_j|_{U_i\cap U_j}$ given by $(s_{jk}|_{U_i\cap U_{jk}})_{k\in K_j}$. As elements of $\mathcal{F}^+(U_i\cap U_j)$, they are equal. We need to show that $\exists!$ section $s \in \mathcal{F}^+(U)$, whose restriction to U_i is equal to $s_i \in \mathcal{F}^+(U_i)$

Consider the refinement $\tilde{\mathfrak{U}}$: $U = \bigcup_{i\in I, k\in K_i} U_{ik}$ and elements $s_{ik} \in \mathcal{F}(U_{ik})$. We can check that $s_{ik}|_{U_{ik}\cap U_{jk'}} = s_{jk'}|_{U_{ik}\cap U_{jk'}}$. Then it follows that (s_{ik}) defines a section in $\check{H}^0(\tilde{\mathfrak{U}}, \mathcal{F})$. The uniqueness follows from $\mathcal{F}$ being seperated.

Indeed, if $i=j$, we are done by assumption (2)

If $i \neq j$, $\tilde{\mathfrak{U}}_{ij}: U_i \cap U_j = \bigcup_{k\in K_i, k'\in K_j} U_{ik}\cap U_{jk'}$ is a common refinement of the open covers that define $s_i|_{U_i\cap U_j}$ & $s_j|_{U_i\cap U_j}$, namely $\mathfrak{U}_{ij}': U_i\cap U_j = \bigcup_{k\in K_i} U_{ik}\cap U_j$ and $\mathfrak{U}_{ij}'': U_{ij} = \bigcup_{k'\in K_j} U_i \cap U_{jk'}$. Thus $(s_{ik}|_{U_{ik}\cap U_{jk'}})$, $(s_{jk'}|_{U_{ik}\cap U_{jk'}})$ are two sections of $\check{H}^0(\tilde{\mathfrak{U}}_{ij}, \mathcal{F})$, which define the same element under further refinements, by our assumption (3). It follows from the next lemma that $(s_{ik}|_{U_{ik}\cap U_{jk'}}) = (s_{jk'}|_{U_{ik}\cap U_{jk'}}) \in \check{H}^0(\mathfrak{U}_{ij}, \mathcal{F})$.

Lemma. If $\mathcal{F}$ is seperated, then all maps coming from refinements are injective. i.e. $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$, then $\check{H}^0(\mathfrak{U}, \mathcal{F}) \longrightarrow \check{H}^0(\mathfrak{U}', \mathcal{F})$ is injective.

Pf: Note that $\mathfrak{U}'': U = \bigcup_{i\in I, i'\in I'} U_i \cap U_{i'}'$ is another refinement of $\mathfrak{U}$, and moreover $\mathfrak{U}''$, $\mathfrak{U}'$ are refinements of each other, since $\mathfrak{U}'$ is a refinement of $\mathfrak{U}$. It follows that $\check{H}^0(\mathfrak{U}', \mathcal{F}) = \check{H}^0(\mathfrak{U}'', \mathcal{F})$.

Now given two sections $(s_i)_{i\in I}$, $(t_i)_{i\in I} \in \check{H}^0(\mathfrak{U}, \mathcal{F})$, which have the same image in $\check{H}(\mathfrak{U}'', \mathcal{F})$, then $s_i, t_i \in \mathcal{F}(U_i)$ have the same image in $\prod_i \mathcal{F}(U_i \cap U_{i'}')$ $\Rightarrow$ $s_i = t_i \in \mathcal{F}(U_i)$, by seperatedness of $\mathcal{F}$. $\square$

(iii). follows from our definition of $\mathcal{F}^+$.

(iv). The functoriality follows from definition. For the second statement, note that by functoriality, we have:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\varphi"] \arrow[d] & \mathcal{G} \arrow[d, "\cong"] \\ \mathcal{F}^+ \arrow[r] & \mathcal{G}^+ \end{tikzcd}$$

The result follows.

(i). Given $s, s' \in \mathcal{F}^+(U)$, whose restriction to an open cover $\mathfrak{U}: U = \bigcup_{i\in I} U_i$ are the same, we need to show that, $s = s'$ in $\mathcal{F}^+(U)$. In each $\mathcal{F}^+(U_i)$, $s|_{U_i} = s'|_{U_i} \Rightarrow$ $\exists$ open refinement $\mathfrak{U}_i: U_i = \bigcup_{k\in K_i} U_{ik}$, s.t. $s|_{U_{ik}} = s'|_{U_{ik}} \in \mathcal{F}(U_{ik})$. Since $\hat{\mathfrak{U}}$: $U =$

$\bigcup_{i\in I, k\in K_i} U_{ik}$ is also an open cover of U and refines $\mathfrak{U}$, it follows that $s=s'$. $\square$

Def. We call $\mathcal{F}^\# \triangleq (\mathcal{F}^+)^+$, with the canonical morphism $\mathcal{F} \xrightarrow{\theta} \mathcal{F}^+ \xrightarrow{\theta^+} \mathcal{F}^\#$ the sheafification of $\mathcal{F}$. It is the unique sheaf s.t.

$$\mathrm{Mor}_{PSh(X)}(\mathcal{F}, \mathcal{G}) = \mathrm{Mor}_{Sh(X)}(\mathcal{F}^\#, \mathcal{G})$$

(Indeed, the last statement follows from the commutative diagram:

$$\begin{tikzcd} \mathcal{F} \arrow[r, "\theta"] \arrow[d] & \mathcal{F}^+ \arrow[r, "\theta^+"] \arrow[d] & \mathcal{F}^\# \arrow[d] \\ \mathcal{G} \arrow[r, "\cong"] & \mathcal{G}^+ \arrow[r, "\cong"] & \mathcal{G}^\# \end{tikzcd}$$

).

Why sheafification?

- There are operations on sheaves whose "direct" outcome is not a sheaf.

For instance, ($X=\mathbb{C}$ with $\mathcal{O}_X \cong \underline{\mathbb{Z}}_X$, $\mathcal{F}, \mathcal{G}$ are sky-scraper sheaves supported at distinct points of X, then $\mathcal{F}(X)\otimes_{\mathcal{O}_X(X)}\mathcal{G}(X) \cong \mathcal{F}(X)\otimes_{\mathbb{Z}}\mathcal{G}(X)$. But choosing a cover $\mathfrak{U}: \mathbb{C}=U_p\cup U_q$ where $U_p=\mathbb{C}\setminus\{q\}$, $U_q=\mathbb{C}\setminus\{p\} \Rightarrow \mathcal{F}(U_p)\otimes_{\mathcal{O}_X(U_p)}\mathcal{G}(U_p)\cong 0$, and so is $\mathcal{F}(U_q)\otimes_{\mathcal{O}_X(U_q)}\mathcal{G}(U_q)=0$

$\Rightarrow \mathcal{F}(X)\otimes_{\mathcal{O}_X(X)}\mathcal{G}(X) \not\hookrightarrow (\mathcal{F}(U_p)\otimes_{\mathcal{O}_X(U_p)}\mathcal{G}(U_p))\times(\mathcal{F}(U_q)\otimes_{\mathcal{O}_X(U_q)}\mathcal{G}(U_q))$.

Def. (Stalk). $x\in X$ a point, $\mathcal{F}$ a presheaf on X. The stalk of $\mathcal{F}$ at x

$$\mathcal{F}_x \triangleq \operatorname{colim}_{x\in U} \mathcal{F}(U)$$

where U runs through open neighborhoods of x in X.

Equivalently, $\mathcal{F}_x = \{(U,s) \mid s\in\mathcal{F}(U),\ U \text{ open}\}/\sim$, where $(U,s)\sim(U',s')$ iff $\exists\, W$ open, $W\subseteq U\cap U'$ s.t. $s|_W = s'|_W \in \mathcal{F}(W)$.

Rmk: The set of open neighborhoods of x in X forms a directed POSet. It follows that if $\mathcal{F}$ is a presheaf of abelian groups (rings, $\mathcal{O}_X$-modules), then $\mathcal{F}_x$ is an abelian group. (ring, $\mathcal{O}_{X,x}$-module).

Fact: $\mathcal{F}^\#_x = \mathcal{F}_x$ ($=\mathcal{F}^+_x$), and $\mathcal{F}\longrightarrow\mathcal{F}_x$ is functorial.

Lemma: Let $\varphi: \mathcal{F}\longrightarrow\mathcal{G}$ be a morphism of sheaves. Then:

(a). φ is an isomorphism iff $\forall x\in X$, $\varphi_x: \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is an isomorphism.

iff $\forall U\subseteq X$ open, $\varphi: \mathcal{F}(U) \xrightarrow{\sim} \mathcal{G}(U)$

(b). φ is a monomorphism iff $\forall x\in X$, $\varphi_x: \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is injective.

iff $\forall U\subseteq X$ open, $\mathcal{F}(U) \hookrightarrow \mathcal{G}(U)$.

(c). φ is an epimorphism iff $\forall x\in X$, $\varphi_x: \mathcal{F}_x \longrightarrow \mathcal{G}_x$ is surjective.

iff $\forall U\subseteq X$ open and $s\in \mathcal{G}(U)$, $\exists \mathcal{U}: U=\bigcup_{i\in I} U_i$

s.t. $s|_{U_i}$ lifts to $\mathcal{F}(U_i)$ for each $i\in I$.

Pf: (a). "$\Leftarrow$" is easy. "$\Rightarrow$":

Pick $U\subseteq X$ open, we need to show that $\varphi: \mathcal{F}(U) \xrightarrow{\sim} \mathcal{G}(U)$

Injectivity: $s, s'\in \mathcal{F}(U)$ s.t. $\varphi(s)=\varphi(s')$. Then the images of $\varphi(s)$ and $\varphi(s')$ in $\mathcal{G}_x$ are same, $\forall x\in U$. By injectivity of φ_x, $s_x = s'_x \in \mathcal{F}_x$, $\forall x\in U$. By definition of stalk, we know that $s|_{U_x} = s'|_{U_x}$ for some open neighborhood U_x of x. But $\mathcal{U}: U=\bigcup_{x\in U} U_x$ forms an open cover of U. $\Rightarrow s=s'$ by the sheaf property.

Surjectivity: Pick $t\in \mathcal{G}(U)$. By assumption, $\forall x\in U$, $\exists s\in \mathcal{F}(\tilde{U}_x)$, s.t. $\varphi(s_x) = t_x \in \mathcal{G}_x$. By definition of stalk, $\varphi(s)$ and t agrees on some open neighborhood $U_x \subseteq \tilde{U}_x$. If s' is another such section in $\mathcal{F}(U_{x'})$, then $\varphi(s'_x) = \varphi(s')_x = \varphi(s)_x = \varphi(s_x)$, $\forall x\in U_{x'}\cap U_x$. By injectivity above, $s'=s$ on $U_x\cap U_{x'}$. By sheaf property $\{(U_x, s)\}$ glues to be a section in $\mathcal{F}(U)$.

(2), (3). Note that in the category of sheaves, we have fiber products and push-outs. $\mathcal{F}\longrightarrow\mathcal{G}$ is monomorphism iff $\mathcal{F} \xrightarrow{\cong} \mathcal{F}\times_{\mathcal{G}}\mathcal{F}$ iff $\mathcal{F}_x \cong \mathcal{F}_x\times_{\mathcal{G}_x}\mathcal{F}_x$ (by part (i)) iff $\mathcal{F}_x \longrightarrow \mathcal{G}_x$ is injective. (Taking stalks commutes with taking fiber products and push-outs). $\square$

Lemma. $PAb(X)$, $Ab(X)$, $Mod(\mathcal{O}_X)$ are abelian categories. Given $\varphi: \mathcal{F}\longrightarrow\mathcal{G}$ in these categories, we have:

$\ker\varphi: U \longmapsto \ker(\mathcal{F}(U)\longrightarrow\mathcal{G}(U))$ lies in each of them

$\mathrm{Pcoker}\varphi: U \longmapsto \mathrm{coker}(\mathcal{F}(U)\longrightarrow\mathcal{G}(U))$ lies in $PAb(X)$

$\mathrm{coker}\varphi = (\mathrm{Pcoker}\varphi)^{\#}$ lies in $Ab(X)$ or $Mod(\mathcal{O}_X)$

Furthermore, taking ker, coker commutes with taking stalks, and

$$0\longrightarrow\mathcal{F}\longrightarrow\mathcal{G}\longrightarrow\mathcal{H}\longrightarrow 0$$

is a s.e.s. in $Ab(X)$ or $Mod(\mathcal{O}_X)$ iff $\forall x \in X$

$$0 \longrightarrow \mathcal{F}_x \longrightarrow \mathcal{G}_x \longrightarrow \mathcal{H}_x \longrightarrow 0$$

is s.e.

Example: $X \cong S^1 = \mathbb{R}/\mathbb{Z}$. Let C^∞_X be the sheaf of C^∞-functions on X. Then we have a s.e.s.

$$0 \longrightarrow \mathbb{Z}_X \longrightarrow C^\infty_X \longrightarrow Q \longrightarrow 0$$

where Q is the sheaf of C^∞-functions valued in $\mathbb{R}/\mathbb{Z} \cong S^1$. In particular, we can treat $\mathrm{id}_{S^1} \in Q(X)$ (which turns out to generate $H^1(X, \mathbb{Z}_X)$).

Skyscraper sheaves

Def. $x \in X$, S: set. The skyscraper sheaf $i_{x*}(S)$ is defined as the sheaf of sets:

$$i_{x*}(S)(U) \triangleq \begin{cases} S & \text{if } x \in U \\ \{*\} & \text{if } x \notin U \end{cases}$$

with the obvious restriction maps.

Lemma. $$(i_{x*}(S))_y = \begin{cases} S & y \in \overline{\{x\}} \\ \{*\} & y \notin \overline{\{x\}} \end{cases}$$ $\square$

Note that if S is an abelian group, then $i_{x*}(S)$ is an abelian sheaf. (so is rings, monoids etc). If $(X, \mathcal{O}_X)$ is a ringed space, $x \in X$ and S is an $\mathcal{O}_{X,x}$-module, then $i_{x*}(S)$ is in a natural way an $\mathcal{O}_X$-module. Indeed:

$$\mathcal{O}_X(U) \times i_{x*}(S)(U) = \begin{cases} \mathcal{O}_X(U) \times S \longrightarrow \mathcal{O}_{X,x} \times S \longrightarrow S & \text{if } x \in U \\ \mathcal{O}_X(U) \times \{0\} \longrightarrow \{0\} & \text{if } x \notin U \end{cases}$$

Adjointness property: $\mathrm{Mor}_{PSh(X)}(\mathcal{F}, i_{x*}(S)) \cong \mathrm{Mor}_{sets}(\mathcal{F}_x, S)$. This also works in the category of $Sh(X)$, $PAb(X)$, $Ab(X)$ and $Mod(\mathcal{O}_X)$.

Aside: This explains why $\mathcal{F}^\#_x = \mathcal{F}_x$ for $\mathcal{F} \in PSh(X)$:

$$\begin{array}{c}\mathrm{Mor}_{psh(X)}(\mathcal{F}, i_{x*}(S)) = \mathrm{Map}(\mathcal{F}_x, S) \\ \| \\ \mathrm{Mor}_{sh(X)}(\mathcal{F}^{\#}, i_{x*}(S)) = \mathrm{Mor}_{psh(X)}(\mathcal{F}^{\#}, i_{x*}(S)) = \mathrm{Map}(\mathcal{F}^{\#}_x, S).\end{array}$$

What's really going on here? $i : \{x\} \hookrightarrow X$ is a continuous map, then the skyscraper is the "push-forward" of the constant sheaf $\underline{S}_{\{x\}}$, (to be defined below) and is adjoint to the "pull-back" i^{-1} (taking stalks)!

Tensor products

Let $(X, \mathcal{O}_X)$ be a ringed space. If $\mathcal{F}$ and $\mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules, we define the presheaf of $\mathcal{O}_X$-modules $\mathcal{F} \otimes_{p\mathcal{O}_X} \mathcal{G}$:

$$\mathcal{F} \otimes_{p\mathcal{O}_X} \mathcal{G}(U) \triangleq \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

with the obvious restriction maps.

Def. The tensor product of $\mathcal{F}$ and $\mathcal{G}$ is the sheaf of $\mathcal{O}_X$-modules:

$$\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G} \triangleq (\mathcal{F} \otimes_{p\mathcal{O}_X} \mathcal{G})^{\#}$$

Fact: $(\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})_x = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \mathcal{G}_x$

Recall that if we have a map of rings: $A \longrightarrow B$, M an A-module, N a B-module, then $\mathrm{Hom}_B(M \otimes_A B, N) = \mathrm{Hom}_A(M, {}_A N)$.

If X is a topological space and $\mathcal{O}_1 \longrightarrow \mathcal{O}_2$ is a map of sheaves of rings, then $\mathcal{F}$ an $\mathcal{O}_1$-module, $\mathcal{G}$ an $\mathcal{O}_2$-module, then

$$\mathrm{Hom}_{\mathcal{O}_1}(\mathcal{F}, {}_{\mathcal{O}_1}\mathcal{G}) = \mathrm{Hom}_{\mathcal{O}_2}(\mathcal{F} \otimes_{\mathcal{O}_1} \mathcal{O}_2, \mathcal{G})$$

Examples

(1). If $X = \{p, q\}$ with discrete topology. $\mathcal{O}_X = \underline{\mathbb{C}}_X$, $U = \{p\}$, $V = \{q\}$, then $\mathcal{O}_X(U) = \mathbb{C}$, $\mathcal{O}_X(V) = \mathbb{C}$. What's an $\mathcal{O}_X$-module?

Answer: $\mathcal{F}(U)$ is a $\mathbb{C}$ vector space K_1 ; $\mathcal{F}(V)$ is a $\mathbb{C}$-vector space K_2
$\mathcal{F}(\phi) = \{0\}$, $\mathcal{F}(X) = K_1 \oplus K_2$ (but now as a $\mathbb{C} \oplus \mathbb{C}$-module!)

Similarly take $\mathcal{G}$ to be another $\mathcal{O}_X$-module, then

$$\mathcal{F} \otimes_{p\mathcal{O}_X} \mathcal{G}(U) \times \mathcal{F} \otimes_{p\mathcal{O}_X} \mathcal{G}(V) = (K_1 \otimes_{\mathbb{C}} L_1) \times (K_2 \otimes_{\mathbb{C}} L_2)$$

and $\mathcal{F} \otimes_{p\mathcal{O}_X} \mathcal{G}(X) = (K_1 \oplus K_2) \otimes_{\mathbb{C} \oplus \mathbb{C}} (L_1 \oplus L_2) = (K_1 \otimes_{\mathbb{C}} L_1) \oplus (K_2 \otimes_{\mathbb{C}} L_2)$

Thus in this case, the presheaf $\mathcal{F}\otimes_{p\mathcal{O}_X}\mathcal{G}$ is a sheaf already.

(2). $X=\mathbb{R}$, $\mathcal{O}_X=\mathbb{Z}_X$, $\mathcal{F}=i_{0*}(\mathbb{Z})\oplus i_{1*}(\mathbb{Z})$

Then $\mathcal{O}_{X,0}=\mathbb{Z}$, $\mathcal{O}_{X,1}\cong\mathbb{Z}$, $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{F}\cong\mathcal{F}$ (by looking at stalks)

thus $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{F}(X)\cong\mathbb{Z}\oplus\mathbb{Z}$. However, $\mathcal{F}\otimes_{p\mathcal{O}_X}\mathcal{F}(X)=(\mathbb{Z}\oplus\mathbb{Z})\otimes_{\mathbb{Z}}(\mathbb{Z}\oplus\mathbb{Z})\cong\mathbb{Z}^4$

In this case, $\mathcal{F}\otimes_{p\mathcal{O}_X}\mathcal{F}$ is not a sheaf.

Functoriality

$f: X\longrightarrow Y$ continuous map between topological spaces.

Def. $f_*: \mathcal{P}Sh(X)\longrightarrow\mathcal{P}Sh(Y)$ by the rule:

$$(f_*\mathcal{F})(V)\triangleq\mathcal{F}(f^{-1}(V))$$

with the obvious restriction maps comming from $\mathcal{F}$. The same def. works in Sh, $\mathcal{P}$Ab, Ab, Rings, ---, by the following:

Lemma. If $\mathcal{F}$ is a sheaf, then so is $f_*\mathcal{F}$.

Pf: If $V=\bigcup_{j\in I}V_j$ is an open covering of $V\overset{\text{open}}{\subseteq}Y$, then $f^{-1}(V)=\bigcup_{j\in I}f^{-1}(V_j)$ is an open covering. The sheaf condition of $\mathcal{F}$

$$\Rightarrow\quad \mathcal{F}(f^{-1}(V))=\ker\big(\textstyle\prod_{j\in I}\mathcal{F}(f^{-1}(V_j))\rightrightarrows\prod_{i,j\in I}\mathcal{F}(f^{-1}(V_i)\cap f^{-1}(V_j))\big)$$

$$\Rightarrow\quad f_*\mathcal{F}(V)=\ker\big(\textstyle\prod_{j\in I}f_*\mathcal{F}(V_j)\rightrightarrows\prod_{i,j\in I}f_*\mathcal{F}(V_i\cap V_j)\big)$$ $\square$

Next, we define the adjoint of f_*: $\mathcal{P}Sh(X)\underset{f_*}{\overset{?}{\rightleftarrows}}\mathcal{P}Sh(Y)$. Note that previously the adjoint of i_{x*} is like taking the stalk, thus this must involve colimit:

Def. Given a presheaf $\mathcal{G}$ on Y, we define $f_p(\mathcal{G})$ in $\mathcal{P}Sh(X)$ by

$$f_p(\mathcal{G})(U)\triangleq\underset{\substack{f(U)\subseteq V\\ V\text{ open in }Y}}{\operatorname{colim}}\mathcal{G}(V)$$

with restriction map given by: $U_1\subseteq U_2\subseteq X$ open subsets.

$$\begin{tikzcd}
f_p(\mathcal{G})(U_2) \arrow[r, dashed] \arrow[d, equal] & f_p(\mathcal{G})(U_1) \arrow[d, equal] \\
\underset{V\supseteq f(U_2)}{\operatorname{colim}}\mathcal{G}(V) \arrow[r] & \underset{V\supseteq f(U_1)}{\operatorname{colim}}\mathcal{G}(V)
\end{tikzcd}$$

, this map given by $id_{\mathcal{G}(V)}$ since $V\supseteq f(U_2)\Rightarrow V\supseteq f(U_1)$.

Easy cor. : $(f_p \mathcal{G})_x = \mathcal{G}_{f(x)}$.

Note that if $\mathcal{G}$ is a sheaf, then $f_p\mathcal{G}$ is generally not a sheaf. For example, take Y to be a point, and $\mathcal{G}$ any sheaf on Y, then $f_p\mathcal{G}$ is a constant presheaf, which is generally not a sheaf.

Lemma. $\mathrm{Mor}_{\mathrm{psh}(X)}(f_p\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{psh}(Y)}(\mathcal{G}, f_*\mathcal{F})$

Pf: Note that a map from a colimit is a compatible collection of maps from objects in the colimit, i.e.

$$\varphi : f_p\mathcal{G}(U) = \operatorname{colim}_{V \supseteq f(U)} \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$$

is given by a collection of $\varphi_{U,V} : \mathcal{G}(V) \longrightarrow \mathcal{F}(U)$ for all :
$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "f"] & X \arrow[d, "f"] \\ V \arrow[r, hook] & Y \end{tikzcd} \quad \text{(as objects)}$$
and compatible with restrictions in the sense that:

$$\begin{tikzcd} U' \arrow[r, hook] \arrow[d, "f"] & U \arrow[r, hook] \arrow[d, "f"] & X \arrow[d, "f"] \\ V' \arrow[r, hook] & V \arrow[r, hook] & Y \end{tikzcd} \quad \Rightarrow \quad \begin{tikzcd} \mathcal{F}(U') & \mathcal{F}(U) \arrow[l] \\ \mathcal{G}(V') \arrow[u, "\varphi_{U',V'}"] & \mathcal{G}(V) \arrow[l] \arrow[u, "\varphi_{U,V}"'] \arrow[ul, phantom, "\circlearrowleft"] \end{tikzcd}$$

On the other hand, a map $\psi : \mathcal{G} \longrightarrow f_*\mathcal{F}$ is given by a collection :

$$\psi_V : \mathcal{G}(V) \longrightarrow f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$$

and compatible with restrictions.

Now, from φ to ψ, define $\psi_V \triangleq \varphi_{f^{-1}(V), V}$. Conversely, from ψ to φ, define $\varphi_{U,V} \triangleq \rho^{f^{-1}(V)}_{U} \circ \psi_V$. Now the compositions :

$$\varphi \longmapsto \psi \longmapsto \varphi' : \quad \varphi'_{U,V} = \rho^{f^{-1}(V)}_{U} \circ \psi_V = \rho^{f^{-1}(V)}_{U} \circ \varphi_{f^{-1}(V),V} = \varphi_{U,V}$$

and $\psi \longmapsto \varphi \longmapsto \psi'$: $\quad \psi'_V \triangleq \varphi_{f^{-1}(V),V} = \rho^{f^{-1}(V)}_{f^{-1}(V)} \circ \psi_V = \psi_V$,

by the compatibilities everywhere. □

Note that $\varphi \in \mathrm{Mor}_{\mathrm{psh}(X)}(f_p\mathcal{G}, \mathcal{F})$ or equivalently, $\psi \in \mathrm{Mor}_{\mathrm{psh}(Y)}(\mathcal{G}, f_*\mathcal{F})$, gives rise to a map $\mathcal{G}_{f(x)} \longrightarrow \mathcal{F}_x$.

Def: If $\mathcal{G}$ is a sheaf on Y, we define $f^{-1}(\mathcal{G}) \triangleq (f_p(\mathcal{G}))^{\#}$.

Prop. $\mathrm{Mor}_{\mathrm{sh}(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{\mathrm{sh}(Y)}(\mathcal{G}, f_*\mathcal{F})$.

Pf: $\mathrm{Mor}_{Sh(X)}(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Mor}_{PSh(X)}(f_p\mathcal{G}, \mathcal{F})$ by def. of sheafification.

$\qquad = \mathrm{Mor}_{PSh(X)}(\mathcal{G}, f_*\mathcal{F})$ by lemma

$\qquad = \mathrm{Mor}_{Sh(X)}(\mathcal{G}, f_*\mathcal{F})$ since $f_*\mathcal{F}$ is already a sheaf. $\square$

Cor. $(f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$ canonically.

Pf: For any skyscraper $i_{x*}S$ in $Sh(X)$.

$$\mathrm{Mor}_{Sh(X)}(f^{-1}\mathcal{G}, i_{x*}S) = \mathrm{Mor}_{Sh(Y)}(\mathcal{G}, (f\circ i_x)_*S)$$
$$= \mathrm{Mor}_{Sh(Y)}(\mathcal{G}, i_{f(x)*}S)$$
$$= \mathrm{Mor}_{Sets}(\mathcal{G}_{f(x)}, S)$$

$\Rightarrow (f^{-1}\mathcal{G})_x \cong \mathcal{G}_{f(x)}$. $\square$

Morphism of ringed spaces.

Goal: Given ring maps $A \longrightarrow B$, we will define morphism between affine schemes (ringed spaces) $\mathrm{Spec}B \longrightarrow \mathrm{Spec}A$, and $\mathrm{Mor}_{Ring}(A, B) = \mathrm{Mor}_{(?)}(\mathrm{Spec}B, \mathrm{Spec}A)$

Motivation: $\psi: M \longrightarrow N$, a C^∞ map between C^∞ manifolds. We can regard it as a morphism between ringed spaces: $(M, C_M^\infty) \longrightarrow (N, C_N^\infty)$.

$U \subseteq M$, $V \subseteq N$ open s.t.

$$\begin{tikzcd} & U \arrow[r, hook] \arrow[d, "\psi"] \arrow[dl, dashed] & M \arrow[d, "\psi"] \\ \mathbb{R} & V \arrow[l, "h"'] \arrow[r, hook] & N \end{tikzcd}$$

$h \in C^\infty(N)(V)$

$\Rightarrow h\circ\psi \in C^\infty(M)(U)$

i.e. we obtain:
$$\begin{tikzcd} \psi_p C_N^\infty \arrow[rr] \arrow[dr] & & C_M^\infty \\ & \psi^{-1}C_N^\infty \arrow[ur] & \end{tikzcd}$$
, the factorization being automatic since C_M^∞ is a sheaf. This equivalently gives rise to $C_N^\infty \longrightarrow \psi_* C_M^\infty$.

Def. A morphism of ringed spaces $(X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ is a pair $(f, f^\#)$ where $f: X \longrightarrow Y$ is continuous, and $f^\#: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$ (or equivalently $f^\#: f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$; or a collection of compatible maps:

$f^\#_{U,V}: \mathcal{O}_Y(V) \longrightarrow \mathcal{O}_X(U)$, for each object:
$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "f"] & X \arrow[d, "f"] \\ V \arrow[r, hook] & Y \end{tikzcd}$$

Def. A morphism of ringed spaces $(X, \mathcal{O}_X) \xrightarrow{(f, f^\#)} (Y, \mathcal{O}_Y)$ gives rises to:

(i) $f_*: \mathrm{Mod}(\mathcal{O}_X) \longrightarrow \mathrm{Mod}(\mathcal{O}_Y) : \mathcal{F} \longmapsto f_*\mathcal{F}$, which is an $f_*\mathcal{O}_X$-module regarded as an $\mathcal{O}_Y$-module via $f^\#: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$.

(ii) $f^*: \mathrm{Mod}(\mathcal{O}_Y) \longrightarrow \mathrm{Mod}(\mathcal{O}_X) : \mathcal{G} \longmapsto f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \triangleq f^*\mathcal{G}$, $f^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_X$ is the canonical one adjoint to $f^\#: \mathcal{O}_Y \longrightarrow f_*\mathcal{O}_X$

- Adjointness: $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) = \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$.

  Pf: $\mathrm{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}) = \mathrm{Hom}_{f^{-1}\mathcal{O}_Y}(f^{-1}\mathcal{G}, {}_{f^{-1}\mathcal{O}_Y}\mathcal{F})$ (as $f^{-1}\mathcal{O}_Y$-modules)

  $= \mathrm{Hom}_{\mathcal{O}_X}(f^{-1}\mathcal{G} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X, \mathcal{F})$ (property of tensor product)

  $= \mathrm{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F})$

Trivial corollaries:

- $f^*\mathcal{O}_Y = \mathcal{O}_X$ $(A \otimes_A B \cong B)$
- $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{Y,f(x)}} \mathcal{O}_{X,x}$.

§2. Schemes

Locally ringed spaces.

Goal: $\mathrm{Mor}_{\text{Ringed Spaces}}(\operatorname{Spec} A, \operatorname{Spec} B) \overset{??}{=} \mathrm{Mor}_{\text{Rings}}(B, A)$. But this doesn't work in general.

E.g. R: a DVR. $\operatorname{Spec} R = \{\bullet, \sim\}$ Let K be its fraction field. $\operatorname{Spec} K = \{\sim\}$.

Now we have a ring map $R \longrightarrow K$ (inclusion). But there are 2 maps as ringed spaces of $\operatorname{Spec} K \longrightarrow \operatorname{Spec} R$. Namely:

$$\sim \;\mapsto\; \bullet \qquad\qquad \text{or} \qquad\qquad \sim \;\mapsto\; \sim$$

$$R = \mathcal{O}_{\operatorname{Spec} R, \bullet} \to \mathcal{O}_{\operatorname{Spec} K, \sim} = K \qquad\qquad K = \mathcal{O}_{\operatorname{Spec} R, \sim} \to \mathcal{O}_{\operatorname{Spec} K, \sim} = K$$

Def. (Locally ringed spaces)

(a). A locally ringed space is a ringed space $(X, \mathcal{O}_X)$ s.t. all stalks $\mathcal{O}_{X,x}$ are local rings.

(b). A morphism of locally ringed spaces is a morphism $(f, f^\#): (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ of ringed spaces s.t. $\forall x \in X$, the map $f_x^\#: \mathcal{O}_{Y, f(x)} \longrightarrow \mathcal{O}_{X,x}$ is a local homomorphism of local rings.

We denote $\kappa(x) \triangleq \mathcal{O}_{X,x}/\mathfrak{m}_{X,x}$ the residue field of $\mathcal{O}_{X,x}$.

Lemma: X, Y ringed spaces. $f: X \longrightarrow Y$ is an isomorphism in the category of ringed spaces. Then f is an isomorphism in the category of locally ringed spaces. $\square$

Open subspaces. $(X, \mathcal{O}_X)$: locally ringed space, $j: U \subseteq X$ open. Then $(U, j^{-1}\mathcal{O}_X \triangleq \mathcal{O}_X|_U)$ is a new locally ringed space.

Affine Schemes.

Lemma: Let R be a ring, M an R-module.

(1). If $f, g \in R$ with $D(g) \subseteq D(f)$, then

(a). f is invertible in R_g

(b). $g^e = af$ for some $e \geq 1$, $a \in R$.

(c). there is a canonical map $R_f \longrightarrow R_g$.

(d). there is a canonical R_f-module homomorphism $M_f \longrightarrow M_g$.

(2). Any open covering of $D(f)$ can be refined by $D(f) = \bigcup_{i=1}^{n} D(g_i)$. If $g_1, \cdots, g_n \in R$ and $D(f) \subseteq \bigcup_{i=1}^{n} D(g_i)$, then $g_1, \cdots, g_n$ generate the unital ideal in R_f. (In particular, this says that the coverings formed by standard opens is cofinal in the coverings). □

Let $\mathcal{B}$ be the collection of standard open sets of Spec$R$. Define a presheaf $\widetilde{M}$ on $\mathcal{B}$ by the rule $\widetilde{M}(D(f)) \triangleq M_f$ (which makes sense if $D(f) = D(g)$, f is invertible in R_g so $M_{gf} = (M_g)_f = M_g$, and it's also equal to M_f.), and if $D(g) \subseteq D(f)$, the restriction map $\rho^{D(f)}_{D(g)} : \widetilde{M}(D(f)) = M_f \longrightarrow \widetilde{M}(D(g)) = M_g$ is the canonical map.

Note that, the sheaf condition says that, w.r.t. a standard open covering of $D(f)$ $D(f) = \bigcup_{i=1}^{n} D(g_i)$, then

$$\begin{array}{ccccccc} 0 \longrightarrow & \widetilde{M}(D(f)) & \longrightarrow & \oplus_i \widetilde{M}(D(g_i)) & \longrightarrow & \oplus_{i,j} \widetilde{M}(D(g_i g_j)) \\ & \| & & \| & & \| \\ & M_f & & M_{g_i} & & M_{g_i g_j} \end{array}$$

This is true by the gluing lemma and $D(g) \subseteq D(f) \Rightarrow f$ is a unit in $R_g \Rightarrow D(g) = D(gf)$ and $M_g = (M_f)_g$, $M_{g_i g_j} = (M_f)_{g_i g_j}$.

Fact: There is an equivalence of categories: $Sh(\mathcal{B}) = Sh(\mathrm{Spec}R)$, where $Sh(\mathcal{B})$ denotes those presheaves on $\mathcal{B}$ satisfying the above sheaf condition w.r.t. standard open covers.

⌜ Explicitly, for each U open in X, let $\Gamma(U, \widetilde{M})$ be the set of elements $\{s_\mathfrak{p}\} \in \prod_{\mathfrak{p} \in U} M_\mathfrak{p}$ for which there is a covering of U by $D(f_\alpha)$'s together with elements $s_\alpha \in M_{f_\alpha}$ such that $s_\mathfrak{p}$ equals s_α under the restriction $M_{f_\alpha} \rightarrow M_\mathfrak{p}$. The restriction map is given by the coordinatewise projection: $\prod_{\mathfrak{p} \in U} M_\mathfrak{p} \longrightarrow \prod_{\mathfrak{p} \in V} M_\mathfrak{p}$. Then it's easy to check that this defines a sheaf. Moreover, by the gluing lemma above, $\Gamma(D(f), \widetilde{M}) = M_f$, showing that this is an equivalence of category.

Hartshorne used this fact implicitly in his construction. ⌟

Conclusion: there exists a unique sheaf of rings $\mathcal{O}_{\mathrm{Spec}R}$ s.t. $\mathcal{O}_{\mathrm{Spec}R}(D(f)) = \widetilde{R}(D(f)) = R_f$ Moreover, for every R-module M, there is a unique sheaf of $\mathcal{O}_{\mathrm{Spec}R}$-module $\mathcal{F} = \widetilde{M}$ s.t. $\mathcal{F}(D(f)) = \widetilde{M}(D(f)) = M_f$ as an $\mathcal{O}_{\mathrm{Spec}R}(D(f)) = R_f$ - module. In particular $\Gamma(\mathrm{Spec}R, \mathcal{O}_{\mathrm{Spec}R}) = R$.

Def. An affine scheme is a locally ringed space isomorphic to $(\operatorname{Spec} R, \mathcal{O}_{\operatorname{Spec} R})$ for some ring R.

Def. A scheme is a locally ringed space s.t. every point has an open neighborhood which is an affine scheme.

Remarks on $\mathcal{F} = \widetilde{M}$

(1). $x \in \operatorname{Spec} R$ corresponds to a prime ideal $\mathfrak{p} \subseteq R$. We have $\mathcal{F}_x = \operatorname{colim}_{x \in D(f)} \mathcal{F}(D(f))$ $= \operatorname{colim}_{f \in R, f \notin \mathfrak{p}} \mathcal{F}(D(f)) = \operatorname{colim}_{f \in R, f \notin \mathfrak{p}} M_f = M_{\mathfrak{p}}$. (the colimit is a direct limit since $f_1, f_2 \in R \setminus \mathfrak{p} \Rightarrow f_1 f_2 \in R \setminus \mathfrak{p}$, and $D(f_1 f_2) \subseteq D(f_i)$, $i=1,2$.). Moreover, $M_{\mathfrak{p}}$ is an $R_{\mathfrak{p}}$-module in an obvious way.

(2). The functor $\mathcal{F} \mapsto \mathcal{F}_x$ or $\widetilde{M} \mapsto M_{\mathfrak{p}}$ is exact.

(3). $\varphi : \widetilde{M} \longrightarrow \widetilde{N}$ an $\mathcal{O}_{\operatorname{Spec} R}$-map $\Rightarrow$ φ on global sections : $\varphi : M \longrightarrow N$.

Why do we define $\widetilde{M}$ on $\mathcal{B}$ instead of all opens?

E.g. $X = \operatorname{Spec} k[x,y]$, $U = X \setminus \{0\}$. 0: the maximal ideal (x,y).

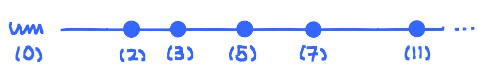
Claim: $\mathcal{O}_X(U) = k[x,y]$.

$$0 \longrightarrow \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(D(x)) \oplus \mathcal{O}_X(D(y)) \longrightarrow \mathcal{O}_X(D(xy))$$

$$\Rightarrow \mathcal{O}_X(U) = \ker\left(k[x,y,\tfrac{1}{x}] \oplus k[x,y,\tfrac{1}{y}] \longrightarrow k[x,y,\tfrac{1}{xy}]\right)$$
$$= k[x,y]$$

This is an analogue of Hartog's thm in complex analysis.

E.g. $(\operatorname{Spec} \mathbb{Z}, \mathcal{O}_{\operatorname{Spec} \mathbb{Z}})$ is an affine scheme.



Any non-empty open set is of the form $D(n)$, which is standard open, thus $\mathcal{O}_{\operatorname{Spec} \mathbb{Z}}(D(n)) = \mathbb{Z}[\tfrac{1}{n}]$.

The Main Lemma.

The main result of this section is the following lemma:

Lemma: Let $(X, \mathcal{O}_X)$, $(Y, \mathcal{O}_Y)$ be locally ringed spaces. Y affine (say, isomorphic to $\operatorname{Spec} R$). Then $\operatorname{Mor}_{L.R.S.}(X, Y) = \operatorname{Hom}_{Rings}(R, \Gamma(X, \mathcal{O}_X))$. Equality holds functorially in X.

Rmk: Together with Yoneda's lemma, this formula determines Y as locally ringed spaces from the $R = \Gamma(Y, \mathcal{O}_Y)$.

Proof of the main lemma. (sketch).

Given $(\psi, \psi^\#) \in \operatorname{Mor}_{L.R.S.}(X, Y)$, $\psi^\#$ induces a ring map:

$$\alpha: R = \Gamma(Y, \mathcal{O}_Y) \xrightarrow{\Gamma(\cdot)} \Gamma(Y, \psi_*\mathcal{O}_X) = \Gamma(X, \mathcal{O}_X).$$

We will express ψ as a set map in terms of α: $\forall x \in X$, $\psi(x) = ?$

$$\begin{tikzcd}
\Gamma(X, \mathcal{O}_X) \arrow[r] & \mathcal{O}_{X,x} \\
R = \Gamma(Y, \mathcal{O}_Y) \arrow[r] \arrow[u] & \mathcal{O}_{Y,\psi(x)} \arrow[u]
\end{tikzcd}$$

The above diagram commutes by definition of morphisms of locally ringed spaces. Hence, if $\psi(x) = \mathfrak{p} \in \operatorname{Spec} R$, we have

$$\begin{tikzcd}
 & \mathcal{O}_{X,x} & & & \operatorname{Spec}\mathcal{O}_{X,x} \arrow[d] \\
R \arrow[r] & R_\mathfrak{p} \arrow[u] & \rightsquigarrow & \operatorname{Spec} R & \operatorname{Spec} R_\mathfrak{p} \arrow[l]
\end{tikzcd}$$

Since $R_\mathfrak{p} \longrightarrow \mathcal{O}_{X,x}$ is a local homomorphism of local rings, the (unique) closed point of $\operatorname{Spec}\mathcal{O}_{X,x}$ must be mapped to the closed point of $\operatorname{Spec} R_\mathfrak{p}$. (This is not true if we only require ringed spaces!) In turn it is mapped to $\mathfrak{p}$ under $\operatorname{Spec} R_\mathfrak{p} \longrightarrow \operatorname{Spec} R$:

- Conclusion: $\psi(x)$ corresponds to $\mathfrak{p} \subseteq R$ which is the kernel of the composite
$$R \xrightarrow{\alpha} \Gamma(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{X,x} \longrightarrow \kappa(x)$$

- Sublemma: Given any locally ringed space $(X, \mathcal{O}_X)$, and any global section $f \in \Gamma(X, \mathcal{O}_X)$, the set $D(f) \triangleq \{x \in X \mid f \notin m_x \subseteq \mathcal{O}_{X,x}\}$ is open in X and $f \in \Gamma(D(f), \mathcal{O}_X^*)$.

Notation: If $\mathcal{O}$ is a sheaf of rings on X, we define $\mathcal{O}^*$ the sheaf of units in $\mathcal{O}$.

which is a sheaf of abelian groups.)

Pf of sublemma: $\forall\, x \in D(f)$, $f \notin m_x \Rightarrow f$ is a unit in $\mathcal{O}_{X,x} \Rightarrow \exists\, g \in \mathcal{O}_{X,x}$ s.t. $fg = 1$ in $\mathcal{O}_{X,x}$. By def. of a stalk, $\exists\, U \ni x$, $fg = 1$ on $U \Rightarrow f \in \Gamma(U, \mathcal{O}_X^*)$. This works for any $x \in D(f) \Rightarrow D(f)$ is open. Moreover, this also shows that $f \in \Gamma(D(f), \mathcal{O}_X^*)$. $\square$

Rmk: The result is not true for ringed spaces. (it doesn't even make sense!)

Now, we can construct from a ring map $\alpha: R \longrightarrow \Gamma(X, \mathcal{O}_X)$ a morphism of locally ringed spaces $(\psi, \psi^\#)$, namely:

- Set $\psi(x) = \ker(R \xrightarrow{\alpha} \Gamma(X, \mathcal{O}_X) \longrightarrow \mathcal{O}_{X,x} \longrightarrow \kappa(x))$

Claim: ψ is continuous. Indeed, $\psi^{-1}(D(f)) = D(\alpha(f))$, which is open by sublemma.

- To construct $\psi^\#: \mathcal{O}_Y \longrightarrow \psi_* \mathcal{O}_X$, it suffices to construct for the standard opens:

$$\begin{tikzcd}
\Gamma(D(f), \mathcal{O}_Y) \arrow[r, dashed] \arrow[d, equal] & \Gamma(D(f), \psi_* \mathcal{O}_X) \arrow[d, equal] \\
R_f \arrow[r, dashed] & \Gamma(D(\alpha(f)), \mathcal{O}_X) \\
R \arrow[r, "\alpha"] \arrow[u] & \Gamma(X, \mathcal{O}_X) \arrow[u] \\
f \arrow[r, mapsto] & \alpha(f)
\end{tikzcd}$$

By sublemma, $\alpha(f)$ is a unit in $\Gamma(D(\alpha(f)), \mathcal{O}_X)$, thus by the universal property of localization, α lifts to a map $R_f \longrightarrow \Gamma(D(\alpha(f)), \mathcal{O}_X)$.

It suffices to check that the functor we constructed is inverse to Γ, which is omitted.

Cor 1. There is an anti-equivalence of affine schemes (as locally ringed spaces) and rings:

$$(\text{Affine Schemes}) \underset{\text{Spec}}{\overset{\Gamma(\cdot)}{\rightleftarrows}} (\text{Rings})$$

The main lemma implies that Spec is a fully faithful functor. $\square$

Cor 2. If Y is an affine scheme and $f \in \Gamma(Y, \mathcal{O}_Y)$, then $(D(f), \mathcal{O}_Y|_{D(f)}) \cong (\operatorname{Spec} R_f, \mathcal{O}_{\operatorname{Spec} R_f})$ as affine schemes. Consequently, any scheme has a basis of topology consisting of affine opens.

Pf: $\mathrm{Mor}_{L.R.S.}(X, D(f)) = \{ (\psi, \psi^{\#}) \in \mathrm{Mor}_{L.R.S}(X, Y) \mid f \text{ is invertible in } \Gamma(X, \mathcal{O}_X) \}$

$$= \{ \alpha \in \mathrm{Hom}(\Gamma(Y, \mathcal{O}_Y), \Gamma(X, \mathcal{O}_X)) \mid \alpha(f) \in \Gamma(X, \mathcal{O}_X^*) \}$$

$$= \mathrm{Hom}(R_f, \Gamma(X, \mathcal{O}_X))$$

$$= \mathrm{Mor}_{L.R.S.}(X, \mathrm{Spec} R_f)$$

$\Rightarrow (D(f), \mathcal{O}_X|_{D(f)}) \cong (\mathrm{Spec} R_f, \mathcal{O}_{\mathrm{Spec} R_f})$. □

Immersions of Locally Ringed Spaces

Let $(X, \mathcal{O}_X)$ be a locally ringed space, $U \subseteq X$, an open subset. Then $(U, \mathcal{O}_X|_U)$ is an open subspace as locally ringed space.

Def. An open immersion of locally ringed spaces is a morphism of locally ringed spaces $j: V \longrightarrow Y$ satisfying:

(a). j is a homeomorphism onto an open subset of Y

(b). $j^{\#}: j^{-1}\mathcal{O}_Y \longrightarrow \mathcal{O}_V$ is an isomorphism.

Lemma: Let $X \xrightarrow{f} Y$ be a morphism of locally ringed spaces. Suppose $U \subseteq X$ and $V \subseteq Y$ are open subsets s.t. $f(U) \subseteq V$. Then in the category of locally ringed spaces, the following diagram commutes:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "f|_U"'] & X \arrow[d, "f"] \\ V \arrow[r, hook] & Y \end{tikzcd}$$

□

Closed immersions are harder to define. The usual definition for schemes (Hartshorne) doesn't work in general for locally ringed spaces:

Example:

Let $X = \mathbb{R}$ with the usual topology. $\mathcal{O}_X = \underline{\mathbb{Z}/2}$, constant sheaf. $Z = \{0\}$. $\mathcal{O}_Z = \underline{\mathbb{Z}/2}$. Let $i: Z \hookrightarrow X$ be the inclusion and $i^{\#}: \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ the obvious map.

This would be a closed immersion if we take the usual definition of closed immersions for schemes. However, this is not so good in the sense that we want closed sets to be cut out by (ideals) of regular function.

Def: Let $i: Z \hookrightarrow X$ be a morphism of locally ringed spaces. We say i is a closed immersion iff:

(a). i is a homeomorphism of Z onto a closed subset of X.

(b). $i^{\#}: \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ is surjective, with kernel $\mathcal{I}$.

(c). As an $\mathcal{O}_X$-module, $\mathcal{I}$ is locally generated by sections, i.e. $\forall x \in X$, $\exists x \in U \overset{\text{open}}{\subseteq} X$, and sections $s_i \in \mathcal{F}(U)$, $i \in I$ s.t. the map:

$$\oplus_{i\in I} \mathcal{O}_X|_U \longrightarrow \mathcal{I}|_U \ ; \ (f_i)_{i\in I} \mapsto \Sigma f_i s_i$$

is surjective.

Example bis.

In the previous example, $\mathcal{I} = \ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ is not locally generated by sections. Indeed, $\forall U \ni x$, U connected, $\mathbb{Z}/2 = \mathcal{O}_X(U) \xrightarrow{\cong} i_*\mathcal{O}_Z(U) = \mathbb{Z}/2 \Rightarrow \mathcal{I}(U) = 0$. If for some U it were generated by (constant) sections, $\mathcal{I}(U) \neq 0$ since $\mathcal{I}_y \cong \mathbb{Z}/2 \neq 0$, $\forall y \in U$, $y \neq 0$.

Upshot. If $i: Z \hookrightarrow X$ is a closed immersion, then $\forall z \in Z$, $\exists U \overset{\text{open}}{\subseteq} X$, $i(z) \in U$, $f_j \in \mathcal{O}_X(U)$ s.t. $i(Z) \cap U$ is "cut out" by the vanising set of $f_j = \bigcap_j \{x \in U \mid f_j = 0 \text{ in } \kappa(x)\}$.

Fact: If $X = \operatorname{Spec} R$ is an affine scheme, then any closed immersion $i: Z \hookrightarrow X$ is of the form $\operatorname{Spec} R/I \xrightarrow{\psi} \operatorname{Spec} R$ for a unique ideal I in R. Moreover,

$$\ker(\mathcal{O}_{\operatorname{Spec} R} \longrightarrow \psi_* \mathcal{O}_{\operatorname{Spec} R/I}) = \tilde{I} \subseteq \tilde{R} = \mathcal{O}_{\operatorname{Spec} R}.$$

Immersions of Schemes

Lemma. X: a scheme, $U \subseteq X$ an open subset, then U is a scheme.

Pf: $\forall x \in U$, and $x \in V$ open affine neighborhood. Then $U \cap V$ is open and $x \in U \cap V$. Now, take $f \in \Gamma(V, \mathcal{O}_X)$ s.t. $x \in D(f)$ and $D(f) \subseteq U \cap V$, then $D(f)$ is affine $\square$

Equivalently:

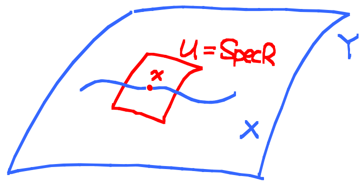
Lemma': If $j: X \longrightarrow Y$ is an open immersion, then Y is a scheme $\Rightarrow X$ is a scheme. $\square$

Note that even Y affine, X need not be, by our previous:

Example: $\operatorname{Spec} k[x,y] \supseteq U = D(x) \cup D(y)$, but U is not affine.

From the fact above:

Lemma'': If $i: X \longrightarrow Y$ is a closed immersion, Y a scheme, then X is a scheme. $\square$



"Fact" above $\Rightarrow$ $i^{-1}(\operatorname{Spec} R) = \operatorname{Spec} R/I$ for some unique I. Hartshorne avoids this complexity by requiring also that X be a scheme.

Def: A morphism $f: X \longrightarrow Y$ of schemes is called an immersion or locally closed immersion if it can be factored as $j \circ i$ where i is a closed immersion and j is an open immersion.

Lemma. An immersion is closed iff its image is closed. $\square$

Lemma. X: a scheme, then any irreducible closed subset has a unique generic point. (i.e. X is a sober topological space).

Pf: Let $Z \subseteq X$ be irreducible, closed. Pick $U = \operatorname{Spec} R \subseteq X$ open affine, s.t. $U \cap Z \neq \phi$. Then $U \cap Z$ is irreducible, closed and by the fact above, corresponds to a unique radical ideal $\mathfrak{p}$. Irreducibility $\Rightarrow \mathfrak{p}$ is prime. Then $\mathfrak{p} \in \operatorname{Spec} R \subseteq X$ satisfies $\overline{\{\mathfrak{p}\}} = Z$ since $\bar{\mathfrak{p}} \cap Z$ contains an open subset of Z, namely $U \cap Z$, and Z is irreducible. Uniqueness follows since any generic point of Z is in U. $\square$

Lemma: The open affines of X form a basis of topology if X is a scheme. $\square$

⚠ In general, U, V affine $\not\Rightarrow$ $U \cap V$ affine. (Separatedness required). However, a useful technical lemma we shall use is:

Lemma: Let X be a scheme. U, V affine opens, $x \in U \cap V$. Then $\exists W \subseteq U \cap V$, W standard open in both U and V.

Pf: $U = \operatorname{Spec} A$, $V = \operatorname{Spec} B$. Choose $f \in A$ s.t. $x \in D(f) \subseteq U \cap V$. Again choose $g \in B$ s.t. $x \in D(g) \subseteq D(f) \subseteq U \cap V$. Now $g \in B = \Gamma(V, \mathcal{O}_X)$ restricts to an element $\frac{a}{f^n}$ in $A_f = \Gamma(U, \mathcal{O}_X)$. It follows that $D(g) = D(af)$ is also standard open in $\operatorname{Spec} A$. $\square$

. $\exists$ non-empty scheme without closed points!

To discuss closed subschemes, we first define a reduced scheme:

Def. A scheme is reduced iff $\forall x \in X$, the local ring $\mathcal{O}_{X,x}$ is reduced.

Lemma: X is reduced iff $\forall U$ open, $\mathcal{O}_X(U)$ is reduced.

Pf: "$\Rightarrow$" $\forall U \subseteq X$ open, $f \in \mathcal{O}_X(U)$, $f^n = 0 \Rightarrow f^n = 0$ in $\mathcal{O}_{X,x}$, $\forall x \in U$. Since $\mathcal{O}_{X,x}$ is reduced by assumption, $f = 0$ in $\mathcal{O}_{X,x} \Rightarrow f = 0$ in $\mathcal{O}_X(U)$, by sheaf properties.

"$\Leftarrow$" colimit of reduced rings are reduced. $\square$

Cor. An affine scheme $X = \operatorname{Spec} R$ is reduced iff R is reduced. $\square$

Cor. X: a scheme. TFAE:

(1). X is reduced

(2). $\exists$ an open covering $X = \bigcup_i U_i$ s.t. each $\Gamma(U_i, \mathcal{O}_X)$ is reduced.

(3). $\forall U$ affine open, $\Gamma(U, \mathcal{O}_X)$ is reduced.

(4). $\forall U$ open, $\Gamma(U, \mathcal{O}_X)$ is reduced. $\square$

This kind of characterization of a property for schemes will occur many times later (Noetherian, quasi-coherent, $\cdots$)

Closed subschemes.

Def. If X is a scheme and $\mathcal{F}$ is a sheaf of $\mathcal{O}_X$-modules. We say $\mathcal{F}$ is quasi-coherent iff $\forall U \subseteq X$ affine open, $U = \operatorname{Spec} R$, we have $\mathcal{F}|_U = \widetilde{M}$ for some R-module M.

Lemma. It's enough to check the above def. for members of an affine open cover of X. $\square$

X: a scheme. Suppose $\mathcal{I}$ is a quasi-coherent sheaf of ideals. Then for every affine open, $\mathcal{I}|_U = \tilde{I}$ for some ideal $I \subseteq R$. Look at the s.e.s. of sheaves:

$$0 \to \mathcal{I} \to \mathcal{O}_X \to \mathcal{O}_X/\mathcal{I} \to 0 \quad \text{on } X$$

and
$$0 \to \tilde{I} \to \tilde{R} \to (\widetilde{R/I}) \to 0 \quad \text{on } U$$

Fact: there is a unique closed subscheme $Z \xhookrightarrow{i} X$ s.t $\mathcal{I} = \ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$ and on each affine open U, we have $Z \cap U = \mathrm{Spec}(R/I)$.

Upshot: There is a 1-1 inclusion reversing bijection between closed subschemes of X and quasi-coherent sheaves of ideals of $\mathcal{O}_X$, given by:

$$(Z \xhookrightarrow{i} X) \mapsto \ker(\mathcal{O}_X \to i_*\mathcal{O}_Z)$$

Lemma. Let X be a scheme, and $T \subseteq X$ a closed subset. Then, $\exists!$ closed subscheme $Z \subseteq X$ s.t.

(a). $Z = T$ as a set

(b). Z is reduced.

Pf: To construct Z, all we need to do is to construct a suitable quasi-coherent sheaf of (radical) ideals, by the upshot above.

Given T, define $\mathcal{I}(U) \triangleq \{ f \in \mathcal{O}_X(U) \mid f(t) = 0 \mod m_t,\ \forall t \in U \cap T \}$, i.e. f maps to in each $k(t) = \mathcal{O}_{X,t}/m_t$.

$\mathcal{I}$ is automatically a subsheaf of $\mathcal{O}_X$, because the definition is local in nature. We just need to check that $\mathcal{I}$ is quasi-coherent. Pick $U = \mathrm{Spec} R$ open affine in X, Since T is closed in X, $T \cap U = V(I)$ for some unique radical ideal I of R. In fact, $I = \bigcap_{\beta \in T \cap U} \beta = \Gamma(U, \mathcal{I}) \subseteq R$, we just have to show that $\mathcal{I}|_U = \tilde{I}$. It now suffices to check for standard opens: $\forall f \in R$,

$$\Gamma(D(f), \mathcal{I}|_U) = \Gamma(D(f), \mathcal{I})$$
$$= \{ h \in \mathcal{O}_X(D(f)) \mid h(t) = 0 \mod m_t,\ \forall t \in D(f) \cap T \}$$

$$= \bigcap_{\mathfrak{p}\in D(f)\cap V(I)} \mathfrak{p}$$
$$= I_f$$
$$= \tilde{I}(D(f)).$$

Finally, take Z to be the closed subscheme associated to $\tilde{I}$. $\square$

Def. Given a closed subset $Z \subseteq X$, we will say "let $(Z, \mathcal{O}_Z)$ be the reduced induced scheme structure on Z" to indicate the above reduced scheme structure.

Def. A scheme is called integral if for all $U \subseteq X$ open, the ring $\mathcal{O}_X(U)$ is a domain. (Also assume $X \neq \phi$, $U \neq \phi$).

Lemma: X integral $\Leftrightarrow$ X is irreducible and reduced.

Pf: "$\Rightarrow$" If there were two non-empty opens U, V, s.t. $U \cap V = \phi$, then $\Gamma(U\cup V, \mathcal{O}_X) = \Gamma(U, \mathcal{O}_X) \times \Gamma(V, \mathcal{O}_X)$ is not a domain. It's reduced by a previous lemma.

"$\Leftarrow$" $\forall U \subseteq X$, $f, g \in \Gamma(U, \mathcal{O}_X)$. If $fg = 0 \Rightarrow U = V(f) \cup V(g)$. Since U is irreducible, $U = V(f)$ or $V(g)$, say $V(f)$. Hence $f \equiv 0 \mod m_x$, $\forall x \in U \Rightarrow f$ is nilpotent in any affine open in U. By the reduced assumption, $f = 0$. $\square$

We summarize all the equivalent definitions of a closed immersion for schemes:

Lemma. Let $i: Z \hookrightarrow X$ be a morphism of schemes. TFAE.

(1). i is a closed immersion.

(2). $\forall\, U = \operatorname{Spec} R$ open affine in X, we have $i^{-1}(U)$ $(\triangleq i|_{i^{-1}(U)})$: $\operatorname{Spec} R/I \to \operatorname{Spec} R$, the canonical morphism defined by some ideal I of R.

(3). $\exists$ open affine covering $X = \bigcup_{j\in J} U_j$, $U_j = \operatorname{Spec} R_j$ s.t. $i^{-1}(U_j) = \operatorname{Spec} R_j/I_j$ as in (2)

(4). (Hartshorne's definition): (a). i is a homeomorphism onto a closed subset of X and (b). $i^\#: \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ is onto.

(5). (a) + (b) + (c): $\operatorname{Ker} i^\#: \mathcal{O}_X \longrightarrow i_*\mathcal{O}_Z$ is a quasi-coherent sheaf of ideals.

(5'). (a) + (b) + (c'): $\operatorname{Ker} i^\# \subseteq \mathcal{O}_X$ is a sheaf of ideals, locally generated by sections.

Moreover, $\forall$ quasi-coherent sheaf of ideals $\mathcal{I} \subseteq \mathcal{O}_X$, $\exists$ a closed immersion $i: Z \longrightarrow X$ s.t $\mathcal{I} = \ker(i^\#)$.

Given an immersion $i: Z \hookrightarrow X$, there doesn't always exist a factorization of i s.t. $i: Z \xhookrightarrow{\text{open}} \overline{Z} \xhookrightarrow{\text{closed}} X$. (It does exist if Z is reduced)

E.g. $X = \operatorname{Spec}\mathbb{C}[x_1, x_2, \cdots]$, $U = \bigcup_{i=1}^{\infty} D(x_i)$. Take $Z \hookrightarrow U$, and Z defined on each $D(x_i) = \operatorname{Spec}\mathbb{C}[x_1, x_2, \cdots, \frac{1}{x_i}]$ by the corresponding ideal:

$$I_i \triangleq (x_1^i, x_2^i, \cdots, x_{i-1}^i, x_i - 1, x_{i+1}, x_{i+2}, \cdots)$$

(the closed point $(0, \cdots, 0, 1, 0, \cdots)$ with "fatter and fatter" ideal). Then on $D(x_i x_j)$

$$I_i|_{D(x_i x_j)} = \operatorname{Spec}\mathbb{C}[x_1, x_2, \cdots, \tfrac{1}{x_i x_j}] = I_j|_{D(x_i x_j)},$$

and thus I_i's glue to define a closed subscheme in U. On the other hand, there is no closed subscheme structure on $\overline{Z}$ in X that restricts to this scheme structure of Z in U: since $\forall f \in \mathbb{C}[x_1, x_2, \cdots]$, $f|_{D(x_i)} \in I_i \Rightarrow \deg f \geq i$ $\Longrightarrow f = 0$.

§3. Construction of Schemes

Gluing schemes

Let I be a set. For each $i \in I$, we have $(X_i, \mathcal{O}_i)$, a scheme (or l.r.s), and $\forall i, j \in I$, $\exists\, U_{ij} \subseteq X_i$, $U_{ji} \subseteq X_j$ open subschemes, and $\varphi_{ij} : U_{ij} \longrightarrow U_{ji}$ an isomorphism of schemes, ($i=j$, take $U_{ii} = X_i$, $\varphi_{ii} = id_{X_i}$) satisfying: $\forall i, j, k \in I$, (equalities among i, j, k allowed)

(1). $\varphi_{ij}^{-1}(U_{ji} \cap U_{jk}) = U_{ij} \cap U_{ik}$.

(2).
$$\begin{tikzcd} U_{ij} \cap U_{ik} \arrow[rr, "\varphi_{ik}"] \arrow[dr, "\varphi_{ij}"'] & \circlearrowleft & U_{ki} \cap U_{kj} \arrow[dl, "\varphi_{kj}"] \\ & U_{ji} \cap U_{jk} & \end{tikzcd}$$
(cocycle condition)

The above setting is called a gluing data.

Lemma. Given a gluing data, $\exists !$ scheme X, with open subschemes $U_i \subseteq X$, and isomorphisms $\varphi_i : X_i \longrightarrow U_i$ s.t.

(1). $\varphi_i(U_{ij}) = U_i \cap U_j$

(2). $\varphi_{ij} = \varphi_j^{-1}|_{U_i \cap U_j} \circ \varphi_i|_{U_{ij}}$

Moreover, $\mathrm{Mor}_{Sch}(X, Y) = \{ (f_i)_{i \in I} \mid f_i : X_i \longrightarrow Y,\ f_j \circ \varphi_{ij} = f_i|_{U_{ij}} \}$ $\square$

A special case of the lemma is when there are only 2 pieces to glue, in which case the cocycle condition is trivial.

E.g. Affine line with zero doubled. ($k = \bar{k}$)

$0_1 \in X_1 = \operatorname{Spec} k[x]$, $\quad 0_2 \in X_2 = \operatorname{Spec} k[y]$

$X_1 \supseteq U = D(x) = \operatorname{Spec} k[x, \frac{1}{x}]$, $\quad X_2 \supseteq V = D(y) = \operatorname{Spec} k[y, \frac{1}{y}]$.

Let $\varphi : U \longrightarrow V$ be the isomorphism defined by the ring map:

$$k[x, \tfrac{1}{x}] \longrightarrow k[y, \tfrac{1}{y}], \quad x \mapsto y.$$

Write $X = X_1 \cup_{U=V} X_2$. Let's calculate $\Gamma(X, \mathcal{O}_X)$:

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X) \longrightarrow \Gamma(X_1, \mathcal{O}_X) \times \Gamma(X_2, \mathcal{O}_X) \longrightarrow \Gamma(U, \mathcal{O}_X)$$

$$\begin{array}{ccccc} & \| & & \| & \\ & k[x] & \times & k[y] & \longrightarrow & k[x, \tfrac{1}{x}] \\ & (f(x), & & g(y)) & \longmapsto & f(x) - g(x) \end{array}$$

$\Rightarrow \Gamma(X, \mathcal{O}_X) \cong k[x]$. It follows that X is not affine, since $\forall f \in \Gamma(X, \mathcal{O}_X)$, $f(0_1) = f(0_2) \in k \Rightarrow X$ is not T_0.

Fiber products

Def. Given $f: X \longrightarrow S$ and $g: Y \longrightarrow S$ morphism of schemes, a fiber product is a scheme $X \times_S Y$ together with morphisms $p: X \times_S Y \longrightarrow X$ and $q: X \times_S Y \longrightarrow Y$, fitting into a commutative diagram:

$$\begin{tikzcd} X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\ X \arrow[r, "f"] & S \end{tikzcd}$$

s.t. given a scheme T (test scheme) and morphisms $a: T \longrightarrow X$, $b: T \longrightarrow Y$ s.t. the diagram commutes:

$$\begin{tikzcd} T \arrow[dr, dashed] \arrow[drr, bend left, "b"] \arrow[ddr, bend right, "a"'] & & \\ & X \times_S Y \arrow[r, "q"] \arrow[d, "p"] & Y \arrow[d, "g"] \\ & X \arrow[r, "f"] & S \end{tikzcd}$$

Then $\exists!$ (dotted) morphism making the whole diagram commute.

Rmk: In the category of sets, vector spaces, fiber product exists.

Thm. Fiber products exist in the category of schemes.

Sketch of proof: Working backwards, assuming $X \times_S Y$ exist.

(1). If $U \subseteq X$ and $V \subseteq Y$, $W \subseteq S$ are opens s.t. $f(U), g(V) \subseteq W$, then

$$U \times_W V = p^{-1}(U) \cap q^{-1}(V)$$

is an open subscheme of $X \times_S Y$. This follows from categorical non-sense.

(2). If $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$, $S = \operatorname{Spec} R$, then $X \times_S Y = \operatorname{Spec}(A \otimes_R B)$.

Pf: $\mathrm{Mor}_{sch}(T, \mathrm{Spec}(A\otimes_R B)) = \mathrm{Hom}(A\otimes_R B, \mathcal{O}_T(T))$
$= \mathrm{Hom}(A, \mathcal{O}_T(T)) \times_{\mathrm{Hom}(R, \mathcal{O}_T(T))} \mathrm{Hom}(B, \mathcal{O}_T(T))$
$= \mathrm{Mor}(T, \mathrm{Spec}A) \times_{\mathrm{Mor}(T,\mathrm{Spec}R)} \mathrm{Mor}(T, \mathrm{Spec}B)$

(3). For general X, Y, S, glue affine pieces together. $\square$

E.g. The affine n-space over a ring R is $\mathbb{A}^n_R = \mathrm{Spec}R[x_1,\cdots,x_n]$ (with the structure morphism $\mathbb{A}^n_R \longrightarrow \mathrm{Spec}R$.
Then $\mathbb{A}^n_R \times_{\mathrm{Spec}R} \mathbb{A}^m_R = \mathbb{A}^{n+m}_R$, since $R[x_1,\cdots,x_n]\otimes_R R[y_1,\cdots,y_m] = R[x_1,\cdots,x_n,y_1,\cdots,y_m]$.

Is the set of points $X\times_S Y$ the same as $|X|\times_{|S|}|Y|$? Not true in general!

E.g. $\mathbb{A}^2_{\mathbb{C}} \overset{?}{=}$ $\underset{\{(x-\alpha),\alpha\in\mathbb{C}\}\cup\{(0)\}}{\text{Points of } \mathbb{A}^1_{\mathbb{C}}}$ $\times$ $\underset{\{(y-\beta),\beta\in\mathbb{C}\}\cup\{(0)\}}{\text{Points of } \mathbb{A}^1_{\mathbb{C}}}$

Then the product set of points consists of $\{(x-\alpha, y-\beta), (x-\alpha), (y-\beta), (0)\}$.
There are many points of $\mathbb{A}^2_{\mathbb{C}}$ not in this set, for instance (x^2-y^3). However, for closed points, they are the same.

Aside: Let K be a field, and S a scheme. What's $\mathrm{Spec}K \longrightarrow S$?
$\mathrm{Mor}_{sch}(\mathrm{Spec}K, S) = \{(s, \kappa(s) \hookrightarrow K)\}$. In particular, for every $s\in S$, we get a canonical morphism $s = (\mathrm{Spec}\,\kappa(s) \longrightarrow S)$.

Points of $X\times_S Y \Leftrightarrow$ quadruples (x, y, s, β) with $x\in X$, $y\in Y$, $s\in S$, $f(x)=s=g(y)$, and $\beta \subseteq \kappa(x)\otimes_{\kappa(s)}\kappa(y)$.

In this notation, $(x^2-y^3) \subseteq \mathbb{C}(x)\otimes_{\mathbb{C}}\mathbb{C}(y)$ is a point of $\mathbb{A}^2_{\mathbb{C}}$.

Def. Given a morphism of schemes $f: X \longrightarrow S$ and a point $s\in S$, the fiber of f at s is a scheme X_s fitting into the fiber product diagram.

$$\begin{tikzcd} X_s = \mathrm{Spec}\,\kappa(s)\times_S X \arrow[r] \arrow[d] & X \arrow[d] \\ \mathrm{Spec}\,\kappa(s) \arrow[r] & S \end{tikzcd}$$

(always think of X_s as over $\kappa(s)$!)

E.g.

$$\begin{array}{ccccc} X = \mathbb{A}^2_{\mathbb{C}} & (x,y) & & & \mathbb{C}[x,y] \\ \downarrow & \downarrow & \text{corresponding to} & & \uparrow \\ S = \mathbb{A}^1_{\mathbb{C}} & (x) & & & \mathbb{C}[x] \end{array}$$

There are two types of fibers:

(1). Over a closed point $(x-\alpha) \subseteq \mathbb{C}[x]$, $\mathbb{C}[x,y] \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x-\alpha) \cong \frac{\mathbb{C}[x,y]}{(x-\alpha)} \cong \mathbb{C}[y]$

i.e. $X_s \cong \mathbb{A}^1_{\mathbb{C}}$

(2). Over the generic point $s=(0)$, $\mathbb{C}[x,y] \otimes_{\mathbb{C}[x]} \mathbb{C}(x) = \mathbb{C}(x)[y]$. In this case $X_s \cong \mathbb{A}^1_{\mathbb{C}(x)}$.

Both kinds of fibers are of the same form except the base field being algebraically closed or not (which is a minor problem in AG).

E.g. $X = \operatorname{Spec}(\mathbb{Z}[x]/(x^2-100)) \longrightarrow \operatorname{Spec}\mathbb{Z}$.

Over a general point $s \neq (2), (5), (0)$, say, $s=(13)$, we have

$$X_s = \operatorname{Spec}(\mathbb{Z}[x]/(x^2-100, 13)) = \operatorname{Spec}\mathbb{F}_{13}[x]/(x-10)(x+10) \cong \operatorname{Spec}\mathbb{F}_{13} \times \operatorname{Spec}\mathbb{F}_{13}.$$

which is a scheme of two reduced points over $\mathbb{F}_{13}$.

Over $s=(2)$ (or (5))

$$X_s = \operatorname{Spec}(\mathbb{Z}[x]/(x^2-100, 2)) = \operatorname{Spec}\mathbb{F}_2/(x^2),$$ a scheme with an unreduced point over $\mathbb{F}_2$ (or $\mathbb{F}_5$).

Over $s=(0)$, $X_s = \operatorname{Spec}(\mathbb{Q}[x]/(x-10)(x+10))$, a scheme of two reduced points over $\mathbb{Q}$. This an example of generically reduced scheme with some unreduced fibers.

E.g. Families of plane curves.

$$\begin{array}{ccc} \operatorname{Spec}\mathbb{C}[x,y,t]/(ty-x^2) & \quad & \operatorname{Spec}\mathbb{C}[x,y,t]/(xy-t) \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C}[t] & & \operatorname{Spec}\mathbb{C}[t] \end{array}$$

generically reduced, with an unreduced fiber over 0.

generically irreducible, with a reducible fiber over 0.

Terminology:

(1). Let S be a scheme. A scheme over S or an S-scheme is just a scheme X together with a structural morphism $X \longrightarrow S$.

(2). A scheme over a ring R (R-scheme) is just a scheme over $\operatorname{Spec} R$.

(3). Base change. Given a scheme $X \longrightarrow S$ and a morphism $S' \longrightarrow S$, the base change of X is just $X' = X \times_S S' \longrightarrow S'$ which is a scheme over S'.

E.g. The fiber of $X \longrightarrow S$ at $s \in S$ is just the base change of X to s.

(4). A morphism of schemes X to Y over S is just a commuting diagram:

$$\begin{tikzcd} X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

Lemma: Suppose $f: X \longrightarrow Y$ is an open immersion (resp. closed, locally closed) of schemes over S. Let $S' \longrightarrow S$ be a morphism of schemes. Then the base change $f': X_{S'} \longrightarrow Y_{S'}$ is an open (resp. closed, locally closed) immersion.

This lemma follows from the next sublemma:

Sublemma:
$$\begin{tikzcd} X \times_S Y \arrow[r, "g"] \arrow[d] & Y \arrow[d] \\ X \arrow[r, "f"] & S \end{tikzcd}$$
f: open (resp. closed) immersion, then g is an open immersion.

Proof of lemma. By categorical non-sense, the top square is a fiber product square:

$$\begin{tikzcd} X_{S'} \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y_{S'} \arrow[r] \arrow[d] & Y \arrow[d] \\ S' \arrow[r] & S \end{tikzcd}$$

$$\begin{aligned} \operatorname{Mor}(T, X_{S'}) &= \operatorname{Mor}(T, X) \times_{\operatorname{Mor}(T,S)} \operatorname{Mor}(T, S') \\ &= \operatorname{Mor}(T, X) \times_{\operatorname{Mor}(T,Y)} (\operatorname{Mor}(T,Y) \times_{\operatorname{Mor}(T,S)} \operatorname{Mor}(T, S')) \\ &= \operatorname{Mor}(T, X) \times_{\operatorname{Mor}(T,Y)} \operatorname{Mor}(T, Y_{S'}) \end{aligned}$$

Hence by the sublemma, f open immersion $\Rightarrow$ so is f'. (resp. closed. Locally closed needs to be factored one step further.) $\square$

Pf of sublemma (In the closed immersion case, open immersion is easy).

A closed immersion is given by a quasi-coherent sheaf of ideals: $X \xrightarrow{i} S$

closed immersion $\Leftrightarrow$ $0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O}_S \longrightarrow i_* \mathcal{O}_X \longrightarrow 0$

Now $Im(g^*\mathcal{I} \longrightarrow \mathcal{O}_Y) = Im(g^*\mathcal{I} \longrightarrow g^*\mathcal{O}_S = \mathcal{O}_Y)$ is locally generated by sections, hence cuts out a closed subscheme $Z \subseteq Y$. Then $Z = X \times_S Y$. On the ring level, this is to say that:

$$\begin{tikzcd} & Y \arrow[d, "g"] \\ X \arrow[r, "i"] & S \end{tikzcd}$$

$$\begin{tikzcd} A/IA & A \arrow[l] \\ R/I \arrow[u] \arrow[ur, phantom, "?"] & R \arrow[l] \arrow[u] \end{tikzcd}$$

$\square$

Def. A morphism of schemes $f: X \longrightarrow S$ is called quasi-compact iff the map on topological spaces is quasi-compact. ($\Leftrightarrow$ $\forall U$ q.c. open in S, $f^{-1}(U)$ is q.c.).

Characterization of quasi-compact morphisms:

Prop. Let $f: X \longrightarrow S$ be a morphism. TFAE:

(1). f is q.c.

(2). $\forall U \subseteq S$ open affine, $f^{-1}(U)$ is q.c.

(3). $\exists$ an open affine covering $S = \bigcup_{i \in I} U_i$, s.t $f^{-1}(U_i)$ is q.c. for all $i \in I$.

(4). $\exists$ an open affine covering $S = \bigcup_{i \in I} U_i$, s.t. $f^{-1}(U_i)$ is a finite union of open affines.

Pf: (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4), easy. Note that affine schemes are q.c.

(4) $\Rightarrow$ (1). $\forall U$ q.c. open in S. Since the affines $D(h)$, $h \in A_i$, $U_i = \operatorname{Spec} A_i$ form a basis of topology for S, U is a finite union of such open affines. Now let $f^{-1}(U_i) = \bigcup_{j=1}^{n} \operatorname{Spec} B_j$. Then $f^{-1}(D(h)) = \bigcup_{j=1}^{n} \operatorname{Spec} B_{j\bar{h}_j}$, where $\bar{h}_j$ is the image of h under $A_i \longrightarrow B_j$. It follows that $f^{-1}(U)$ is a finite union of open affines, thus q.c. $\square$

Lemma.

(1). A base change of a q.c. morphism is a q.c.

(2). Composition of q.c. morphism is q.c.

(3). A closed immersion is q.c.

Pf: (1) Consider the fiber product. $\forall\, s' \in S$, $\exists$ an affine open neighborhood U of $g(s)$ and an affine neighborhood V of s s.t. $g(V) \subseteq U$. Now $f^{-1}(U)$ is covered by finitely many affine opens, say $W_1, \cdots, W_n$. Then $V \times_U W_i$ form a finite affine cover of $f'^{-1}(V)$.

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

(2). $X \xrightarrow{f} Y \xrightarrow{g} Z$. $\forall\, U$ open in Z, $g^{-1}(U) = \bigcup_{i=1}^{n} V_i$, V_i open affine in Y. $f^{-1}(V_i) = \bigcup_{j=1}^{N_i} W_{ij}$, W_{ij} open affine $\Rightarrow (g \circ f)^{-1}(U) = \bigcup_{i,j} W_{ij}$ is a finite union of open affines.

(3). Follows by def. since it's locally of the form $\operatorname{Spec} A/I \longrightarrow \operatorname{Spec} A$. □

An open immersion need not be q.c. in general. A conterexample is given by taking the locally closed subscheme in $U = \bigcup_{i=1}^{\infty} D(x_i) \subseteq \operatorname{Spec} \mathbb{C}[x_1, x_2, \cdots]$, defined locally on each $D(x_i) = \operatorname{Spec} \mathbb{C}[x_1, x_2, \cdots][\frac{1}{x_i}]$ as the closed subscheme $I_i = (x_1^i, \cdots, x_{i-1}^i, x_i - 1, x_{i+1}, \cdots)$. Then on $D(x_i x_j)$, $i \neq j$,

$$I_i \cdot \mathbb{C}[x_1, x_2, \cdots][(x_i x_j)^{-1}] = I_j \cdot \mathbb{C}[x_1, x_2, \cdots][(x_i x_j)^{-1}] = \mathbb{C}[x_1, x_2, \cdots][(x_i x_j)^{-1}]$$

Thus this sheaf of ideals glue to define a scheme on U. However, there is no closed subscheme on $\operatorname{Spec} \mathbb{C}[x_1, x_2, \cdots]$ giving rise to this closed subscheme structure on U. If there were, it would correspond to an ideal I of $\mathbb{C}[x_1, x_2, \cdots]$, and $\forall\, f \in I$, $f \notin I_N$ when $N > \deg f$.

§4. Valuative Criterion

Lemma. (algebra). $R \longrightarrow A$: a ring map. $T \subseteq \operatorname{Spec} A$ is closed. If $f(T)$ is closed under specialization (notation: $x \rightsquigarrow x'$ iff $x' \in \overline{\{x\}}$), where $f: \operatorname{Spec} A \longrightarrow \operatorname{Spec} B$. then $f(T)$ is closed.

Pf: Write $T = V(I)$, $I \subseteq A$. Set $J = \ker(R \to A \to A/I)$. Then we have

$$\begin{array}{ccccc} \operatorname{Spec}(A/I) = V(I) = & T & \subseteq & \operatorname{Spec} A \\ & \downarrow & & \downarrow \\ \operatorname{Spec}(R/J) = & V(J) & \subseteq & \operatorname{Spec} R \\ & \cup\!\text{I} & \leftarrow \text{want this be "="} & \\ & f(T) & & \end{array}$$

Thus we are reduced to the situation:

(1). $R \hookrightarrow A$ (2). $T = \operatorname{Spec} A$ (3). $f(T)$ is closed under specialization.

and we want to show $f(T) = \operatorname{Spec} R$.

Take $q \subseteq R$ any minimal prime. then R_q is a local ring with only 1 prime ideal. Furthermore $R_q \subseteq A_q \Rightarrow A_q \neq 0 \Rightarrow q \in \operatorname{Im}(f)$.

Now any prime of R is a specialization of some minimal prime of R. By (3), we get $f(T) = R$. $\square$

Def. $f: X \longrightarrow S$: map of topological spaces. We say specializations lift along f if $\forall\, f(x) = s$, and $s \rightsquigarrow s'$ in S, $\exists\, x' \in X$. $x \rightsquigarrow x'$ and $f(x') = s'$.

Lemma. (topology). (1). If specializations lift along f and $T \subseteq X$ is closed under specializations, so is $f(T)$.

(2) Specializations lift along closed maps between topological spaces. $\square$

Lemma. Let $f: X \longrightarrow S$ be a quasi-compact morphism of schemes. Then f is closed iff specializations lift along f.

Pf: "$\Rightarrow$" easy by the above lemma.

"$\Leftarrow$" Take T closed in X. We may cover S by affine opens U_i and try to show that $f(T) \cap U_i = f(T \cap f^{-1}(U_i))$ is closed in U_i. This reduces us to the case where S is affine. Since f is q.c., $X = \bigcup_{i=1}^n X_i$, and $X_i = \operatorname{Spec} A_i$ affine open. Set $T_i = X_i \cap T$. Now we know that $S = \operatorname{Spec} R$ and A_i is an R-algebra. $f(T) \subseteq \operatorname{Spec} R$ is closed under specialization and it's the image:

$$\begin{tikzcd} \coprod_{i=1}^n T_i \subseteq \coprod_{i=1}^n X_i = \operatorname{Spec}(\prod_{i=1}^n A_i) \arrow[d] \arrow[dr] & \\ f(T) \arrow[r, hook] & S \end{tikzcd}$$

We are done by the algebra lemma. $\square$

Lemma. Let $R \to A$ be a ring map, $f: \operatorname{Spec} A \to \operatorname{Spec} R$. Then $R \to A$ satisfies going up (GU) $\Leftrightarrow$ specializations lift along f. In particular f is closed as a map of topological spaces. $\square$

Rmk: If A is integral over $\operatorname{im}(R)$ then it satisfies GU. In particular, finite maps and surjections satisfy GU.

Def. Let K be a field. $A, B \subseteq K$ are local domains (not fields). We say A dominates B iff $B \subseteq A$ and $m_B = B \cap m_A$. This gives a partial ordering on the set of local domains contained in K. Valuation rings are the maximal elements under this relation.

Lemma. Given any local domain $R \subseteq K$, then $\exists$ a valuation ring $A \subseteq K$ s.t. A dominates R and $\operatorname{f.f.} A = K$. $\square$

Valuative Criterion: $f: X \to S$: morphism of schemes.
(E). We say f satisfies the existence part of the valuative criterion if given any solid diagram

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] \arrow[ur, dashed] & S \end{tikzcd}$$

where $K = f.f.(A)$ and A is a valuation ring, then the dotted arrow exists.
(U). Uniqueness part: if the dotted arrow exists, it is then unique.

Rmk: How to map $\operatorname{Spec} A$ into a scheme S if A is a local ring?

$$\operatorname{Mor}_{Sch}(\operatorname{Spec} A, S) = \{ (s, \mathcal{O}_{S,s} \xrightarrow{\psi} A),\ \psi: \text{loc. hom. of loc. rings} \}.$$

The inverse map is given as follows: $\forall\ (s, \psi: \mathcal{O}_{S,s} \to A)$, take an open affine nhd $\operatorname{Spec} R$ of s. Then $s \leftrightarrow \mathfrak{p} \subseteq R$ and $R \to R_{\mathfrak{p}} = \mathcal{O}_{S,s} \xrightarrow{\psi} A \Rightarrow \operatorname{Spec} A \to \operatorname{Spec} R \subseteq S$.
A special case is when $A = \mathcal{O}_{S,s}$, which gives a canonical morphism of schemes $\operatorname{Spec} \mathcal{O}_{S,s} \to S$, whose image is exactly those $s' \in S$ which specialize to s.

Lemma. Let $f: X \to S$ be a morphism of schemes. TFAE:
(1). f satisfies (E).
(2). Specializations lift along any base change of f.
Pf: "E $\Rightarrow$ universally E": Given a solid diagram:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X_{S'} \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} R \arrow[r] \arrow[ur, dashed, red] \arrow[urr, dashed, blue] & S' \arrow[r] & S \end{tikzcd}$$

The blue dotted arrow exists by assumption $\Rightarrow$ the red dotted arrow exists by the universal property of fiber product. Therefore to prove (1) $\Rightarrow$ (2), it suffices to prove specializations lift along f.

Let $s \rightsquigarrow s'$ in S, $x \in X$, $f(x) = s$ (assuming $s \neq s'$), we have:

$$\begin{tikzcd} \operatorname{Spec} \kappa(x) \arrow[rr] \arrow[d] & & X \arrow[d, "f"] \\ \operatorname{Spec} \kappa(s) \arrow[r] & \operatorname{Spec} \mathcal{O}_{S,s'} \arrow[r] & S \end{tikzcd}$$

On the algebra level, let R be the image ring of $\mathcal{O}_{S,s'}$ in K. Then R is not equal to the image of $\kappa(s)$ in K by our assumption that $s \neq s'$. Thus R is dominated by

$$\begin{tikzcd} \kappa(x) = K & R \arrow[l, hook'] \\ \kappa(s) \arrow[u, hook] & \mathcal{O}_{S,s'} \arrow[l] \arrow[u] \end{tikzcd}$$

a valuation ring A whose $f.f.(A) = K$:

$$\begin{tikzcd} \kappa(x) = K & A \arrow[l, hook'] & R \arrow[l, hook'] \\ \kappa(s) \arrow[u, hook] & & \mathcal{O}_{S,s'} \arrow[ll] \arrow[u] \end{tikzcd} \Rightarrow \begin{tikzcd} \operatorname{Spec} K \arrow[rrr] \arrow[d] & & & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[urrr, dashed] \arrow[r] & \operatorname{Spec} R \arrow[r] & \operatorname{Spec} \mathcal{O}_{S,s'} \arrow[r] & S \end{tikzcd}$$

(E) $\Rightarrow$ dotted arrow exists. Let x' be the image of the closed point of $\operatorname{Spec} A$ in X. Then $f(x') = s'$ by the commutativity of the diagram.

Conversely, given a solid diagram as below, we obtain, by definition of fiber product:

$$\begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & S \end{tikzcd} \Rightarrow \begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & X_A \triangleq X \times_S A \arrow[r] \arrow[d, "f'"] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "="] & \operatorname{Spec} A \arrow[r] & S \end{tikzcd}$$

Since specializations lift along f', the base change of f, $\exists\, x' \in X_A$ s.t. $x \rightsquigarrow x'$, where x is the image of $\operatorname{Spec} K$, and $f'(x')$ the closed point of $\operatorname{Spec} A$.

Now we have algebraically:

$$\begin{tikzcd} K & \mathcal{O}_{x', X_A} \arrow[l] \\ A \arrow[u] \arrow[r, equal] & A \arrow[u] \end{tikzcd}$$

$\Rightarrow$ The image ring R of $\mathcal{O}_{x',X_A}$ in K dominates A, thus must equal A. It follows that $\mathcal{O}_{x',X_A} \longrightarrow K$ factors through $\mathcal{O}_{x',X_A} \longrightarrow A$, which is a loc. hom. of loc. rings. This gives a desired section of $\operatorname{Spec} A \longrightarrow X_A$. By further composing with the projection $X_A \longrightarrow X$, we are done. $\square$

Def. A morphism is called universally closed (u.c.) iff $\forall\ S' \longrightarrow S$, the base change $X_{S'} = X \times_S S' \longrightarrow S'$ is closed.

Combining the topology lemma with the above one, we obtain:

Prop: Let $f: X \longrightarrow S$ be quasi-compact. TFAE:

(1). f is u.c.

(2). (E) holds for f. $\square$

Seperation Axioms

Motivation: In a topological space X, X is Hausdorff iff $\Delta: X \to X\times X$ is closed. ($X\times X$ with the product topology).

Lemma. For any morphism of schemes $f: X \to S$, the diagonal $\Delta: X \to X\times_S X$ is an immersion.

Pf: Define $W = \bigcup_{(U,V) \text{ with } *} U\times_V U \subseteq X\times_S X$ an open set, where $*$ is the condition: "$U \subseteq X$ open affine, $V\subseteq S$ open affine and $f(U) \subseteq V$".

Claim: $\Delta(X) \subseteq W$.

Indeed, $\forall x\in X$, $V\subseteq S$ open affine with $f(x)\in V$, then we may take $U\ni x$ open affine in X s.t. $f(U)\subseteq V$. Then $(*)$ is satisfied for (U,V) and $\Delta(x) = (x,x) \in U\times_V U$.

It suffices to show that $\Delta: X \to W$ is closed. Now whenever $(*)$ holds for (U,V), $U = \operatorname{Spec} A$, $V = \operatorname{Spec} R$,

$$\Delta: U \longrightarrow U\times_V U$$

is a closed immersion since it's associated with the ring map $A\otimes_R A \to A \to 0$.

$\square$

Cor 1. Δ is closed iff $\Delta(X) \subseteq X\times_S X$ is a closed subset.
(An immersion is closed iff the image is closed). $\square$

Cor 2. Given a commutative diagram:

$$\begin{tikzcd} X \arrow[rr, shift left, "a"] \arrow[rr, shift right, "b"'] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd}$$

The equalizer Z of a, b (exists since fiber product exists in schemes) is a locally closed subscheme of X. It's closed iff $\Delta_{Y/S}$ is closed.

Pf: a, b give $X\times_S X \xrightarrow{(a,b)} Y\times_S Y$. Then the equalizer is just the fiber product

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & Y \arrow[d, "\Delta"] \\ X\times_S X \arrow[r, "(a{,}b)"] & Y\times_S Y \end{tikzcd}$$

The result follows since immersions are stable under base change. $\square$

Def. Let $f: X \longrightarrow S$ be a morphism of schemes.

(1). We say f is seperated iff $\Delta_{X/S}$ is closed

(2). We say f is quasi-seperated iff $\Delta_{X/S}$ is quasi-compact.

(3). We say a scheme S is (quasi-) seperated if $\Delta_{S/\mathrm{Spec}\,\mathbb{Z}}$ is (quasi-) seperated.

Lemma. (Characterization of quasi-seperated morphisms).

Given $f: X \longrightarrow S$. TFAE:

(1). f is quasi-seperated.

(2). $\forall\, U, V$ open affines mapping into a common affine open in S, the open $U \cap V$ is quasi-compact.

(3). $\exists$ affine open covering $S = \bigcup_{i\in I} U_i$, $f^{-1}(U_i) = \bigcup_{j\in J_i} V_j$ affine open covering and $\forall\, j_1, j_2 \in J_i$, $V_{j_1} \cap V_{j_2}$ is quasi-compact.

$\square$

Lemma. (Characterization of seperated morphisms).

Given $f: X \longrightarrow S$. TFAE:

(1). f is seperated

(2). $\forall\, U, V$ open affines mapping into a common affine open in S, we have

(a). the open $U \cap V$ is affine.

(b). the map $\mathcal{O}_X(U) \otimes_{\mathbb{Z}} \mathcal{O}_X(V) \longrightarrow \mathcal{O}_X(U \cap V)$ is onto.

(3). For all $x, x' \in X$, $f(x) = f(x')$, $\exists$ affine opens $U \ni x$, $V \ni x'$ in X, mapping into a common affine open W of S, s.t. (a), (b) above holds.

Pf: (1) $\Rightarrow$ (2). Assume f seperated, and $U = \mathrm{Spec}\,A$, $V = \mathrm{Spec}\,B$ mapping into $W = \mathrm{Spec}\,R$, open affine in S. Then $\mathrm{Spec}(A \otimes_R B) = U \times_W V = p^{-1}(U) \cap q^{-1}(V) \subseteq X \times_S X$ is an affine open. f seperated $\Rightarrow$ Δ is a closed immersion and $U \cap V = \Delta^{-1}(U \times_W V)$ is closed thus equals $\mathrm{Spec}(A \otimes_R B / I)$ for some ideal $I \subseteq A \otimes_R B$.

Hence $A \otimes_{\mathbb{Z}} B \twoheadrightarrow A \otimes_R B \twoheadrightarrow A \otimes_R B / I$.

(2) ⇒ (3). Trivial

(3) ⇒ (1). Since such $U\times_W V$'s form an affine open cover of $X\times_S X$. It suffices to show that $\Delta^{-1}(U\times_W V) = U\cap V \longrightarrow U\times_W V$ is closed. But by our assumption, $U\cap V = \operatorname{Spec} C$ and $A\otimes_{\mathbb{Z}} B \longrightarrow A\otimes_R B \longrightarrow C$ is surjective, since $A\otimes_{\mathbb{Z}} B \twoheadrightarrow A\otimes_R B$, $A\otimes_R B \twoheadrightarrow C$ and Δ is a closed immersion. □

Cor. Any affine scheme is seperated.

Pf: $R\otimes_{\mathbb{Z}} R \twoheadrightarrow R$. □

Remarks: If $X\longrightarrow S$ is seperated and S is seperated, then the intersection of any two open affine in X is affine. Indeed, the composition of 2 seperated morphisms is seperated: $X\longrightarrow S \longrightarrow T$, two seperated morphisms, then:

$$\begin{tikzcd} X \arrow[r, "\Delta_S"] \arrow[dr, "\Delta_T"'] & X\times_S X \arrow[r] \arrow[d] & S \arrow[d] \\ & X\times_T X \arrow[r] & S\times_T S \end{tikzcd}$$

the right-hand-side diagram is a fiber diagram, thus the composition $X\longrightarrow X\times_T X$ is closed. Hence $X\longrightarrow \operatorname{Spec}\mathbb{Z}$ is seperated, and $U\cap V = \Delta_{\mathbb{Z}}^{-1}(U\times_{\mathbb{Z}} V)$, which is a closed subscheme of an affine scheme, thus affine.

Thm. (Valuative criterion of seperatedness).

Let $f: X\longrightarrow S$ be a morphism. Suppose

1). f is quasi-seperated

2). f satisfies $\mathcal{U}$.

Then f is seperated.

Rmk: We will see that later if S is locally noetherian, f is locally of finite type, then f is automatically quasi-seperated.

Pf: Need to check $X \xrightarrow{\Delta} X\times_S X$ is closed. Now given

$$\begin{tikzcd} \operatorname{Spec}K \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ \operatorname{Spec}A \arrow[r, "g"] & X\times_S X \end{tikzcd}$$

then $g=(a,b)$, which gives

$$\begin{tikzcd} \operatorname{Spec}K \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec}A \arrow[r] \arrow[ur, shift left, "a"] \arrow[ur, shift right, "b"'] & S \end{tikzcd}.$$

Now $U \Rightarrow a=b$, thus $\Delta\circ a=(a,a)=(a,b)=g$.

$$\begin{tikzcd} \operatorname{Spec}K \arrow[r] \arrow[d] & X \arrow[d, "\Delta"] \\ \operatorname{Spec}A \arrow[r, "g"] \arrow[ur, "a"] & X\times_S X \end{tikzcd}$$

$\square$

E.g. Construction of $\mathbb{P}^1_R$ by gluing:

$$\mathbb{A}^1_R = \operatorname{Spec}R[x] \supseteq D(x) \xleftrightarrow{\text{glue}} D(y) \subseteq \mathbb{A}^1_R = \operatorname{Spec}R[y]$$

$$x \leftrightarrow y^{-1}$$

$$x^{-1} \leftrightarrow y$$

$\mathbb{P}^1_R$ constructed in this way is seperated: we just check that for the open covering U, V, $\mathcal{O}(U)\otimes_{\mathbb{Z}}\mathcal{O}(V) \twoheadrightarrow \mathcal{O}(U\cap V)$:

$\mathcal{O}(U)$,	$\mathcal{O}(V)$	$\mathcal{O}(U\cap V)$	
$R[x]$,	$R[x] \longrightarrow$	$R[x]$	
$R[x]$,	$R[y] \longrightarrow$	$R[x,x^{-1}]$	$(y\mapsto x^{-1})$
$R[y]$,	$R[x] \longrightarrow$	$R[x,x^{-1}]$	$(y\mapsto x)$.
$R[y]$,	$R[y] \longrightarrow$	$R[y]$	

Claim: $\mathbb{P}^1_R \longrightarrow R$ is universally closed.

We just need to check E:

$$\begin{tikzcd} \operatorname{Spec}K \arrow[r] \arrow[d] & \mathbb{P}^1_R \arrow[d] \\ \operatorname{Spec}A \arrow[r] & \operatorname{Spec}R \end{tikzcd}$$

Suppose $\mathrm{Im}(\operatorname{Spec} K) \subseteq \operatorname{Spec} R[x]$:

$$\begin{tikzcd} K & R[x] \arrow[l, "\xi^{\#}"'] \\ A \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd}$$

If $\xi^{\#}(x) \in A$, we are done. Otherwise, $\xi^{\#}(x)^{-1} = \xi^{\#}(y) \in A$ since A is a valuation ring. Thus:

$$\begin{tikzcd} K & R[y] \arrow[l, "\xi^{\#}"'] \arrow[dl] \\ A \arrow[u] & R \arrow[l] \arrow[u] \end{tikzcd} \Rightarrow \begin{tikzcd} \operatorname{Spec} K \arrow[r] \arrow[d] & \operatorname{Spec} R[y] \arrow[d] \\ \operatorname{Spec} A \arrow[r] \arrow[ur] & \operatorname{Spec} R \end{tikzcd}$$

§5. Properties of Schemes

We first define quasi-coherent sheaves.

Def. (EGA) X: ringed space. An $\mathcal{O}_X$-module $\mathcal{F}$ is called quasi-coherent (QC) iff $\forall x \in X$, $\exists U \ni x$, open in X and an exact sequence:

$$\oplus_{j\in J}\mathcal{O}_U \longrightarrow \oplus_{i\in I}\mathcal{O}_U \longrightarrow \mathcal{F}|_U \longrightarrow 0$$

This def., being general in nature, is not that easy to use. However, in the category of schemes, we have:

Lemma. Let X be a scheme, $\mathcal{F}$ an $\mathcal{O}_X$-module. TFAE:

(1). $\mathcal{F}$ is Q.C.

(2). For all affine opens $U = \operatorname{Spec}R \subseteq X$, we have $\mathcal{F}|_U \cong \tilde{M}$ for some R-module M.

(3). $\exists$ an affine open covering $X = \bigcup_{i\in I}\operatorname{Spec}R_i$ s.t. $\mathcal{F}|_{U_i} \cong \tilde{M}_i$.

Lemma. (Mapping properties of $\tilde{M}$). $X = \operatorname{Spec}R$, M: an R-module. $\mathcal{G}$ an $\mathcal{O}_X$ module. Then $\operatorname{Mor}_{\mathcal{O}_X}(\tilde{M}, \mathcal{G}) = \operatorname{Hom}_R(M, \Gamma(X,\mathcal{G}))$.

$$\overset{\in}{\beta} \quad\mapsto\quad \beta_X : M = \Gamma(X,\tilde{M}) \longrightarrow \Gamma(X,\mathcal{G}).$$

Lemma. X: a scheme.

(a). Kernel, cokernel of maps between Q.C. sheaves are Q.C.

(b). If $0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$ is a s.e.s. of $\mathcal{O}_X$-modules, and 2 out of 3 are Q.C., then so is the third.

Pf: (a). It reduces to the case where X is affine by the first lemma. Now suppose $\tilde{\varphi}: \tilde{M} \longrightarrow \tilde{N}$ is an $\mathcal{O}_X$-module morphism, then by the previous lemma, $\tilde{\varphi}$ arises as some R-module map $\varphi: M \longrightarrow N$. Now, we will show that $\ker\tilde{\varphi} = \widetilde{(\ker\varphi)}$ and $\operatorname{coker}\tilde{\varphi} = \widetilde{(\operatorname{coker}\varphi)}$. All we need to do is to show that

$$0 \longrightarrow \widetilde{(\ker\varphi)} \longrightarrow \tilde{M} \longrightarrow \tilde{N} \longrightarrow \widetilde{(\operatorname{coker}\varphi)} \longrightarrow 0$$

is exact. But on stalks, it's just:

$$0 \longrightarrow (\ker\varphi)_{\mathfrak{p}} \longrightarrow M_{\mathfrak{p}} \longrightarrow N_{\mathfrak{p}} \longrightarrow (\operatorname{coker}\varphi)_{\mathfrak{p}} \longrightarrow 0$$

Thus it's exact since *localization* is exact.

Rmk: The above lemma says that "$\sim$" is a *fully-faithful functor from R-mod* to $\mathcal{O}_X$-mod. The proof says that it's exact.

(b). It suffices to show, on Spec$R$. if we have a s.e.s. of $\mathcal{O}_X$-modules:

$$0 \longrightarrow \widetilde{M}_1 \longrightarrow \mathcal{F} \longrightarrow \widetilde{M}_2 \longrightarrow 0$$

then $\mathcal{F} = \widetilde{M}$ for some M. It suffices to show that

$$0 \longrightarrow \underset{\substack{\| \\ M_1}}{\Gamma(X, \widetilde{M}_1)} \longrightarrow \Gamma(X, \mathcal{F}) \xrightarrow{\beta_X} \underset{\substack{\| \\ M_2}}{\Gamma(X, \widetilde{M}_2)} \longrightarrow 0$$

is exact, or, β_X is surjective. (Later we will see that this is true since for any Q.C. sheaf on $X = \operatorname{Spec} R$, $H^i(X, \widetilde{M}) = 0$, $\forall i > 0$).

Now $\forall m_2 \in M_2$, set $I = \{f \in R \mid f \cdot m_2 \in \operatorname{Im}\beta_X\}$, which is an ideal of R, and we will show that $I = R$. We have $X = \cup_{i=1}^{n} D(f_i)$ standard open covering, s.t. m_2 locally lifts, i.e. $\exists\, s_i \in \mathcal{F}(D(f_i))$, $\beta(s_i) = m_2|_{D(f_i)}$. Then:

$$s_i|_{D(f_if_j)} - s_j|_{D(f_if_j)} \in \ker\beta|_{D(f_if_j)} = \operatorname{Im} \widetilde{M}_1(D(f_if_j))$$

$$\Rightarrow \qquad s_i|_{D(f_if_j)} - s_j|_{D(f_if_j)} = \frac{m_{ij}}{(f_if_j)^A}$$

where by finiteness of i, j, we can choose one A large enough for all i, j. Fix i_0, set $s_{i_0}' = f_{i_0}^A s_{i_0}$, and $s_i' = f_{i_0}^A s_i + m_{i_0 i}/f_i^A \in \mathcal{F}(U_i)$ for $i \neq i_0$. We compute:

$$s_i' - s_{i_0}' = f_{i_0}^A s_i + m_{i_0 i}/f_i^A - f_{i_0}^A s_{i_0} = -f_{i_0}^A (s_{i_0} - s_i) + m_{i_0 i}/f_i^A = -m_{i_0 i}/f_i^A + m_{i_0 i}/f_i^A = 0$$

Now if $i \neq j$ both not equal to i_0, we have:

$$s_i' - s_j' = f_{i_0}^A(s_i - s_j) - m_{i_0 i}/f_i^A + m_{i_0 j}/f_j^A = f_{i_0}^A(m_{ij}/(f_if_j)^A) - m_{i_0 i}/f_i^A + m_{i_0 j}/f_j^A.$$

Note that as a section of $\Gamma(D(f_{i_0}f_if_j), \widetilde{M}_1) = (M_1)_{f_{i_0}f_if_j} = (M_{1 f_if_j})_{f_{i_0}}$

$$m_{ij}/(f_if_j)^A + m_{i_0 i}/(f_{i_0}f_i)^A - m_{i_0 j}/(f_{i_0}f_j)^A = (s_i - s_j) + (s_{i_0} - s_i) - (s_{i_0} - s_j) = 0$$

$\Rightarrow$ By multiplying a large enough power of $f_{i_0}^B$, the above elt on the l.h.s. will be killed in $M_{f_if_j}$, where again we can take one $B \gg 0$ to work for all i, j. Hence we have if $s_{i_0}'' = f_{i_0}^{A+B} s_i$, $s_i'' = f_{i_0}^{A+B} s_i + f_{i_0}^B m_{i_0 i}/f_i^A$ $(i \neq i_0)$, they would glue to give a section s of $\Gamma(X, \mathcal{F})$, i.e. $f_{i_0}^{A+B} m_2 = \beta_X(s)$. We may do this for any $i_0 \in \{1, \cdots, n\}$. Thus $R = \langle f_1^N, \cdots, f_n^N \rangle \subseteq I$ $(N \gg 0)$ $\Rightarrow$ $R = I$. $\square$

Pull-back of QC.

Let $f: X \longrightarrow S$ be a morphism of schemes, $\mathcal{F}$ Q.C. sheaf of $\mathcal{O}_S$-module. Then $f^*\mathcal{F}$ is Q.C. on X. In particular, if $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} R$, $\mathcal{F} = \tilde{M}$ on S, then $f^*\mathcal{F} = \widetilde{A \otimes_R M}$.

Pf: $\forall x \in X$, $\exists\, U$ open affine on X, $f(U) \subseteq V$ open affine in S. Then

$$f^*: \ \oplus_{i\in J} \mathcal{O}_V \longrightarrow \oplus_{i\in I} \mathcal{O}_V \longrightarrow \mathcal{F} \longrightarrow 0$$

$$\Rightarrow \ \oplus_{i\in J} \mathcal{O}_U \longrightarrow \oplus_{i\in I} \mathcal{O}_U \longrightarrow f^*\mathcal{F} \longrightarrow 0$$

(It's exact since on stalks $(f^*\mathcal{G})_x = \mathcal{G}_{f(x)} \otimes_{\mathcal{O}_{f(x),S}} \mathcal{O}_{x,X}$, and tensor is right exact).

In the affine case:

$$\begin{aligned} \operatorname{Mor}_{\mathcal{O}_X}(f^*\mathcal{F}, \mathcal{G}) &= \operatorname{Mor}_{\mathcal{O}_S}(\mathcal{F}, f_*\mathcal{G}) \\ &= \operatorname{Hom}_R(M, \Gamma(S, f_*\mathcal{G})) \\ &= \operatorname{Hom}_R(M, \Gamma(X, \mathcal{G})) \\ &= \operatorname{Hom}_A(M \otimes_R A, \Gamma(X, \mathcal{G})) \\ &= \operatorname{Mor}_{\mathcal{O}_X}(\widetilde{M \otimes_R A}, \mathcal{G}). \end{aligned}$$

□

Pushforward of Q.C.

If $f: \operatorname{Spec} A \longrightarrow \operatorname{Spec} R$, and N an A-module, then $f_*(\tilde{N}) = \tilde{N}_R$ where N_R is N considered as an R module via $R \longrightarrow A$. This is some sort of forgetful functor (forgetting its "bigger" A-module structure).

E.g. Suppose k is a field (or any ring), $X = \coprod_{n=1}^{\infty} \operatorname{Spec} k[x] \xrightarrow{f} \operatorname{Spec} k[x]$. Let $\mathcal{F}$ be $\mathcal{O}_X$. Then:

$$\Gamma(S, f_*\mathcal{F}) = \Gamma(X, \mathcal{F}) = \prod_{n=1}^{\infty} k[x]$$

$$\Gamma(D(x), f_*\mathcal{F}) = \Gamma(f^{-1}(D(x)), \mathcal{F}) = \prod_{n=1}^{\infty} k[x]_{(x)}.$$

There is a natural map:

$$\Gamma(S, f_*\mathcal{F}) \longrightarrow \Gamma(D(x), f_*\mathcal{F})$$

which induces $(\prod_{n=1}^{\infty} k[x])_{(x)} \longrightarrow (\prod_{n=1}^{\infty} k[x]_{(x)})$, which would be an isomorphism if $f_*\mathcal{F}$ were Q.C. This is not true since, for instance $(1, \frac{1}{x}, \frac{1}{x^2}, \cdots, \frac{1}{x^n}, \cdots)$ is not in the image.

Prop. If $f: X \longrightarrow S$ is quasi-seperated and quasi-compact, then f_* preserves Q.C. sheaves.

Pf: It reduces to S affine immediately. Now let $\mathcal{F}$ be a Q.C. $\mathcal{O}_X$-module. Write $X = \cup_{i=1}^{n} X_i$, X_i open affine and since X is quasi-seperated, we have $X_i \cap X_j = \cup_{k=1}^{N_{ij}} X_{ijk}$, X_{ijk} open affine. Now

$$0 \longrightarrow f_*\mathcal{F} \longrightarrow \oplus_i (f|_{X_i})_* (\mathcal{F}|_{X_i}) \longrightarrow \oplus_{ijk} (f|_{X_{ijk}})_* (\mathcal{F}|_{X_{ijk}})$$

is exact. Thus $f_*\mathcal{F}$ is the kernel of a morphism of Q.C. sheaves, previous lemma applies. □

Remark: f: being quasi-seperated is usually free, but quasi-compactness is not usually automatic.

Properties of Schemes

Let X be a scheme, P a property of rings.

Def. X is said "locally P" iff $\forall x \in X$, $\exists U \ni x$, affine open s.t. $P(\mathcal{O}(U))$.

Def. We say "P is local" iff

(a). $P(R) \Rightarrow P(R_f)$, $\forall f \in R$.

(b). If $f_1, \cdots, f_n$ generate R, $P(R_{f_i})$ $i = 1, \cdots, n \Rightarrow P(R)$.

Meta-lemma.

Let P be a local property of rings, and X a scheme. TFAE:

(1). X is locally P.

(2). $\forall U \subseteq X$ open affine, $P(\mathcal{O}(U))$.

(3). $\exists$ an open affine covering, $X = \cup_i U_i$ s.t. $P(\mathcal{O}(U_i))$.

(4). $\exists$ open covering $X = \cup_i X_i$, each X_i locally P.

Moreover, if this holds, then any open subscheme is locally P.

Pf: The only non-trivial part is $(3) \Rightarrow (2)$.

$\forall X = \cup_i U_i$ affine open s.t. $P(\mathcal{O}(U_i))$, $\forall i$. Now by a previous lemma, $\exists$

$U = \bigcup_{j=1}^{m} W_j$, W_j standard open in U and in some U_{ij}. Thus:

$$P(\mathcal{O}(U_{ij})) \Rightarrow P(\mathcal{O}(W_j)) \Rightarrow P(\mathcal{O}(U)). \qquad \square$$

Lemma. Being Noetherian is a *local property*.

Pf: (a). R Noetherian $\Rightarrow R_f$ Noetherian.

(b). $0 \longrightarrow R \longrightarrow \prod_{i=1}^{n} R_{f_i} \longrightarrow \prod_{i,j=1}^{n} R_{f_i f_j}$. Now R_{f_i} Noetherian implies $R_{f_i f_j}$ Noetherian. Thus R is Noetherian. being the kernel of Noetherian ring maps

$\square$

Def. A scheme X is called Noetherian iff X is *locally Noetherian* and quasi-compact.

Lemma: If $j: U \hookrightarrow X$ is an immersion and X is *locally* Noetherian, then j is quasi-compact.

Pf: X is covered by affine opens which are spectrum of Noetherian rings, and these opens are thus quasi-compact as topological spaces, and so are their subspaces. $\square$

Classes of morphisms associated to properties of ring maps.

Def: Let P be a property of ring maps.

(1). We say P is *local* if

(a). $\forall f \in R$, $P(R \longrightarrow A) \Rightarrow P(R_f \longrightarrow A_f)$

(b). $\forall f \in R$, $a \in A$ and $R_f \longrightarrow A$, then $P(R_f \longrightarrow A) \Rightarrow P(R \longrightarrow A_a)$

(c). $\forall R \longrightarrow A$, if $P(R \longrightarrow A_{a_i})$ with $(a_1, \cdots, a_n) = A \Rightarrow P(R \longrightarrow A)$.

(usually (a) & (b) are easy, and (c) needs some work).

(2). We say that P is stable under base change if $\forall R \longrightarrow A$, and $R \longrightarrow R'$ $P(R \longrightarrow A) \Rightarrow P(R' \longrightarrow R' \otimes_R A)$.

(3). We say that P is stable under composition if $\forall A \longrightarrow B$, $B \longrightarrow C$ ring maps, with $P(A \longrightarrow B)$, $P(B \longrightarrow C) \Rightarrow P(A \longrightarrow C)$.

(4). Let P be a property of ring maps and $f: X \longrightarrow S$ a morphism of schemes

We say f is locally of type P if $\forall x \in X$, $\exists x \in U \subseteq X$ affine open, $V \subseteq S$ affine open s.t. $f(U) \subseteq V$, and $P(\mathcal{O}_S(V) \longrightarrow \mathcal{O}_X(U))$.

Rmk: Usually we won't define a morphism to be locally of type P unless P is local!

E.g. local properties of ring maps.

- $P(R \longrightarrow A)$: finite type : "A is a finite type R-algebra"
- $P(R \longrightarrow A)$: finite presentation : "A is of finite presentation over R"
- $P(R \longrightarrow A)$: flat : "A is flat over R"
- $P(R \longrightarrow A)$: smooth : "A is a smooth R-algebra"

Pf of finite type (algebra):

(a): $P(R \longrightarrow A) \Leftrightarrow A = R[x_1, \cdots, x_n]/I \Rightarrow A_f = R_f[x_1 \cdots, x_n]/IR_f[x_1 \cdots x_n]$
$\Leftrightarrow P(R_f \longrightarrow A_f)$

(b): $P(R_f \longrightarrow A) \Leftrightarrow A = R_f[x_1, \cdots, x_n]/I \Leftrightarrow A = R[x_1, \cdots, x_n, y]/(I, yf-1)$
$\Rightarrow A_a = R[x_1 \cdots, x_n, y, z]/(I, yf-1, za-1) \Leftrightarrow P(R \longrightarrow A_a)$.

(c): $P(R \longrightarrow A_{a_i})$, $i = 1, \cdots, n$ $\Rightarrow A_{a_i} = R[x_{i1}, \cdots, x_{ik_i}]/I_i$. By definition, $\bar{x}_{ij}$ in A_{a_i} is of the form h_{ij}/a_i^N, where we take N large enough to work for all $i = 1, \cdots, n$ $j = 1, \cdots, k_i$. Since $D(a_i)$'s cover $\operatorname{Spec} A$, we have $1 = \sum a_i g_i$.

Claim:
$$R[u_i, v_{ij}, z_k] \longrightarrow A$$
$$u_i \mapsto a_i, \ v_{ij} \mapsto h_{ij}, \ z_k \mapsto g_k$$
is surjective. Indeed, $\forall a \in A$, $\exists N, M >> 0$
$$a = a \cdot 1 = \sum a_i^{N+M} \tilde{g}_i \cdot a \quad (\tilde{g}_i : \text{combination of } g_i, a_j\text{'s})$$
$$= \sum a_i^M h_{ij} \tilde{g}_i \quad (\text{where } M>>0 \text{ so that } a_i^M(a a_i^N - h_{ij}) = 0)$$
and the claim follows. $\Rightarrow P(R \longrightarrow A)$. $\square$

Lemma: Let $f: X \longrightarrow S$ be a morphism of schemes. TFAE:

(1). f is locally of type P.

(2). For every open affine $U \subseteq X$ and $V \subseteq S$ with $f(U) \subseteq V$, we have $P(\mathcal{O}_S(V) \longrightarrow \mathcal{O}_X(U))$.

(3). $\exists$ open covering $S=\bigcup_{i\in I} V_i$ and open coverings $f^{-1}(V_i)=\bigcup_{j\in I_i} U_j$ s.t. $f|\,U_j \longrightarrow V_i$ is locally of type P.

(4). $\exists$ affine open covering $S=\bigcup_{i\in I} V_i$ and affine open coverings $f^{-1}(V_i)=\bigcup_{j\in I_i} U_j$ s.t. $P(\mathcal{O}_S(V_i)\longrightarrow \mathcal{O}_X(U_j))$.

Moreover if f is locally of type P, then so is $f|_U : U \longrightarrow V$, where $U\subseteq X$, $V\subseteq S$ are open subschemes s.t. $f(U)\subseteq V$.

Pf: $(2)\Rightarrow(4)\Rightarrow(3)\Rightarrow(1)$ is trivial.

$(1)\Rightarrow(2)$: By def, $\exists\ U=\bigcup_i U_i$, $U_i=\operatorname{Spec}A_i$, $f(U_i)\subseteq V_i \overset{\text{open}}{\subseteq} S$, $V_i=\operatorname{Spec}R_i$ s.t. $P(R_i\longrightarrow A_i)$. But V_i is not necessarily in V. $\forall x\in U_i$, $f(x)\in V\cap V_i$. Thus $\exists$ $(f(x)\in)\, V_{ij}\subseteq V\cap V_i$, standard open in both V and V_i, say $V_{ij}=\operatorname{Spec}(R_i)_{h_j}$, $h_j\in R_i$. Now $f^{-1}(V_{ij})\cap U_i = U_i\times_{V_i} V_{ij} = \operatorname{Spec}(A_i\otimes_{R_i}(R_i)_{h_j})$, and by def (1).(a). we have $P((R_i)_{h_j}\longrightarrow A_i\otimes_{R_i}(R_i)_{h_j})$. Next, we may take $U_i'=\operatorname{Spec}(\mathcal{O}_X(U)_{a_i})\ni x$, standard open affine in both U and $U_i\cap f^{-1}(V_{ij})$, and since V_{ij} is also of the form $V_{ij}=\operatorname{Spec}(\mathcal{O}_S(V)_{t_j})$, by def (1).(b). $P(\mathcal{O}_S(V)_{t_j}=(R_i)_{h_j}\longrightarrow A_i\otimes_{R_i}(R_i)_{h_j})\Rightarrow P(\mathcal{O}_S(V)\longrightarrow \mathcal{O}_X(U)_{a_i})$. Since U is quasi-compact, finitely such $\mathcal{O}_X(U)_{a_i}$ will do. Finally, def (1).(c) $\Rightarrow P(\mathcal{O}_S(V)\longrightarrow\mathcal{O}_X(U))$. $\square$

Def. (1). We say a morphism of schemes is locally of finite type if it's locally of type P: finite type as in the example above.

(2). We say f is of finite type if it's locally of finite type and quasi-compact.

Def. (Variety) Let k be a field. A variety over k is an integral, seperated scheme of finite type over k.

Rmk: Here we don't require $k=\bar{k}$ as Hartshorne does. Note that varieties are not stable under base changes $k'\longrightarrow k$.

E.g. $\operatorname{Spec}\mathbb{Q}(i)$ is a variety over $\mathbb{Q}$, yet:

$$\begin{array}{ccc} \underbrace{\operatorname{Spec}\mathbb{C}\times\operatorname{Spec}\mathbb{C}}_{\text{Not a variety!}} = \operatorname{Spec}\mathbb{Q}(i)\times_{\operatorname{Spec}\mathbb{Q}}\operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{Q}(i) \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{C} & \longrightarrow & \operatorname{Spec}\mathbb{Q} \end{array}$$

Lemma. Let P be a property of ring maps.

(1). If P is local and stable under base change, then a morphism locally of type P is stable under base change.

(2). If P is local and stable under composition, then a morphism locally of type P is stable under composition.

Pf: (1) Let $S' \longrightarrow S$ be a base change map:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

Then $\forall s' \in S'$, $\exists U' \subseteq S'$ affine open, $U \subseteq S$ affine open s.t. $g(U') \subseteq U$. Now $f^{-1}(U) = \bigcup_{i\in I} V_i$, and $f'^{-1}(U') = \bigcup_i U' \times_U V_i$ is an affine open cover, and $P(\mathcal{O}_S(U) \longrightarrow \mathcal{O}_X(V_i))$

$$\Rightarrow P(\mathcal{O}_{S'}(U') \longrightarrow \mathcal{O}_{X'}(U' \times_U V_i) = \mathcal{O}_{S'}(U') \times_{\mathcal{O}_S(U)} \mathcal{O}_X(V_i)).$$

(2) Let $X \xrightarrow{f} Y \xrightarrow{g} Z$ be morphisms locally of type P. $\forall U \subseteq Z$ affine open $g^{-1}(U) = \bigcup_{i\in I} V_i$, $P(\mathcal{O}_Z(U) \longrightarrow \mathcal{O}_Y(V_i))$. Moreover $f^{-1}(V_i) = \bigcup_{j\in I_i} W_{ij}$, and we have $P(\mathcal{O}_Y(V_i) \longrightarrow \mathcal{O}_X(W_{ij}))$. Now P is stable under composition

$$\Rightarrow P(\mathcal{O}_Z(U) \longrightarrow \mathcal{O}_X(W_{ij}))$$

and $(g\circ f)^{-1}(U) = \bigcup_{i\in I, j\in I_i} W_{ij}$. $\square$

E.g. P: "finite type" is stable under base change and composition. Thus so are morphisms locally of finite type.

Lemma. If $f: X \longrightarrow S$ is locally of finite type, and S is locally Noetherian, then X is locally Noetherian. Consequently, f is quasi-seperated.

Pf: $\forall x \in X$, $\exists\ x \in U \subseteq X$, affine open, $V \subseteq S$ affine open s.t. $f(U) \subseteq V$. Now S locally Noetherian $\Rightarrow \mathcal{O}_S(V)$ is Noetherian. Now $\mathcal{O}_S(V) \longrightarrow \mathcal{O}_X(U)$ is of finite type $\Rightarrow \mathcal{O}_X(U)$ is Noetherian.

For the second statement, consider:

$$\begin{tikzcd} X \arrow[r, "\Delta"] \arrow[dr] & X\times_S X \arrow[d, "pr_1"'] \arrow[dr] & \\ & X \arrow[r, "f"] & S \end{tikzcd}$$

It suffices to show that $X \times_S X \longrightarrow S$ is locally Noetherian since Δ is an immersion. Now since $X \times_S X \longrightarrow X$ is locally of finite type by base change, and $f: X \longrightarrow S$ is of finite type, $X \times_S X \longrightarrow S$ is locally of finite type by composition. $\square$

Rmk: In general, the above proof shows that if f, g :
$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$
are locally of finite type, so is $X \times_S Y \longrightarrow S$.

Upshot: If f is of finite type and S is locally Noetherian, then $f_*(QC) = QC$.

§6. Projective Schemes

Notation:

$S = \bigoplus_{d\geq 0} S_d$: graded ring. $S_+ = \bigoplus_{d>0} S_d$: the irrelevant ideal.

$\mathrm{Proj}(S) \triangleq \{ \mathfrak{p} \subseteq S \mid \mathfrak{p} : \text{graded s.t. } S_+ \not\subseteq \mathfrak{p} \}$.

M: a graded S-module, $M = \bigoplus_{d\in\mathbb{Z}} M_d$, $S_a \cdot M_b \subseteq M_{a+b}$.

Note that $\mathrm{Proj}(S) \subseteq \mathrm{Spec}\, S$ is a subset. Give it the induced topology.

Let $f \in S_+$ be a homogeneous polynomial, $D_+(f) \triangleq D(f) \cap \mathrm{Proj}\, S$, and $M_{(f)} \triangleq \{ \frac{x}{f^n} \mid x \in M \text{ homogeneous}, \deg x = n \deg f \}$ $(\subseteq M_f)$. Then $D_+(f)$ is open in $\mathrm{Proj}(S)$.

Easy facts:

(a). $D_+(f)$ form a basis of topology of $\mathrm{Proj}\, S$.

(b). There is a natural bijection of sets $D_+(f) \longleftrightarrow \mathrm{Spec}\, S_{(f)}$, where $S_{(f)}$ is the subring of S_f consisting of elements of degree 0:

$$S_{(f)} = \{ \tfrac{x}{f^k} \mid x \text{ homogeneous, and } \deg x = k \cdot \deg f \}$$

$$\begin{tikzcd}
\mathrm{Spec}\, S \arrow[r, phantom, "\supset"] & D(f) \arrow[r, leftrightarrow] & \mathrm{Spec}\, S_f \arrow[drr] & & \\
\mathrm{Proj}\, S \arrow[u, phantom, "\cup\!\!\mathsf{I}"] \arrow[r, phantom, "\supseteq"] & D_+(f) \arrow[u, phantom, "\cup\!\!\mathsf{I}"] \arrow[r, leftrightarrow] & ?? \arrow[u, phantom, "\cup\!\!\mathsf{I}"] \arrow[rr, leftrightarrow, "1:1"] & & \mathrm{Spec}\, S_{(f)}
\end{tikzcd}$$

This diagram is explained by the following:

Lemma. If S is a $\mathbb{Z}$-graded ring, $S = \bigoplus_{d\in\mathbb{Z}} S_d$, and assume $\exists\, d>0$, $f \in S_d$ s.t. f is invertible. Then:

$$\begin{tikzcd}
\mathfrak{q} \arrow[d, mapsto] & \sqrt{\mathfrak{p}S} & \mathrm{Spec}\, S \arrow[d] \arrow[r, phantom, "\supset"] & \{\mathbb{Z}\text{-graded prime ideals of } S\} \\
\mathfrak{q} \cap S_0 & \mathfrak{p} \arrow[u, mapsto] & \mathrm{Spec}\, S_0 \arrow[ur, "\varphi"] &
\end{tikzcd}$$

φ is 1:1 and a homeomorphism. $\square$

Note that in the special case $S = S_0[x, x^{-1}]$, then $\mathrm{Spec}\, S \cong \mathrm{Spec}\, S_0 \times \mathbb{G}_m$

where $\mathbb{G}_m$ is the group scheme $\operatorname{Spec}\mathbb{Z}[x,\frac{1}{x}]$ $(\operatorname{Hom}(T,\mathbb{G}_m)\cong\Gamma(T,\mathcal{O}_T^*))$

If $\mathfrak{p}\subseteq S$ is a homogeneous prime, then we can define

$$M_{(\mathfrak{p})}\triangleq\{\frac{x}{f}\mid x,f \text{ homog. of the same degree, } f\notin\mathfrak{p}\}\subseteq M_{\mathfrak{p}}$$

$S_{(\mathfrak{p})}$ is defined by regarding S as a homog. S-module.

(c). If $D_+(g)\subseteq D_+(f)$ then:

- $g^e=af$ for some $e\geq 1$ and a homog.
- The diagram is defined and commutes by the above fact:

$$\begin{tikzcd} S_f \arrow[r] & S_g \quad (\text{localization w.r.t. } g) \\ S_{(f)} \arrow[u, hook] \arrow[r] & S_{(g)} \arrow[u, hook] \quad (\text{localization w.r.t. } \frac{g^{\deg f}}{f^{\deg g}}) \end{tikzcd}$$

- Similar diagrams exists with :

$$\begin{tikzcd} M_f \arrow[r] & M_g \\ M_{(f)} \arrow[u, hook] \arrow[r] & M_{(g)} \arrow[u, hook] \end{tikzcd}$$

- The diagram commutes:

$$\begin{tikzcd} D_+(f) \arrow[r, phantom, "\supseteq"] & D_+(g) \\ \operatorname{Spec}S_{(f)} \arrow[u, "\varphi_f"'] & \operatorname{Spec}S_{(g)} \arrow[l] \arrow[u, "\varphi_g"'] \end{tikzcd}$$

- For every $h\in S_{(f)}$, $\exists\, g\in S_+$ homog. s.t. $D_+(g)=\varphi_f(D(h))$. Here $D(h)$ is taken in $\operatorname{Spec}S_{(f)}$

Prop. / Def. S: graded ring, M: graded S-module. Then:

(a). The structure sheaf $\mathcal{O}_{\operatorname{Proj}S}$ is the unique sheaf of rings on $\operatorname{Proj}S$ s.t.

$$\mathcal{O}_{\operatorname{Proj}S}(D_+(f))=S_{(f)}.$$

and with restriction maps given by:

$$\begin{tikzcd} \mathcal{O}_{\operatorname{Proj}(S)}(D_+(f))=S_{(f)} \arrow[r] \arrow[d, hook] & S_{(g)}=\mathcal{O}_{\operatorname{Proj}(S)}(D_+(g)) \arrow[d, hook] \\ S_f \arrow[r] & S_g \end{tikzcd}$$

In particular, $\mathcal{O}_{\mathrm{Proj}S,\mathfrak{p}} = S_{(\mathfrak{p})}$.

(b). The pair $(\mathrm{Proj}\,S, \mathcal{O}_{\mathrm{Proj}S})$ is a scheme, and the opens $D_+(f)$ are affine and isomorphic to $\mathrm{Spec}\,S_{(f)}$.

(c). $\exists$ a unique sheaf of $\mathcal{O}_{\mathrm{Proj}S}$-modules $\widetilde{M}$ with $\widetilde{M}(D_+(f)) = M_{(f)}$ and restrictions given by:

$$\begin{tikzcd} \widetilde{M}(D_+(f)) = M_{(f)} \arrow[r] \arrow[d, hook] & M_{(g)} = \widetilde{M}(D_+(g)) \arrow[d, hook] \\ M_f \arrow[r] & M_g \end{tikzcd}$$

(d). $\widetilde{M}$ is a quasi-coherent sheaf of $\mathcal{O}_{\mathrm{Proj}S}$-modules, i.e. $\widetilde{M}|_{D_+(f)} \cong \widetilde{M_{(f)}}$

(e). There is a canonical map : $M_0 \longrightarrow \Gamma(\mathrm{Proj}\,S, \widetilde{M})$, which when restricted to $D_+(f)$ gives the map $M_0 \longrightarrow M_{(f)}$, $x \mapsto \frac{x}{1}$.

(f). There is a canonical morphism of schemes:

$$\mathrm{Proj}(S) \longrightarrow \mathrm{Spec}\,S_0$$

coming from $S_0 \longrightarrow \Gamma(\mathrm{Proj}\,S, \widetilde{S}) = \Gamma(\mathrm{Proj}\,S, \mathcal{O}_{\mathrm{Proj}S})$.

Def. For $n \in \mathbb{Z}$, let $M(n)$ be the graded S-module with $M(n)_d = M_{d+n}$.

$\mathcal{O}_{\mathrm{Proj}S}(n) \triangleq \widetilde{S(n)}$ (twist of structure sheaf)

Note that $S_n = S(n)_0 \longrightarrow \Gamma(\mathrm{Proj}\,S, \mathcal{O}_{\mathrm{Proj}S}(n))$

Rmk: In general, the above map is neither surjective nor injective, and $\mathcal{O}_{\mathrm{Proj}S}(n)$ need not be invertible.

Constructions with $\mathcal{O}_{\mathrm{Proj}S}(n)$:

Given graded S modules M, N, we have a canonical $\mathcal{O}_{\mathrm{Proj}S}$-module map

$$\widetilde{M} \otimes_{\mathcal{O}} \widetilde{N} \longrightarrow \widetilde{M \otimes_S N}$$

On each $D_+(f)$, $M_{(f)} \otimes_{S_{(f)}} N_{(f)} \longrightarrow (M \otimes_S N)_{(f)}$ is given by:

$$\frac{m}{f^k} \otimes \frac{n}{f^l} \mapsto \frac{m \otimes n}{f^{k+l}}$$

This gives multiplication maps:

$$\mathcal{O}_{\mathrm{Proj}S}(n) \otimes_{\mathcal{O}_{\mathrm{Proj}S}} \mathcal{O}_{\mathrm{Proj}S}(m) \longrightarrow \mathcal{O}_{\mathrm{Proj}S}(m+n). \qquad (m_1)$$

$$\mathcal{O}_{\mathrm{Proj}S}(n) \otimes_{\mathcal{O}_{\mathrm{Proj}S}} \widetilde{M} \longrightarrow \widetilde{M(n)} \qquad (m_2)$$

E.g. Proj$S$ need not be quasi-cpt.
Take $S = \mathbb{C}[x_1, x_2, \cdots]$, then Proj$S$ is not cpt. $D_+(x_i)$ $(i=1,2,\cdots)$ form an open cover, but no finite subset of $D_+(x_i)$'s cover it.

Lemma. $\mathrm{Proj}\, S \longrightarrow \mathrm{Spec}\, S_0$ is seperated.
Pf: By our previous results, it suffices to show that

(i). $D_+(f) \cap D_+(g) = D_+(fg)$ is affine (true since it's $\mathrm{Spec}\, S_{(fg)}$)

(ii). $S_{(f)} \otimes_{\mathbb{Z}} S_{(g)} \twoheadrightarrow S_{(fg)}$.

But $\forall \frac{a}{f^n g^m} \in S_{(fg)}$, then $\deg a = n\deg f + m \deg g$. thus it comes from:

$$\frac{ag^{\ell}}{f^{n+k}} \otimes \frac{f^k}{g^{m+\ell}} \longmapsto \frac{a}{f^n g^m}$$

(for instance, we may take $k = \deg g \cdot (\deg f)^r$, $\ell = (\deg f)^{r+1} - m$. for $r \gg 0$). □

Def. For R a ring, $R[x_0, x_1, \cdots, x_n]$ the graded algebra with $\deg x_i = 1$. Set $\mathbb{P}^n_R \triangleq \mathrm{Proj}\, R[x_0, \cdots, x_n] \longrightarrow \mathrm{Spec}\, R$, the projective n-space.

Lemma. Let $Y = \mathrm{Proj}\, S$, and assume $Y = \bigcup_{f \in S_1} D_+(f)$. Then each $\mathcal{O}_Y(n)$ is invertible and the multiplication maps m_1 and m_2 are isomorphisms.
Pf: Pick $f \in S_1$. $\mathcal{O}_Y(n)|_{D_+(f)} = \widetilde{S(n)_{(f)}} = \widetilde{(S_f)_n}$ ($(S_f)_n$: as an $S_{(f)}$-mod).
But $f^n \in (S_f)_n$ and

$$\begin{aligned} S_{(f)} &\longrightarrow (S_f)_{(n)} \\ x &\longmapsto f^n . x \end{aligned}$$

is an isomorphism.

For m_1 and m_2

$$(S_f)_n \otimes_{S_{(f)}} (S_f)_{(m)} \longrightarrow (S_f)_{(n+m)} \ : x f^n \otimes y f^m \mapsto xy f^{n+m}$$
$$(M_f)_n \otimes_{S_{(f)}} (S_f)_{(m)} \longrightarrow (M_f)_{(n+m)} \ : m f^n \otimes y f^m \mapsto ym f^{n+m}$$

are isomorphisms. □

Rmk: 1) In the situation as in the lemma, ($\bigcup_{f \in S_1} D_+(f) = Y$), we have:

$$S \longrightarrow \oplus_{d\geq 0}\Gamma(Y, \mathcal{O}_Y(d)) \triangleq \Gamma_*(Y, \mathcal{O}_Y(1))$$

is a graded ring map.

2). In this case, $\mathcal{O}_Y(n) \cong \mathcal{O}_Y(1)^{\otimes n}$. Since m_1 & m_2 are isomorphisms,

$$\widetilde{M(n)} \cong \tilde{M} \otimes_{\mathcal{O}} \mathcal{O}(1)^n$$

(note that $M \otimes_S S(n) \cong M(n)$).

Def. $\Gamma_*(Y, \tilde{M}) \triangleq \oplus_{n\in\mathbb{Z}} \Gamma(Y, \tilde{M}(n))$, which is a $\Gamma_*(Y, \mathcal{O}_Y(1))$-module.

There is a natural map:

$$\begin{tikzcd} M \arrow[r, "??"] & \Gamma_*(Y, \tilde{M}) \\ M_d \arrow[u] \arrow[r] & \Gamma(Y, \tilde{M}(d)) \arrow[u] \end{tikzcd}$$

(← an S-module map)

Question: Is the map ?? an isomorphism? or how close to being an iso?
We know that Q.C. sheaves on Spec $R \longleftrightarrow$ modules over R. Is every Q.C. $\mathcal{O}_Y$-module of the form $\tilde{M}$?

Lemma. The morphism $\mathbb{P}^n_R \longrightarrow \operatorname{Spec} R$ is quasi-cpt, of finite type, seperated, and universally closed.

Pf: quasi-cpt: $\mathbb{P}^n_R = \cup_{i=0}^n D_+(x_i) = \cup_{i=0}^n \operatorname{Spec} R[\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_0}]$. ($\mathfrak{p} \in \operatorname{Proj} S \Rightarrow \mathfrak{p} \not\supseteq S_+ \Rightarrow \mathfrak{p} \not\ni x_i$ for some i).

Also $R[\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_0}]$ is a finitely generated R-algebra.

Seperatedness followes from a lemma above.

Universally closed-ness uses valuative criterion. □

Def. A morphism of schemes is called proper iff it's of finite type, seperated, and universally closed.

Projective space as an example of Proj.

Consider $\mathbb{P}^n_R = \operatorname{Proj}(R[x_0, \cdots, x_n]) \longrightarrow \operatorname{Spec} R$.

Lemma: (a). $\mathbb{P}^n_R = \bigcup_{i=0}^n D_+(X_i)$

(b). $D_+(X_i) \cong \mathbb{A}^n_R$.

(c). $D_+(X_iX_j)$ is affine.

Pf: (a). $S_+ = (X_0, \cdots, X_n)$ so $D_+(X_i)$'s cover $\mathbb{P}^n_R$.

(b). $D_+(X_i) \cong \mathbb{A}^n_R \cong \operatorname{Spec} R[\frac{X_1}{X_i}, \cdots \frac{X_n}{X_i}]$.

(c).
$$\begin{tikzcd}
\mathbb{A}^n_R \cong \operatorname{Spec} R[\frac{X_\alpha}{X_i}] & \mathbb{A}^{n-1}_R \times_{\operatorname{Spec} R} \mathbb{G}_{m,R} \arrow[l, hook'] \arrow[r, hook] & \operatorname{Spec} R[\frac{X_\alpha}{X_j}] \cong \mathbb{A}^n_R \\
D_+(X_i) \arrow[u, equal] & D_+(X_iX_j) \arrow[u, equal] \arrow[l, hook'] \arrow[r, hook] \arrow[d, equal] & D_+(X_j) \arrow[u, equal] \\
 & \operatorname{Spec}[\frac{X_\alpha}{X_i}][(\frac{X_j}{X_i})^{-1}] \arrow[d, equal] & \\
 & \operatorname{Spec}[\frac{X_\alpha}{X_j}][(\frac{X_i}{X_j})^{-1}] &
\end{tikzcd}$$
□

Rmk: We can redefine $\mathbb{P}^n_R$ as the scheme one gets by gluing $(n+1)$ standard affine spaces along the opens $D_+(X_iX_j)$ above.

Lemma. The canonical maps $R[X_0, \cdots, X_n]_d \longrightarrow \Gamma(\mathbb{P}^n_R, \mathcal{O}(d))$ are isomorphisms for all $d \in \mathbb{Z}$. ($n > 0$!).

Pf: By the sheaf condition, we have.

$$0 \longrightarrow \Gamma(\mathbb{P}^n, \mathcal{O}(d)) \longrightarrow \bigoplus_{i=0}^n \Gamma(D_+(X_i), \mathcal{O}(d)) \xrightarrow{\varphi} \bigoplus_{i,j=0}^n \Gamma(D_+(X_iX_j), \mathcal{O}(d))$$

$$\operatorname{Ker}\left(\bigoplus_{i=0}^n (R[X_0, \cdots, X_n]_{x_i})_d \longrightarrow \bigoplus_{i,j=0}^n (R[X_0, \cdots, X_n]_{x_ix_j})_d\right) = ?$$

Given $(F_i/x_i^{n_i})_{i=0,\cdots,n}$, $X_i \nmid F_i$, $\deg F_i - n_i = d$, we have $F_i/x_i^{n_i} - F_j/x_j^{n_j} = 0$, or equivalently, $X_j^{n_j} F_i = X_i^{n_i} F_j \Rightarrow X_j \mid F_j$ or $n_j = 0$. By assumption, $X_i \nmid F_i$, thus $n_i = 0$ for all i. Hence we have polynomials F_i's to start with, $F_i - F_j = 0 \Rightarrow F_i = F$ for all $i = 0, \cdots, n$. $F \in R[X_0, \cdots, X_n]_{(d)}$. □

How to map into $\mathbb{P}^n_R$?

- Motivation: From topology, we know that $\mathbb{P}^\infty \cong B(\mathbb{C}^*)$. $[X, \mathbb{P}^\infty] \cong \{$line bundles on $X\}$.

Recall that: on $\mathrm{Proj}(S)$, where S is generated by degree 1 elts over R.

- $\mathcal{O}_{\mathbb{P}^n}(1)$ is an invertible $\mathcal{O}_{\mathbb{P}^n}$ - module
- $\mathcal{O}_{\mathbb{P}^n}(n) = \mathcal{O}_{\mathbb{P}^n}(1)^{\otimes n}$.
- $\Gamma(\mathbb{P}^n, \mathcal{O}(1)) = R\langle X_1, \cdots, X_n \rangle$, where $\{X_1, \cdots, X_n\}$ generate $\mathcal{O}_{\mathbb{P}^n}(1)$ over $\mathcal{O}_{\mathbb{P}^n}$.

Def. Let X be a scheme, $\mathcal{L}$ an invertible $\mathcal{O}_X$-module.

1). Given $s \in \Gamma(X, \mathcal{L})$, we set $X_s = \{x \in X \mid s_x \notin m_x \mathcal{L}_x\}$. We have shown that X_s is an open set of X.

2). Given sections $s_0, \cdots, s_n \in \Gamma(X, \mathcal{L})$, we say they generate $\mathcal{L}$ over X iff $X = \bigcup_{i=0}^{n} X_{s_i}$.

Trivial observations.

- If $F \in R[X_0, \cdots, X_n]_d$, $d > 0$, then think of it as a global section, we have $(\mathbb{P}^n_R)_F = D_+(F)$.
- If $f: Y \longrightarrow X$ is a morphism of schemes, then $f^{-1}(X_s) = Y_{f^*(s)}$, $f^*(s) \in \Gamma(Y, f^*\mathcal{L})$.
- Let $\varphi: X \longrightarrow \mathbb{P}^n_R$ be a morphism, then we get $\mathcal{L} \triangleq \varphi^*(\mathcal{O}_{\mathbb{P}^n}(1))$ an invertible sheaf on X and $s_i = \varphi^*(X_i)$ $i = 0, \cdots, n$ sections of $\Gamma(X, \mathcal{L})$ which generate $\mathcal{L}$ over X.

Converse Thm. Given a scheme X over R, an invertible sheaf $\mathcal{L}$ on X and $(n+1)$ global sections $s_0, \cdots, s_n$ of $\mathcal{L}$ which generate $\mathcal{L}$, then $\exists!$ morphism

$$\varphi_{(\mathcal{L}, s_0, \cdots, s_n)} : X \longrightarrow \mathbb{P}^n_R$$

which is characterized by:

(i). $\varphi^*_{(\mathcal{L}, s_0, \cdots, s_n)}(\mathcal{O}_{\mathbb{P}^n}(1)) = \mathcal{L}$

(ii). $\varphi^*(X_i) = s_i$.

Pf: Let $p \in X$, choose i s.t. $p \in X_{s_i}$. Choose an affine open $p \in U \subseteq X_{s_i}$, write $U = \mathrm{Spec} A$, where A is an R-module. Then $\mathcal{L}|_U \cong \widetilde{M}$, where $s_j|_U = m_j \in M$. Since $U \subseteq X_{s_i}$, $M = A m_i$. Write $m_j = f_j \cdot m_i$ for some unique $f_j \in A$. Define:

$$\operatorname{Spec} A = U \longrightarrow D_+(X_i) = \operatorname{Spec} R[\tfrac{X_j}{X_i}] \subseteq \mathbb{P}^n_R$$

by the ring map:

$$\begin{aligned} R[\tfrac{X_j}{X_i}] &\longrightarrow A \\ \tfrac{X_j}{X_i} &\mapsto f_j \end{aligned}$$

Then it's routine to check the rest of the thm. $\square$

Rmk: This thm says that $\operatorname{Mor}(X, \mathbb{P}^n) \dashrightarrow \operatorname{Pic}(X)$.

Particular cases.

(i). $X = \operatorname{Spec} B$ where B is an R-algebra, $\mathcal{L} = \mathcal{O}_X$. Then $s_0, \cdots, s_n$ correspond to $f_0, \cdots, f_n \in B = \Gamma(X, \mathcal{O}_X)$ s.t. $\langle f_1, \cdots, f_r \rangle = B$. Then by our thm, $\exists \operatorname{Spec} B \longrightarrow \mathbb{P}^n_R$. and we shall describle this morphism.

Note that we have:

$$\begin{array}{ccc} \mathbb{A}^{n+1}_R \setminus V(X_0, \cdots, X_n) & \supseteq & D(X_i) \\ \downarrow \pi & & \downarrow \pi \\ \mathbb{P}^n_R & \supseteq & D_+(X_i) \end{array}$$

Then $f_0, \cdots, f_n$ give rise to a morphism of $\operatorname{Spec} B$ to $\mathbb{A}^n_R$ avoiding $V(X_0, \cdots, X_n)$. Since $(f_0, \cdots, f_n) = B$. Then the morphism in the theorem corresponds to the composition of this morphism with π.

(ii). How to map $\mathbb{P}^1_k$ into $\mathbb{P}^n_R$ (k: a field)

$\mathcal{L} = \mathcal{O}_{\mathbb{P}^1}(d)$:

1). $d < 0$, not possible, since $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d)) = 0$ in this case.

2). $d = 0$. $s_0, \cdots, s_n \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = k$. (they generate k over $\operatorname{Spec} k$!)

Claim: $\varphi(\mathcal{O}_{\mathbb{P}^1}, s_0, \cdots, s_n)$ is constant.

Indeed this morphism factors through
$$\begin{tikzcd} \mathbb{P}^1 \arrow[rr] \arrow[dr] & & \mathbb{P}^n \\ & \operatorname{Spec} k \arrow[ur, "(s_0 : \cdots : s_n)"'] & \end{tikzcd}$$

3). $d > 0$. $s_0, \cdots, s_n \longleftrightarrow F_0, \cdots, F_n \in k[X_0, \cdots, X_n]_d$, with $F_0, \cdots, F_n$ having no common zeros in $\mathbb{A}^2_k$ except $(0,0)$. We denote this morphism by $(X_0 : X_1) \longmapsto (F_0 : \cdots : F_n)$.

Let's desribe this morphism on $D_+(X_0)$: write $f_i = F_i/X_0^d = F_i(1, \frac{X_i}{X_0}) \in k[\frac{X_i}{X_0}]$.
The above condition implies that: $(f_0, \cdots, f_n) = k[\frac{X_i}{X_0}] \Rightarrow$

$$\begin{tikzcd} & \mathbb{A}^{n+1}_R \setminus \{0\} \arrow[d, "\pi"] \\ \operatorname{Spec} k[\frac{X_i}{X_0}] \arrow[ur] \arrow[r, "\varphi|_{D_+(X_0)}"'] & \mathbb{P}^n_R \end{tikzcd}$$

Closed subschemes of $\mathbb{P}^n_R$

Prop. Let $Z \hookrightarrow \mathbb{P}^n_R$ be a closed subscheme, then there exists a graded ideal $I \subseteq R[X_0, \cdots, X_n]$ s.t.

(1). $Z = V_+(I)$ as subsets.

(2). $Z \cong \operatorname{Proj}(R[X_0, \cdots, X_n]/I)$

(3). The ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n_R}$ of Z equals $\tilde{I} \subseteq R[X_0, \cdots X_n]^{\sim} = \mathcal{O}_{\mathbb{P}^n_R}$.

This will follow from a slightly more general result about quasi-coherent sheaves on $\mathbb{P}^n_R$.

Prop. (Hartshorne, II 5.15). Let $\mathcal{F}$ be a q.c. sheaf on $\mathbb{P}^n_R$. Then $\mathcal{F} = \tilde{M}$ for some graded $R[X_0, \cdots, X_n]$-module M. In fact, we can take

$$M = \Gamma_*(\mathbb{P}^n_R, \mathcal{F}) = \bigoplus_{d \in \mathbb{Z}} \Gamma(\mathbb{P}^n_R, \mathcal{F}(d)).$$

Rmk: Suppose $\varphi: S_1 \longrightarrow S_2$ is a homomorphism of graded rings, then

$$\operatorname{Proj}(S_2) \dashrightarrow \operatorname{Proj}(S_1)$$

is only well-defined as a morphism on an open set $U(\varphi)$:

$$U(\varphi) = \bigcup_{f \in (S_1)_+} D_+(\varphi(f))$$

where on each $D_+(\varphi(f))$ the morphism is given by

$$\begin{tikzcd} D_+(\varphi(f)) \arrow[r] \arrow[d, equal] & D_+(f) \arrow[d, equal] \\ \operatorname{Spec} S_{2(\varphi(f))} \arrow[r] & \operatorname{Spec} S_{1(f)} \end{tikzcd}$$

with the bottom arrow induced from $S_{1(f)} \longrightarrow S_{2(\varphi(f))}$.

Note that $U(\varphi) = \mathrm{Proj}(S_2)$ iff $(S_2)_+ \subseteq \sqrt{\varphi((S_1)_+)S_2}$. A special case where this is automatically true is when $S_1 \twoheadrightarrow S_2$, in which case the morphism

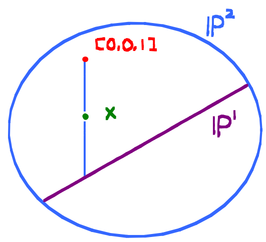
$$\mathrm{Proj}(S_2) \hookrightarrow \mathrm{Proj}(S_1)$$

is a closed immersion.

E.g. $\mathbb{C}[X,Y] \overset{\varphi}{\hookrightarrow} \mathbb{C}[X,Y,Z]$, the standard inclusion.

$$\Rightarrow \quad \begin{array}{ccc} \mathbb{P}^2 = \mathrm{Proj}(\mathbb{C}[X,Y,Z]) & \dashrightarrow & \mathrm{Proj}(\mathbb{C}[X,Y]) = \mathbb{P}^1 \\ \cup\!\!\mid & \nearrow & \\ U(\varphi) = D_+(X) \cup D_+(Y) & & \end{array}$$

On $D_+(X)$, the morphism is induced from $\mathbb{C}[\frac{Y}{X}] \hookrightarrow \mathbb{C}[\frac{Y}{X}, \frac{Z}{X}]$, i.e. $\mathbb{A}^2 \longrightarrow \mathbb{A}^1$ projection onto the y-axis.



E.g. $\mathbb{C}[X,Y,Z] \longrightarrow \mathbb{C}[X,Y]$, $Z \longmapsto 0$.
This is nothing but the closed immersion of $\mathbb{P}^1$ into $\mathbb{P}^2$.

E.g. By writing "Fermat's hypersurface of deg d in $\mathbb{P}^n_{\mathbb{C}}$: $X_0^d + \cdots + X_n^d = 0$", people mean $\mathrm{Proj}\ \mathbb{C}[X_0, \cdots, X_n]/(X_0^d + \cdots + X_n^d)$.

Now we show that the 2nd prop. $\Rightarrow$ the 1st prop.
Denote for short, $\mathcal{O}_{\mathbb{P}^n_R}$ by $\mathcal{O}$ and $\mathcal{O}_{\mathbb{P}^n_R}(d)$ by $\mathcal{O}(d)$. We have the defining s.e.s.

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathcal{O} \longrightarrow i_*\mathcal{O}_Z \longrightarrow 0$$

where $i: Z \hookrightarrow \mathbb{P}^n_R$ is the closed immersion.

By the 2nd prop, we know that, since $\mathcal{I}$ is a q.c. sheaf of ideals, $\mathcal{I} = \tilde{I}$ and $I = \oplus_{d\in\mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{I}(d)) \subseteq \oplus_{d\in\mathbb{Z}} \Gamma(\mathbb{P}^n, \mathcal{O}(d)) = R[X_0, \cdots, X_n]$. (Here the inclusion comes from tensoring the above s.e.s with $\mathcal{O}(d)$ (invertible!) and taking Γ.)

Moreover, since I is an $R[X_0,\cdots,X_n]$-module, I is then an ideal.

Now $i: Z \hookrightarrow \mathbb{P}^n_R$ is a closed immersion, we have $\mathcal{I}|_{D_+(X_j)} = \tilde{I}_j$ for some ideal I_j in $R[\frac{X_i}{X_j}]$, $Z \cap D_+(X_j) = \operatorname{Spec}(R[\frac{X_i}{X_j}]/I_j)$. By def. of $\tilde{I}$, we have $I_j = I_{(X_j)} = \{ f/X_j^d \mid f \in I,\ \deg f = d \}$. Hence we obtain

$$(R[X_0,\cdots,X_n]/I)_{(X_i)} = R[\tfrac{X_i}{X_j}]/I_j$$

$\Rightarrow$ $D_+(X_j) \cap Z = D_+(\bar{X}_j)$, where $\bar{X}_j$ is the image of X_j in $R[X_0,\cdots,X_n]/I$. Hence on an affine open cover, we have:

$$\begin{array}{ccccc} D_+(X_j) \cap Z & \subseteq & Z & \subseteq & \operatorname{Proj}(R[X_0,\cdots,X_n]) \\ \| & & & & \uparrow \\ D_+(\bar{X}_j) & & \subseteq & & \operatorname{Proj}(R[X_0,\cdots,X_n]/I) \end{array}$$

It follows that $Z \cong \operatorname{Proj}(R[X_0,\cdots,X_n]/I)$. $\square$

E.g. $\mathbb{P}^2_k = \operatorname{Proj}(k[X,Y,Z])$. $I_1 = (X^2+XY+XZ,\ XY+Y^2+YZ,\ ZX+ZY+Z^2)$
$I_2 = (X+Y+Z)$.

Claim: $\operatorname{Proj}(k[X,Y,Z]/I_1) = \operatorname{Proj}(k[X,Y,Z]/I_2)$, as closed subschemes of $\mathbb{P}^2_k$.

We check this on each affine $D_+(X)$, $D_+(Y)$, $D_+(Z)$. For instance, on $D_+(Z)$:

$$(I_1)_{(Z)} = \left(\frac{X^2+XY+XZ}{Z^2}, \frac{XY+Y^2+YZ}{Z^2}, \frac{XZ+YZ+Z^2}{Z^2} \right) = (x^2+xy+x,\ xy+y^2+y,\ x+y+1) = (x+y+1).$$

$$(I_2)_{(Z)} = \left(\frac{X+Y+Z}{Z} \right) = (x+y+1) = (I_1)_Z.$$

In general, given any homogeneous $I \subseteq k[X_0,\cdots,X_n]$, I and $I\,k[X_0,\cdots,X_n]_+$ define the same closed subscheme.

Next we prove the 2nd prop.

We need to show that if $\mathcal{F}$ is q.c. on $\mathbb{P}^n_R$, then

$$\begin{array}{c} \Gamma_*(\mathbb{P}^n_R, \mathcal{F})_{(X_0)} \xrightarrow{\cong} \Gamma(D_+(X_0), \mathcal{F}) \\ \| \\ \{ \frac{s}{X_0^d} \mid s \in \Gamma(\mathbb{P}^n, \mathcal{F}(d)) \} / \sim \end{array}$$

This turns into the following 2 statements.

(a). Given $s_1, s_2 \in \Gamma(\mathbb{P}^n_R, \mathcal{F})$ s.t. $s_1|_{D_+(x_0)} = s_2|_{D_+(x_0)}$, then $\exists N \gg 0$, s.t. $x_0^N s_1 = x_0^N s_2 \in \Gamma(\mathbb{P}^n_R, \mathcal{F}(N))$.

(b). Given $s \in \Gamma(D_+(x_0), \mathcal{F})$, $\exists d \geq 0$, and $\tilde{s} \in \Gamma(\mathbb{P}^n_R, \mathcal{F}(d))$ s.t. $\tilde{s}|_{D_+(x_0)} = x_0^d s$.

More general versions can be found in Hartshome, II.5.14:

$(X, \mathcal{F}, \mathcal{L})$, $\mathcal{L}$ invertible sheaf on X, $\mathcal{F}$ q.c. on X, $f \in \Gamma(X, \mathcal{L})$, then

$$\Gamma(X_f, \mathcal{F}) \xleftarrow{\cong} \Gamma_*(X, \mathcal{L}, \mathcal{F})_{(f)} = (\oplus_{d \in \mathbb{Z}} \Gamma(X, \mathcal{F} \otimes \mathcal{L}^d))_{(f)}$$

under reasonable conditions (R Noetherian or X quasi-cpt and quasi-seperated).

Pf of (a).

It suffices to show that $s \in \Gamma(\mathbb{P}^n, \mathcal{F})$, $s|_{D_+(x_0)} = 0$, then $0 = x_0^N s \in \Gamma(\mathbb{P}^n, \mathcal{F}(N))$.

Now on each $D_+(x_i)$, $\mathcal{F}|_{D_+(x_i)} \cong \widetilde{M_i}$ for some $R[\frac{x_j}{x_i}]$-module M_i, $s|_{D_+(x_i)} = m_i \in M_i$

$D_+(x_0) \cap D_+(x_i) = \operatorname{Spec}(R[\frac{x_0}{x_i}, \cdots, \frac{x_n}{x_i}, (\frac{x_0}{x_i})^{-1}])$. Thus $m_i|_{D_+(x_0) \cap D_+(x_i)} = 0 \Rightarrow (\frac{x_0}{x_i})^{N_i} m_i = 0$ or $x_0^{N_i} m_i / x_i^{N_i} = 0$. Take $N \geq \max\{N_j, j = 1, \cdots, n\}$. Then $x_0^N s|_{D_+(x_j)} = \frac{x_0^N m_j}{x_j^N} = 0$. □

Pf of (b).

Now $\mathcal{F}|_{D_+(x_j)} = \widetilde{M_j}$ for some $R[\frac{x_0}{x_j}, \cdots, \frac{x_n}{x_j}]$ module M_j. We have:

$$\begin{array}{ccccc}
\mathcal{F}|_{D_+(x_0)} & \longrightarrow & \mathcal{F}|_{D_+(x_0 x_j)} & \longleftarrow & \mathcal{F}|_{D_+(x_j)} \\
\Gamma \updownarrow \sim & & \Gamma \updownarrow \sim & & \Gamma \updownarrow \sim \\
M_0 & \xrightarrow{\alpha} & (M_0)_{\frac{x_j}{x_0}} \cong (M_j)_{\frac{x_0}{x_j}} & \xleftarrow{\beta} & M_j
\end{array}$$

Now s corresponds to an element $m_0 \in M_0$, denote the image of $s|_{D_+(x_0 x_j)}$ by $\alpha(m_0)$, then $\alpha(m_0) \cdot (\frac{x_0}{x_j})^{d'} = \beta(m_j)$ for some m_j and d', by def. of localization. Choose $s_j = x_j^{d'} m_j \in \Gamma(D_+(x_j), \mathcal{F}(d))$, $s_0 = x_0^{d'} m_0 \in \Gamma(D_+(x_j), \mathcal{F}(d))$, d' large enough so that the above equation holds for all $D_+(x_j)$. If these sections glue to an element $\tilde{s} \in \Gamma(D_+(x_j), \mathcal{F}(d))$, then we are done. But now we only know that $s_j|_{D_+(x_0 x_j)} = s_0|_{D_+(x_0 x_j)}$, we still need $s_i|_{D_+(x_i x_j)} = s_j|_{D_+(x_i x_j)}$. However, on $D_+(x_0 x_i x_j)$

$$s_i|_{D_+(x_0 x_i x_j)} - s_j|_{D_+(x_0 x_i x_j)} = s_0|_{D_+(x_0 x_i x_j)} - s_0|_{D_+(x_0 x_i x_j)} = 0$$

By part (a), we may multiply $x_0^{d''}$ to all $s_0, \cdots, s_n$ so that

$$x_0^{d''} s_i|_{D_+(x_i x_j)} - x_0^{d''} s_j|_{D_+(x_i x_j)} = 0$$

Take $d = d' + d''$, $\tilde{s}|_{D_+(x_i)} = x_0^{d''} s_i$, the claim follows. □

Question : $\mathcal{F}$: q.c. on $\mathbb{P}^n_R$, R Noetherian, then $\Gamma_*(\mathbb{P}^n_R, \mathcal{F})$ is an $R[x_0, \cdots, x_n]$ module. If $\mathcal{F}$ is further coherent, is this module always finitely generated?

The "correct" generality of the finiteness of H^0 is:

R: Noetherian ring, $S = \operatorname{Spec} R$, $\pi : X \longrightarrow S$ a proper morphism, and $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module (e.g. $\mathcal{F} = \mathcal{O}_X$, invertible / locally free sheaves, etc.). Then $H^0(X, \mathcal{F})$ is a finite R-module.

The "correct" proof consists of:

(1). Use cohomology and prove it for $H^i(X, \mathcal{F})$, $\forall i \geq 0$.

(2). Use Chow's lemma to reduce to the projective case.

(3). In the projective case, $\mathbb{P}^n_S \longrightarrow S$

(3.a). Prove $\mathcal{F} = \widetilde{M}$ (as done above for M of finite type over $R[x_0, \cdots, x_n]$)

(3.b). Reduce to $M = R[x_0, \cdots, x_n](d)$

(3.c). Explicitly compute cohomology of $\mathcal{O}_{\mathbb{P}^n}(d) = R[x_0, \cdots, x_n](d)^{\sim}$.

§7. Chow's Lemma

Ref. [Hartshorne, II. ex 4.10], [EGA, II.5.6.1], [Limits, § Chow's lemma].

Thm. (Chow)

Let S be a Noetherian scheme, $f: X \longrightarrow S$: finite type, seperated. Then there exists a diagram:

$$\begin{tikzcd}
& \pi^{-1}(U) \arrow[dl] \arrow[dr] & & \\
U \arrow[dr] & & X' \arrow[dl, "\pi"'] \arrow[dr, hook] \arrow[dd] & \\
& X \arrow[dr] & & \mathbb{P}^n_S \arrow[dl] \\
& & S &
\end{tikzcd}$$

s.t.
- π is proper and surjective
- $X' \hookrightarrow \mathbb{P}^n_S$ is an immersion
- There exists some open dense $U \subseteq X$ s.t. $\pi^{-1}(U) \longrightarrow U$ is an isomorphism.

Discussions about the lemma.

Assume X, S are reduced, then we can take X' to be reduced. Let $\overline{X'} \longrightarrow \mathbb{P}^n_S$ be the closure. We have the diagram:

$$\begin{tikzcd}
X' \arrow[r, hook, "\text{open}"] \arrow[d, "\pi"'] & \overline{X'} \arrow[r, hook, "\text{closed}"] \arrow[d] & \mathbb{P}^n_S \arrow[dl] \\
X \arrow[r] & S &
\end{tikzcd} \qquad (*)$$

Denote the composite $X' \xhookrightarrow{\text{open}} \overline{X'} \xhookrightarrow{\text{closed}} \mathbb{P}^n_S$ by h.

Def: We call a morphism of schemes $X \longrightarrow S$ "H-projective" if $\exists$ a closed immersion $X \hookrightarrow \mathbb{P}^n_S$ over S. ("H" stands for Hartshorne's definition, which is not totally the same as in EGA).

Lemma.

(a). Closed immersions are proper and H-projective.

(b). H projective $\Rightarrow$ proper.

(c). Composition of H-projective (resp. proper) is H-projective (resp. proper)

(d). Base change of H-projective (resp. proper) is H-projective (resp. proper)

(e). Fiber product of H-projective (resp. proper) is H-projective (resp. proper).

Pf: (a) Obvious. (b) Proven before

(c). Given $Y \longrightarrow X$, $X \longrightarrow S$ H-projective morphisms, we have, by def.

$$\begin{tikzcd}
Y \arrow[r] \arrow[d, hook] & X \arrow[r] \arrow[d, hook] & S \\
\mathbb{P}^n_X \arrow[ur] & \mathbb{P}^m_S \arrow[ur] &
\end{tikzcd}$$

Then by base change we obtain:

$$\begin{tikzcd}
\mathbb{P}^n_X \arrow[r, hook] \arrow[d] & \mathbb{P}^n_{\mathbb{P}^m_S} = \mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} (\mathbb{P}^m_{\mathbb{Z}} \times_{\mathbb{Z}} S) = (\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^m_{\mathbb{Z}}) \times_{\mathbb{Z}} S \arrow[d] \\
X \arrow[r, hook] & \mathbb{P}^m_S = \mathbb{P}^m_{\mathbb{Z}} \times_{\mathbb{Z}} S
\end{tikzcd}$$

and $\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^m_{\mathbb{Z}}$ is projective over $\operatorname{Spec}\mathbb{Z}$ by the Segre embedding:

$$\mathbb{P}^n_{\mathbb{Z}} \times \mathbb{P}^m_{\mathbb{Z}} \longrightarrow \mathbb{P}^{nm+n+m}_{\mathbb{Z}}$$

$$[x_0 : \cdots : x_n], [y_0 : \cdots : y_m] \mapsto [x_0 y_0 : \cdots : x_n y_m]$$

Thus $Y \hookrightarrow \mathbb{P}^n_X \hookrightarrow (\mathbb{P}^n_{\mathbb{Z}} \times_{\mathbb{Z}} \mathbb{P}^m_{\mathbb{Z}}) \times_{\mathbb{Z}} S \hookrightarrow (\mathbb{P}^{nm+n+m}_{\mathbb{Z}}) \times_{\mathbb{Z}} S = \mathbb{P}^{nm+n+m}_S$ is a closed immersion. as required.

(d). $X \longrightarrow Y$: H-proj, $Y' \longrightarrow Y$. Then by def.

$$\begin{tikzcd}
X \arrow[r] \arrow[d, hook] & Y \\
\mathbb{P}^n_Y \arrow[ur] &
\end{tikzcd}$$

Thus take the fiber product, we have:

$$\begin{tikzcd}
X \times_Y Y' \arrow[r, hook] \arrow[d] & \mathbb{P}^n_{Y'} = \mathbb{P}^n_Y \times_Y Y' \arrow[d] \\
X \arrow[r, hook] & \mathbb{P}^n_Y
\end{tikzcd}$$

(e). If $X \longrightarrow S$, $Y \longrightarrow S$ are H-proj. then so is $X \times_S Y \longrightarrow X$ by (d), and

$$\begin{tikzcd}
X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
X \arrow[r] & S
\end{tikzcd}$$

thus so is the composition $X \times_S Y \longrightarrow X \longrightarrow S$. $\square$

Lemma. Given a diagram:
$$\begin{tikzcd} X \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$
, where f is proper and g is seperated, then $h(X)$ is closed.

Pf: The diagram gives rise to:
$$\begin{tikzcd} & X \times_S Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[ur, bend right, "h'"'] \arrow[rr, "h"] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & S & \end{tikzcd}$$
where h' is a section given by h. g seperated $\Rightarrow$ $X \times_S Y \xrightarrow{\pi_X} X$ is seperated.
$\Rightarrow$ $h'(X)$ is closed ($h'(X)$ being the equalizer of the morphisms $X \times_S Y \underset{h \circ \pi_X}{\overset{\pi_Y}{\rightrightarrows}} Y$).
Moreover, f proper $\Rightarrow$ π_Y is closed. Hence $h(X) = \pi_Y \circ h'(X)$ is closed $\square$

An interesting application of the lemma is the following:

If X is a proper variety over $k = \bar{k}$, then $\Gamma(X, \mathcal{O}_X) = \bar{k}$.

Indeed, if $X \longrightarrow \operatorname{Spec} k$ is proper, $\Gamma(X, \mathcal{O}_X) \cong \operatorname{Mor}(X, \mathbb{A}^1_k)$ $\quad$ ($\mathbb{A}^1_k \cong \mathbb{G}_a(k)$)

$\Rightarrow$
$$\begin{tikzcd} f: X \arrow[r] \arrow[dr] & \mathbb{A}^1_k \arrow[r, hook] \arrow[d] & \mathbb{P}^1_k \arrow[dl] \\ & \operatorname{Spec} k & \end{tikzcd}$$

Lemma $\Rightarrow$ $f(X)$ is closed in both $\mathbb{A}^1_k$ and $\mathbb{P}^1_k$. thus must be a closed point of $\mathbb{A}^1_k$ $\Rightarrow$ $\Gamma(X, \mathcal{O}_X) = \bar{k} = k$.

Now we can discuss about the diagram $(*)$.

If we further assume X is proper over S, then Chow's lemma $\Rightarrow$ π' is proper and thus so is the composite $X' \longrightarrow X \longrightarrow S$. The lemma above $\Rightarrow$ $X' \xrightarrow{h} \overline{X'} \hookrightarrow \mathbb{P}^n_S$ has closed image. Hence $X' \hookrightarrow \mathbb{P}^n_S$ is a closed immersion, since we have assumed X, S reduced.

Conversely, if $X' = \overline{X'}$ $\Rightarrow$ X' is H-projective over S $\Rightarrow$ X' is proper over S $\Rightarrow$ $\forall T$ closed in X, we have $f(T) = f \circ \pi(\pi^{-1}(T))$ is closed. This also holds for any base change $S' \longrightarrow S$: $X'_S \longrightarrow X_S$. Thus $X \longrightarrow S$ is proper.

Upshot: (Refinement of Chow's lemma)

Given $f: X \longrightarrow S$ seperated, finite type. $X \longrightarrow S$ is proper iff $\exists$ H-proj $X' \longrightarrow S$ with a surjective S-morphism: $X' \longrightarrow X$.

$\square$ of discussion

Proof of Chow's lemma.

We only prove it in the special case where $S = \operatorname{Spec} k$, $k = \bar{k}$ and X is a variety.

Write $X = U_1 \cup \cdots \cup U_k$, where $\phi \neq U_i \subseteq X$ is affine open and $U_i = \operatorname{Spec} A_i$, and $A_i = \operatorname{Spec} k[x_{i0}, \cdots, x_{in_i}]/I_i$ is the affine coordinate ring. Then we have:

$$\begin{tikzcd} & \mathbb{A}^{n_i} \arrow[dr, "\text{open}"] & \\ U_i \arrow[ur, "\text{closed}"] \arrow[rr, "j_i"] \arrow[dr, "\text{open}"'] & & \mathbb{P}^{n_i} \\ & Z_i \arrow[ur, "\text{closed}"'] & \end{tikzcd}$$

where Z_i is the closure of U_i in $\mathbb{P}^{n_i}$.

Set $U = U_1 \cap \cdots \cap U_n \subseteq X$, which is dense open, and $j = (j_1, \cdots, j_n)$:

$$\begin{tikzcd} j: U \arrow[rr] \arrow[dr, "\text{open}"'] & & \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \\ & Z \arrow[ur, "\text{closed}"'] & \end{tikzcd}$$

Let Z be the closure of the image $j(U)$. We have the diagram:

$$\begin{tikzcd} U \arrow[r, hook, "\text{open}"] \arrow[d, "\text{open}"'] & Z \arrow[r, hook] \arrow[dr, "P_i"] & Z_1 \times \cdots \times Z_k \arrow[d, "Pr_i"] \\ U_i \arrow[rr, hook, "\text{open}"] & & Z_i \end{tikzcd}$$

Then Pr_i is proper since it's the restriction of $\mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_k} \longrightarrow \mathbb{P}^{n_i}$ to a closed subscheme. Consequently P_i is proper.

Let $V_i = P_i^{-1}(U_i)$, and $X' = P_1^{-1}(U_1) \cup \cdots \cup P_k^{-1}(U_k) = V_1 \cup \cdots \cup V_k$. We will try to map $X' \longrightarrow X$.

Claim 1: $P_i|_{V_i \cap V_j} = P_j|_{V_i \cap V_j}$, and hence they glue to give a morphism of schemes. $\pi: X' \longrightarrow X$.

In fact, the locally closed subscheme in $V_i \cap V_j$ where $P_i : V_i \cap V_j \longrightarrow V_i \hookrightarrow X$ and $P_j : V_i \cap V_j \longrightarrow V_j \hookrightarrow X$ agree is closed since X is seperated and contains U (dense). Hence it's all of $V_i \cap V_j$.

Claim 2: $\pi^{-1}(U_i) = V_i$

Consider the diagram:

$$\begin{tikzcd} V_i \arrow[r, hook] \arrow[dr, "P_i|_{V_i}"'] & \pi^{-1}(U_i) \subseteq Z \arrow[d, "\pi|_{\pi^{-1}(U_i)}"] \\ & U_i \end{tikzcd}$$

Since Z is seperated, $\pi^{-1}(U_i)$ is seperated (or $\pi|_{\pi^{-1}(U_i)}$ is seperated). $P_i|_{V_i}$ is proper since it's the base change of a proper morphism to U_i. Thus the image of V_i is closed in $\pi^{-1}(U_i)$, and thus must be equal since it's dense (contains U).

Claim 3: π is proper.

This is true because $X = U_1 \cup \cdots \cup U_n$ and each restriction of π to $\pi^{-1}(U_i)$ is now identified with $P_i|_{V_i} : V_i \longrightarrow U_i$ is proper. and being proper is local on the base.

Claim 4: $\pi^{-1}(U) \longrightarrow U$ is an isomorphism.

Pf: The same proof as in claim 2 to the diagram:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[dr, "id"'] & \pi^{-1}(U) \arrow[d] \\ & U \end{tikzcd}$$

$\square$

Def. A scheme X of finite type over a field k is called quasi-projective if X has an immersion into $\mathbb{P}^n_k$ for some n.

Lemma. A proper quasi-projective variety is projective.

Pf: Similar as in Chow's lemma.

$$\begin{tikzcd} X \arrow[r] \arrow[dr, "\text{proper}"'] & \mathbb{P}^n_k \arrow[d, "\text{seperated}"] \\ & \operatorname{Spec} k \end{tikzcd}$$

$\Rightarrow Im(X)$ is closed. $\square$

We can reformulate Chow's lemma for varieties.

Thm. For any variety X, $\exists$ a quasi-projective variety X' and a surjective morphism $X' \xrightarrow{\pi} X$ which is an isomorphism over a non-empty open $U \subseteq X$. Moreover, X proper $\Leftrightarrow$ X' projective.

Application to curves

Def. A curve over k is a variety of dimension 1. (i.e. we have one generic point and every other point, infinitely many, is closed)

From dimension theory, we know that, for a variety X:

$$\begin{aligned} \dim(X) = d &\Leftrightarrow \forall x \in X. \text{ closed point, } \dim \mathcal{O}_{X,x} = d \\ &\Leftrightarrow tr.deg_k k(X) = d \end{aligned}$$

Def. S: an integral scheme, we set the function field $k(S)$ equal to $f.f.\mathcal{O}_S(U)$ for any non-empty open $U \subseteq S$. ($k(S) = \mathcal{O}_{S,\eta}$, where η is the generic point of S.)

Def: A morphism $f: X \longrightarrow Y$ of varieties over k is called:

(1). Dominant $\Leftrightarrow$ $f(X)$ is dense in Y.

$\Leftrightarrow$ $f(\eta_X) = \eta_Y$

$\Leftrightarrow$ $f(X)$ contains a non-empty open subset of Y

(This uses Chevalley's thm, which says that $f(X)$ is constructible. Thus it's dense iff it contains a non-empty U)

(2). Birational $\Leftrightarrow$ it's dominant and $\mathcal{O}_{\eta_Y} = k(Y) \longrightarrow k(X) = \mathcal{O}_{\eta_X}$ is an isomorphism.

$\Leftrightarrow \exists\, \phi \neq U \subseteq Y$ open s.t. $f^{-1}(U) \xrightarrow{\cong} U$ is an isomorphism.

Lemma. Suppose $f: X \longrightarrow Y$ is a proper, birational morphism of curves. Assume Y is regular, i.e. $\mathcal{O}_{Y,\eta}$ is regular for all $y \in Y$. Then f is an isomorphism.

Pf: Algebra: A: Noetherian local ring of dim 1, then A is regular iff A is a DVR.

Pick $x \in X$. Then:

$$\begin{tikzcd}
\mathcal{O}_{X,x} \arrow[d, hook] & \mathcal{O}_{Y,y} \arrow[l, "f^*"'] \arrow[d, hook] \\
k(X) & k(Y) \arrow[l, "\cong"']
\end{tikzcd}$$

$\Rightarrow f^*$ is an isomorphism, by def. of a valuation ring (maximal).

Furthermore, suppose $x, x' \in X$ are closed points with $f(x) = f(x')$. Then $\mathcal{O}_x = \mathcal{O}_{x'} = \mathcal{O}_{f(x),Y}$, and a diagram:

$$\begin{tikzcd}
\operatorname{Spec}\mathcal{O}_x \arrow[rr, "i_1"] \arrow[dd] & & X \arrow[dd] \\
& \operatorname{Spec}\mathcal{O}_{x'} \arrow[ur, "i_2"] \arrow[dl] & \\
\operatorname{Spec}\mathcal{O}_{f(x)} \arrow[rr] \arrow[uu, bend left, "s_1"] \arrow[ur, bend left, "s_2"] & & Y
\end{tikzcd}$$

Since $\mathcal{O}_{f(x)} \cong \mathcal{O}_x, \mathcal{O}_{x'}$, we have sections s_1, s_2, and composing with i_1, i_2 resp., we have two morphisms, $i_1 \circ s_1, i_2 \circ s_2$ from $\operatorname{Spec}\mathcal{O}_{f(x)}$ to X. The valuative criterion of properness $\Rightarrow i_1 \circ s_1 = i_2 \circ s_2 \Rightarrow x = x'$.

Finally, since f is proper, we see that $f(X)$ is closed in Y. Since it contains the generic point of Y, $f(X) = Y$. Summing up, we have shown f is 1-1, onto and closed, $f_x : \mathcal{O}_x \xrightarrow{\cong} \mathcal{O}_{f(x)}$, and thus $X \cong Y$. $\square$

Rmk: A curve is regular iff it's normal. This follows from the algebraic fact that a local Noetherian domain of dimension 1 is regular iff it's normal, iff it's a DVR.

Prop. A regular curve is quasi-projective.
(The correct generality of this result is that, a 1-dimensional seperated scheme of finite type over a field is quasi-projective)

Pf: Let X be one such curve. Chow's lemma gives: $\pi: X' \longrightarrow X$ proper, birational and X' quasi-projective. The lemma above show that $\pi: X' \cong X$. $\square$

Lemma. X: regular curve; Y: proper variety. Any morphism $f: U \longrightarrow Y$, where $\phi \neq U$ open in X extends to a morphism $X \longrightarrow Y$.

Pf: Take Z = closure of $i_U \times f: U \longrightarrow X \times Y$. Then Z is a variety, and $U \subseteq Z$ is open dense:

$$\begin{tikzcd} Z \arrow[r, hook] \arrow[d] & X \times Y \arrow[r] \arrow[d] & Y \\ X \arrow[r, equal] \arrow[u, bend right] & X & \end{tikzcd}$$

Lem. above $\Rightarrow Z \cong X$, and thus we obtain an inverse $X \longrightarrow Z$. Composing with $Z \longrightarrow Y$, we are done. $\square$

Rmk: Another way to prove is to notice that $X \setminus U = \{$finitely many closed points, and using (E) of the valuative criterion we can extend the morphism over each of these points, whose local rings are DVR's.

Def. An integral scheme S is called normal if $\forall\, U \subseteq S$ affine open, $\mathcal{O}_S(U) \subseteq K(S)$ is integrally closed, i.e. $\mathcal{O}_S(U)$ is a normal domain.

Def. A morphism of schemes $\varphi: S' \longrightarrow S$ is called finite iff $\forall\, U \subseteq S$ affine open, $\varphi^{-1}(U)$ is affine and $\mathcal{O}_{S'}(\varphi^{-1}(U))$ is finite over $\mathcal{O}_S(U)$.

Lemma: For any variety X, $\exists$ a canonical morphism of varieties $\nu: X^\nu \longrightarrow X$, called the normalization of X, which is birational, finite, with X^ν a normal variety.

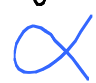
Pf: This basically follows from the algebraic fact that if A is finitely generated domain over k, then the integral closure A^ν of A in $f.f.(A)$ is a finite A module. $\square$

Lemma. A finite morphism is proper. $\square$

E.g.

Consider the singular curves:

$\operatorname{Spec}(k[x,y]/(y^2-x^3))$ $\qquad\qquad$ $\operatorname{Spec}(k[x,y]/(y^2-x^2(1-x)))$

 : a cusp $\qquad\qquad$  : a node

In their respective function fields. $(\frac{y}{x})^2 - x = 0$, $(\frac{y}{x})^2 - (1-x) = 0$ are integral. It can be checked that $A[\frac{y}{x}] \subseteq k(x)$ are normal for both rings. These are the respective normalizations.

⚠ Integral closure of a finitely generated domain over k in its fraction field is finite over itself. This is not true for finitely generated algebras over k. (where there may be nilpotent elements). For example, in $k[x,\varepsilon]/(\varepsilon^2)$. In the total fraction ring (where we invert all non-zero divisors) $k(x)[\varepsilon]/(\varepsilon^2)$, the integral closure is not finitely generated: $(\frac{\varepsilon}{x^n})^2 = 0$ satisfies an integral equation, $\forall n \geq 0$.

Def. (Rational maps).

(a). X, Y: varieties over k, a rational map $X \dashrightarrow Y$ is an equivalence class of morphisms $f: U \longrightarrow Y$, where $U \subseteq X$ is non-empty open, and $(f: U \longrightarrow Y) \sim (g: V \longrightarrow Y)$ iff $\exists\ \phi \neq W \subseteq U \cap V$, non-empty open s.t. $f|_W = g|_W$.

(b). We say a rational map $X \dashrightarrow Y$ is dominant if for any representative $f: U \longrightarrow Y$ of this map, it is dominant, i.e. $f(\eta_U) = \eta_Y$. (Note that $\eta_U = \eta_X$, thus this is independent of representatives chosen).

Rmk: X, Y varieties over k. If $f: U \longrightarrow Y$ and $g: V \longrightarrow Y$ define the same rational map, then $f|_{U\cap V} = g|_{U\cap V}$. Indeed, since Y is seperated / k, we know that where they agree is closed in $U\cap V$. But it's also dense since it contains a non-empty open W. Thus f and g glue to give a morphism $U \cup V \longrightarrow Y$. (thus there is a maximal open where this rational map is defined as a morphism).

Notation: $f: X \dashrightarrow Y$ means a rational map with a chosen representative.
Construction: $f: X \dashrightarrow Y$, $g: Y \dashrightarrow Z$ rational maps and f is dominant. Then we may compose them, defined by:

$$\begin{tikzcd}
U\cap f^{-1}(V) \arrow[r, phantom, "\subseteq"] \arrow[dr, "f|_{U\cap f^{-1}(V)}"] & U \arrow[r, phantom, "\subseteq"] \arrow[dr, "f"] & X \arrow[d, dashed] \\
 & V \arrow[r, phantom, "\subseteq"] \arrow[dr, "g"] & Y \arrow[d, dashed] \\
 & & Z
\end{tikzcd}$$

Rmk: Set $R(X) = \{$ the set of rational maps from $X \dashrightarrow \mathbb{A}^1_k \}$
$= $ the set of rational functions on X.
Then $R(X) = \operatorname{colim}_{U\subseteq X}(\mathrm{Mor}_{Var}(U, \mathbb{A}^1_k)) = \operatorname{colim}_{U\subseteq X}(\mathcal{O}_X(U)) = \mathcal{O}_{X,\eta} = k(X)$.

Thm. The category of varieties with morphisms dominant rational maps is anti-equivalent to the category of finitely generated field extensions $k \subseteq K$ with morphisms k-algebra homomorphisms.

Variety		Fields
X	$\longmapsto$	$k(X)$
$(\varphi: X \dashrightarrow Y)$	$\longmapsto$	$(\varphi^*: k(Y) \longrightarrow k(X))$

Pf: "Surjectivity on objects": $k \subseteq K$, finitely generated field extension, i.e. $\exists\, \lambda_1, \cdots, \lambda_n \in K$ s.t. $K = k(\lambda_1, \cdots, \lambda_n)$. Let A be the k-algebra generated by $\lambda_1, \cdots \lambda_n$ in K. i.e. $A = k[\lambda_1, \cdots, \lambda_n] \subseteq K$, which is a finitely generated domain $/k$. Then $X = \operatorname{Spec} A$ is a variety whose function field is K.

"Surjectivity on morphisms": X, Y: varieties. $\psi: k(Y) \longrightarrow k(X)$ is a k-algebra map. Then we want a non-empty open $U \subseteq X$ and a dominant morphism $f: U \longrightarrow Y$ which induces ψ on function fields.

Pick any non-empty affine opens: $\operatorname{Spec} A \subseteq X$, $\operatorname{Spec} B \subseteq Y$, $A = k[x_1, \cdots, x_n]/I$ $B = k[y_1, \cdots, y_m]/J$. $B \subseteq k(Y) \xrightarrow{\psi} k(X) \Rightarrow \psi(y_i) \in k(X)$. The problem is that $\psi(y_i)$ need not be in A. But $\mathrm{f.f.}(A) = k(X) \Rightarrow \psi(y_i) = \frac{t_i}{a_i}$, $a_i, t_i \in A$, $\forall i$. Then $\psi(y_i) \in A_{a_1 \cdots a_m}$, and $\operatorname{Spec}(A_{a_1 \cdots a_m}) \subseteq \operatorname{Spec} A$ is non-empty affine open. Thus $\psi: B \longrightarrow A_{a_1 \cdots a_m} \Rightarrow \operatorname{Spec}(A_{a_1 \cdots a_m}) \longrightarrow \operatorname{Spec} B$, which induces ψ on function fields. $\square$

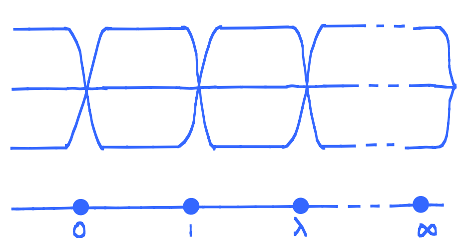
Thm. Let k be a field. Then there is an anti-equivalence of categories:

- The category of regular (i.e. normal) projective curves, with morphisms dominant morphisms of varieties (i.e. non-constant).
- The category of finitely generated field extensions $k \subseteq K$ of $\operatorname{tr.deg}_k K = 1$, with morphisms k-algebra homomorphisms.

Consequently, for every K with $\operatorname{tr.deg}_k K = 1$, there exists a unique regular projective curve X/k up to unique isomorphism.

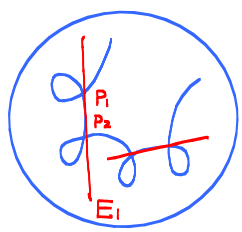
Rmk: (1). One crucial part is the finitely-generatedness of K. Those fields of tr. deg. 1 which are not finitely generated are analogues of Riemann surfaces with infinite genus:

$C_n: y^n = x(x-1)(x-\lambda)$: n-sheeted branch cover of $\mathbb{P}^1$:

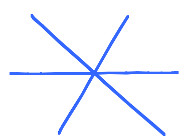


genus $g(C_n) = n-1$. Thus we have a family of field extensions (morphism of curves) $K(C_n) \hookrightarrow K(C_{2n}) \hookrightarrow K(C_{4n}) \hookrightarrow \cdots$ ($\cdots \to C_{4n} \to C_{2n} \to C_n$: $C_{2^{k+1}n} \longrightarrow C_{2^k n}$: $y \mapsto y^2$.) The field $\lim_k K(C_{2^k n})$ is not finitely generated but of transcendence degree 1. It's not the function field of an algebraic curve.

(2). In ancient times, people restrict themselves to curves in $\mathbb{P}^2$ and had to use ingenious ways to create smooth curves.



They take the Cremona transformation of $\mathbb{P}^2$ ($[x,y,z] \longmapsto [yz, zx, xy]$) to blow up p_1, p_2 and blow down E_1. After difficult computations they could only obtain curves with singularity like below in finitely many steps.



Pf of thm.

Assume that we can construct a smooth curve X for every K. Then given $K(Y), K(X)$, tr. deg $= 1$ and $\psi : K(Y) \longrightarrow K(X)$, we have

$$\begin{tikzcd} K(X) & & X \supseteq U \arrow[dl, "i \circ f"'] \arrow[d, "f"] \\ K(Y) \arrow[u] & \Rightarrow & Y & V \arrow[l, hook', "i"'] \end{tikzcd}$$

Since Y is proper and X is regular, $i \circ f$ extends to a morphism $X \longrightarrow Y$. Thus the only thing left to show is that: Given K, tr. deg $K = 1$, find a regular projective curve with $k(X) = K$.

By the previous thm. $\exists$ some curve U s.t. $k(U) \cong K$. By shrinking, we may assume that U is affine. Then take the close of the image of:

$$U \hookrightarrow \mathbb{A}^n_k \hookrightarrow \mathbb{P}^n_k$$

call it Z. $U \hookrightarrow Z$ is open dense. and Z is a projective variety with function field $k(Z) = K$, i.e. Z is a projective curve.

Set $X = Z^\nu \xrightarrow{\nu} Z$ the normalization. Then X is a curve and is regular. Since ν is proper (finite morphism!) $\Rightarrow$ X is proper. Hence X is projective, by a previous lemma. $\square$

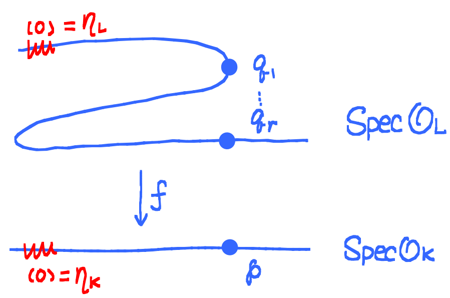
§8. Degrees of Morphisms

[Read: Hartshorne II §6.]

E.g. (Thm / Motivation from number theory)

Let L/K be a finite extension of fields. $\mathcal{O}_K, \mathcal{O}_L$ rings of integers. Let $\mathfrak{p} \in \operatorname{Spec} \mathcal{O}_K$ be a prime, $\mathfrak{q}_1, \cdots, \mathfrak{q}_r$ the primes in $\operatorname{Spec} \mathcal{O}_L$ over $\mathfrak{p}$. Let $f_i = [\kappa(\mathfrak{q}_i) : \kappa(\mathfrak{p})]$, where $\kappa(\mathfrak{p}), \kappa(\mathfrak{q}_i)$ are the residue fields of $\mathcal{O}_{K,\mathfrak{p}}, \mathcal{O}_{L,\mathfrak{q}_i}$. and e_i = the exponent of $\mathfrak{q}_i$ in the expression $\mathfrak{p}\mathcal{O}_L = \mathfrak{q}_1^{e_1} \cdots \mathfrak{q}_r^{e_r}$. Then we have the degree formula for the morphism $\operatorname{Spec}(\mathcal{O}_K) \longrightarrow \operatorname{Spec}(\mathcal{O}_L)$:

$$[L:K] = \sum_{i=1}^{r} e_i f_i$$



Upshot: This formula states that the degree of f above the generic point $\eta_K = (0)$ is the same as above $\mathfrak{p}$.

General statement: A: DVR, with $\operatorname{f.f.}(A) = K$. L: a finite extension of K. Assume the integral closure B of A in L is finite over A. (when the ring is reasonable: Japanese, Nagata, $\cdots$) Then $m_A B = m_1^{e_1} \cdots m_r^{e_r}$ (B is a Dedekind domain), and

$$[L:K] = \sum_{i=1}^{r} e_i [\kappa(m_i) : \kappa(m_A)]$$

Pf: Since B is a finite A-module and torsion free as an A-module, B is free: $B \cong A^{\oplus n}$. Obviously $n = [L:K]$.

Since B is the integral closure of A in L, B is normal. B finite over $A \Rightarrow \dim B = \dim A = 1$. Hence B as a finite (type) algebra over a Noetherian ring is Noetherian. $\Rightarrow$ B is a Dedekind domain. Thus all the local rings at closed points are DVR's, and $mB = m_1^{e_1} \cdots m_r^{e_r}$

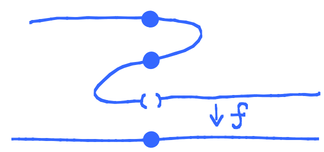
$\Longrightarrow$ $n \overset{*}{=} \mathrm{length}_A(B/m_A B)$ (being finite is crucial here!)

$= \sum_i \mathrm{length}_A(B/m_i^{e_i} B)$ (Chinese remainder thm.)

$= \sum_i e_i \cdot \mathrm{length}_A(B/m_i B)$ (length is additive, and $\frac{m_i^k}{m_i^{k+1}} = \frac{(\pi^k)}{(\pi^{k+1})} \cong \frac{B}{m_i}$)

$= \sum_i e_i [\kappa(m_i) : \kappa(m_A)]$

$*$: Here finite guarantees the following situation won't occur:



$\square$

Def. (Divisors). Let X be a Noetherian scheme.

(1). An effective Cartier divisor on X is a closed subscheme $D \hookrightarrow X$ s.t. $\forall x \in D$, $\exists$ an affine open nhd $\mathrm{Spec} A$ of x in X s.t. $D \cap \mathrm{Spec} A = \mathrm{Spec} A/(f)$, for some non-zero divisor $f \in A$. (f can't be a unit as well: $x \in D \cap \mathrm{Spec} A$ is non-empty).

An equivalent def. is that $\mathcal{I}_D \subseteq \mathcal{O}_X$ the ideal sheaf of D is an invertible sheaf of $\mathcal{O}_X$-modules.

(2). A Weil divisor on X is a finite formal sum $D = \sum n_Z [Z]$, $n_Z \in \mathbb{Z}$ with each $Z \subseteq X$ a prime divisor.

(3). A prime divisor $Z \subseteq X$ is an irreducible, reduced closed subscheme s.t. $\dim \mathcal{O}_{X,\xi} = 1$ where $\xi \in Z$ is the generic point of Z.

- Rule to associate a Weil divisor to an effective Cartier divisor:

Let D be an effective Cartier divisor:

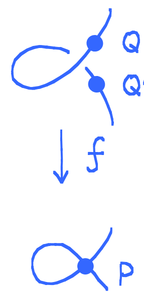
$$D \longmapsto [D] = \sum_{\substack{\xi \in Z \subseteq X, \\ Z \text{ prime divisor}, \, \xi \in D}} \mathrm{length}_{\mathcal{O}_{X,\xi}}(\mathcal{O}_{D,\xi}) [Z]$$

Here $\xi \in D \Leftrightarrow \overline{\{\xi\}} = Z \subseteq D$. This works since if $\xi \in D$, then $\mathcal{O}_{D,\xi} = \mathcal{O}_{X,\xi}/(f)$, where f is not a zero-divisor in $\mathcal{O}_{X,\xi}$ and thus $\dim \mathcal{O}_{D,\xi} = \dim \mathcal{O}_{X,\xi} - 1 = 0$.

- Pulling-back divisors.

Rmk: Not always possible.

E.g.



It's not easy to define the pull-back Weil divisor of the singular point $[P]$.
(The correct definition should be $\frac{1}{2}[Q]+\frac{1}{2}[Q']$, by symmetry of the picture).

E.g. Simple cases where we can.

Suppose $f: X \longrightarrow Y$ is a morphism of Noetherian schemes. $D \hookrightarrow Y$ an effective Cartier divisor. Let $f^{-1}(D)$ be the fiber product:

$$\begin{tikzcd} f^{-1}(D) \arrow[r] \arrow[d] & D \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

If $f^{-1}(D)$ is an effective Cartier divisor, (this always holds if $f: X \longrightarrow Y$ is a dominant morphism of varieties, since $k(Y) \hookrightarrow k(X)$), then

$$f^*D \triangleq f^{-1}(D).$$

Rmk: (Cartier divisors / Invertible sheaves) form a cohomology theory while (Weil divisors) form a homology theory. It's easy to define pull-back of cohomology and push-forward of homology. The other way around is not easy.

Application on curves

Lemma. On a regular (normal) curve Y, any Weil divisor D can be written uniquely as $D = [D_1] - [D_2]$, where $D_1, D_2 \hookrightarrow Y$ are effective Cartier divisors and $D_1 \cap D_2 = \phi$.

Reason: $D=\sum_{y\in Y,\ \text{closed pts}} n_y[y] = \sum_{n_y>0} n_y[y] - \sum_{n_y<0}(-n_y)[y]$. Just let $D_1 = \sum_{n_y>0} n_y[y]$, $D_2 = \sum_{n_y<0}(-n_y)[y]$. It suffices to see that each $[y]$ is associated to an effective Cartier divisor / invertible sheaf $\mathcal{I}(y)$ and then $\sum n_y[y]$ is associated to $\otimes \mathcal{I}(y)^{n_y}$. But we have the invertible sheaf:

$$\mathcal{I}(y)_x = \begin{cases} \mathcal{O}_{Y,x}, & x\neq y \\ m_y, & x=y \end{cases}$$

□

E.g. $\mathbb{P}^1_{\mathbb{C}} = \mathrm{Proj}(\mathbb{C}[T_0,T_1])$, $t=\frac{T_0}{T_1}$.

The Weil divisor $3[t=0]+5[t=17]$ is associated with

$$D = \mathrm{Spec}(k[t]/(t-17)^5t^3) \xhookrightarrow{\text{closed}} \mathrm{Spec}\,k[t] \xhookrightarrow{\text{open}} \mathbb{P}^1_{\mathbb{C}}$$

The composition is a closed immersion since the image set is closed. (This is not true for higher dimensional cases!) Thus this closed subscheme is an effective Cartier divisor.

- Pull-back of Weil divisors for curves.

Let $f: X\longrightarrow Y$ be a non-constant (dominant) morphism of curves and Y regular. Let D be a Weil divisor on Y, and write it as $D=D_1-D_2$ as in the lemma above. Set:

$$f^*D = [f^*D_1] - [f^*D_2]$$

where f^*D_i is the pull-back of Cartier divisors, and $[f^*D_i]$ taking the associated divisor.

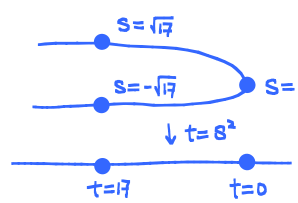
E.g. $\mathbb{P}^1_{\mathbb{C}} \xrightarrow{\varphi} \mathbb{P}^1_{\mathbb{C}}$: $\mathrm{Proj}(\mathbb{C}[S_0,S_1]) \longrightarrow \mathrm{Proj}(\mathbb{C}[T_0,T_1])$, $T_0\mapsto S_0^2$, $T_1\mapsto S_1^2$.

On the affine open $\mathrm{Spec}\,\mathbb{C}[t]$, φ gives a finite morphism:

$$\begin{array}{ccccc} & & \mathbb{P}^1_{\mathbb{C}} & \xrightarrow{\varphi} & \mathbb{P}^1_{\mathbb{C}} \\ & & \cup\!\text{I} & & \cup\!\text{I} \\ \mathrm{Spec}\,\mathbb{C}[s] & = & \mathbb{A}^1_{\mathbb{C}} & \xrightarrow{\varphi} & \mathbb{A}^1_{\mathbb{C}} = \mathrm{Spec}\,\mathbb{C}[t] \\ & & \| & & \\ & & \varphi^{-1}(\mathrm{Spec}(\mathbb{C}[t])). & & \end{array}$$

which is induced from the finite ring extension: $\mathbb{C}[t] \hookrightarrow \mathbb{C}[s]$, $t \mapsto s^2$. $\varphi^{-1}(\operatorname{Spec}\mathbb{C}[t]) = \operatorname{Spec}\mathbb{C}[s]$ because $\mathbb{C}[s]$ is the integral closure of $\mathbb{C}[t]$ in $\mathbb{C}(s)$. Now consider our divisor as before:

$$\begin{aligned}\varphi^*(D) &= [\varphi^{-1}D] \\ &= [\operatorname{Spec}\mathbb{C}[s]/(s^6(s^2-17)^5)] \\ &= 6[s=0] + 5[s=\sqrt{17}] + 5[s=-\sqrt{17}]\end{aligned}$$



Def. Given a Weil divisor $D = \sum n_x[x]$ on a curve X. Set:

$$\deg D = \sum_x n_x [\kappa(x):k]$$

($[\kappa(x):k] < \infty$ by Hilbert Nullstenllensatz.)

Lemma. Any non-constant proper morphism between curves / k is finite.

Rmk: The correct generality of the statement is that any proper morphism with finite fibers is finite. We can prove this curve case by hand, as done in Hartshorne, but it's better to prove this correct generality after some machinary is developed. $\square$

Thm. k: a field. Let $f: X \longrightarrow Y$ be a non-constant morphism of projective regular curves ($\Leftrightarrow$ proper regular curves). Let $n = [k(X):k(Y)]$ be the degree of f. Then $\forall y \in Y$, closed point, we have:

$$\deg(f^*([y])) = n \cdot \deg([y])$$

and by linearity, $\forall D$ Weil divisor on Y,

$$\deg(f^*D) = n \cdot \deg D$$

Pf: Take $y \in Y$ closed and choose an affine open nhd $\operatorname{Spec}A \subseteq Y$ of y. Now by the previous lemma $f^{-1}(\operatorname{Spec}A) = \operatorname{Spec}B$ is affine and $A \longrightarrow B$

is finite, B is integrately closed since X is regular. Now the algebraic result at the beginning applies to:

$$\begin{array}{ccc} A_{m_y} & \hookrightarrow & k(Y) \\ \downarrow & & \downarrow \\ B_{m_y} & \hookrightarrow & k(X) \\ \cup\!\!\!\mid & & \\ m_{x_1}, \cdots, m_{x_r} & \text{all maximal ideals} & \end{array}$$

and $m_y B_{m_y} = m_{x_1}^{e_1} \cdots m_{x_r}^{e_r}$. Now unwind the definitions:

$$\begin{aligned} \deg(f^*([y])) &= \deg[f^{-1}(y)] \\ &= \deg\left(\textstyle\sum_{i=1}^r \mathrm{length}_{\mathcal{O}_{X,x_i}}(\kappa(y) \otimes \mathcal{O}_{X,x_i}) \cdot [x_i]\right) \\ &= \textstyle\sum_{i=1}^r \mathrm{length}_{B_{m_{x_i}}}\left(\frac{B_{m_{x_i}}}{m_y B_{m_{x_i}}}\right) [\kappa(x_i) : k] \\ &= \textstyle\sum_{i=1}^r \dim_{k(x_i)} \frac{B_{m_{x_i}}}{m_{x_i}^{e_i}} [\kappa(x_i) : \kappa(y)][\kappa(y) : k] \\ &= \textstyle\sum_{i=1}^r e_i [\kappa(x_i) : \kappa(y)][\kappa(y) : k] \\ &= [k(X) : k(Y)] \cdot [\kappa(y) : k] \\ &= n \cdot \deg([y]). \end{aligned}$$

$\square$

Before more applications, we need more general results.

Invertible sheaves and effective Cartier divisors

X: noetherian scheme, $\mathcal{L}$: invertible sheaf. $s \in \Gamma(X, \mathcal{L})$.

Def. We call s a regular section if $\mathcal{O}_X \xrightarrow{\cdot s} \mathcal{L}$ is injective.

Let $Z(s) \triangleq$ the largest closed subscheme of X s.t. $s|_Z \equiv 0$, i.e. $i: Z \hookrightarrow X$ then $0 = i^* s \in \Gamma(Z(s), i^*\mathcal{L})$. Locally, if we choose a trivialization: $\varphi_U: \mathcal{L}_U \cong \mathcal{O}_U$, then $\varphi_U(s) = f \in \Gamma(U, \mathcal{O}_X)$, and $Z(s) \cap U = Z(f)$; if $U = \operatorname{Spec} A$, then $Z \cap U = \operatorname{Spec} A/(f)$). s regular means that f is a non-zero divisor. In other words s is a regular section iff $Z(s)$ is an effective Cartier divisor on X. On an integral noetherian scheme, s is regular iff $s \neq 0$.

Lemma. $X, \mathcal{L}, s$ as above and s regular. $D = Z(s)$, then

$$\mathcal{L} \cong \mathcal{I}_D^{-1} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X)$$

Pf: Another way of saying this is that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{I}_D \cong \mathcal{O}_X$. But

$$\begin{aligned} \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{I}_D &\longrightarrow \mathcal{O}_X \\ s', f &\mapsto \text{"}\left(\frac{fs'}{s}\right)\text{"} \end{aligned}$$

where s', f are local sections, is easily seen to be an isomorphism. $\square$

Converse construction: If $D \hookrightarrow X$ is an effective Cartier divisor, then we can define $\mathcal{O}_X(D) \triangleq \mathcal{I}_D^{-1} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X)$ to be the invertible sheaf associated to D, which has a canonical section 1_D s.t. $Z(1_D) = D$.

Question: Given D, D', when is $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$?

A partial answer:

Lemma. If X is integral, then $\mathcal{O}_X(D) \cong \mathcal{O}_X(D')$ iff $\exists f \in k(X)^*$ s.t. $\forall$ $\operatorname{Spec} A \subseteq X$ affine open, $D \cap \operatorname{Spec} A = \operatorname{Spec} A/(a)$, $D' \cap \operatorname{Spec} A = \operatorname{Spec} A/(a')$ we have $f = (\text{unit in } A) \cdot \frac{a}{a'}$.

Pf: Say, if $\varphi: \mathcal{O}_X(D) \xrightarrow{\cong} \mathcal{O}_X(D')$, then $\varphi(1_D) = f \cdot 1_{D'}$. The result follows. $\square$

Def: Given an integral Noetherian scheme X, and $f \in k(X)^*$, we set the Weil divisor of f:

$$\operatorname{div}(f) = \sum_{\xi \in Z \subseteq X} \operatorname{ord}_\xi(f) [Z].$$

where Z is a prime divisor and ξ its generic point, and

$$\operatorname{ord}_\xi(f) \triangleq \operatorname{length} \mathcal{O}_{X,\xi}/(a) - \operatorname{length} \mathcal{O}_{X,\xi}/(b)$$

and $a, b \in \mathcal{O}_{X,\xi}$, $f = \frac{a}{b} \in \text{f.f.}(\mathcal{O}_{X,\xi}) = k(X)$.

Def. The class group $\operatorname{Cl}(X) \triangleq (\text{Weil divisors on } X)/(\text{principal divisors})$

Back to applications for curves

Lemma. On a regular curve X. $Cl(X) \xrightarrow{\cong} Pic(X)$.
$$D \mapsto \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$$
where we write $D = D_1 - D_2$ as a difference of effective Cartier divisors.

Pf: By the previous lemma, the map is well-defined and injective. It suffices to show that it's surjective.

First of all, any $\mathcal{L}$ which has a non-zero section is isomorphic to $\mathcal{O}_X(D_1)$ for some effective D_1. Thus it suffices to show that given any $\mathcal{L}$, there exists some D_2 effective s.t. $\mathcal{L}(D_2)$ has a non-zero global section.

Pick $\phi \neq U \subseteq X$. non-empty affine, s.t. $\Gamma(U, \mathcal{L}) \neq 0$, pick any non-zero s in it. $X \setminus U = \{x_1, \cdots, x_r\}$ finitely many closed points. Then for $N \gg 0$, s extends to a section $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_X(N([x_1] + \cdots + [x_r]))$. (For instance, regard s as a rational section in $\mathcal{L} \otimes_{\mathcal{O}_X} k(X)$, then take $N \geq \max_{i=1}^{r} \{ ord_{x_i}(s) \}$.) $\square$

Lemma. On a non-singular projective curve X, the degree of a principal Weil divisor is 0.

Pf:

Given $f \in k(X)^*$, (w.l.o.g. assume f is not finite over k, in which case $div(f) = 0$). f defines a morphism $f: X \longrightarrow \mathbb{P}^1$. This gives $k(t) \hookrightarrow k(X)$ $t \mapsto f$. Observe that $div(f) = f^*([0] - [\infty])$
$$\begin{aligned} \Rightarrow \deg(div(f)) &= \deg f^*([0]) - \deg f^*([\infty]) \\ &= \deg f - \deg f \\ &= 0 \end{aligned}$$
(both 0 and ∞ are k-points, $\deg[0] = \deg[\infty] = 1$). $\square$

It follows that the degree of an invertible sheaf $\mathcal{L}$ on a regular projective curve is well-defined: $\mathcal{L} \cong \mathcal{O}_X(D_1) \otimes \mathcal{O}_X(D_2)^{-1}$, then
$$\deg \mathcal{L} \triangleq \deg D_1 - \deg D_2.$$

E.g. $Pic(\mathbb{P}^1_k) \cong \mathbb{Z}$.

Pf: Since $\mathrm{Pic}(X) \cong Cl(X)$, it suffices to show that every degree 0 Weil divisor is principal.

In case $k = \bar{k}$: a degree 0 divisor is of the form:

$$\sum a_i [\alpha_i] - \sum b_j [\beta_j]$$

$a_i, b_j > 0$ and $\sum a_i = \sum b_j$. By a linear change of coordinates if necessary, assume $\alpha_i, \beta_j \neq \infty$. Then this is the divisor of the rational function

$$\prod (t-\alpha_i)^{a_i} / \prod (t-\beta_j)^{b_j}$$

which is regular at ∞ since $\sum a_i = \sum b_j$.

If $k \neq \bar{k}$, $k(\alpha_i) \cong k[t]/(f_i)$ for some monic irreducible polynomial $f \in k[t]$. (similar g_j for β_j). Then degree 0 means

$$\sum a_i \deg(f_i) - \sum b_j \deg(g_j) = 0$$

and thus it's $\mathrm{div}(\prod_i f_i^{a_i} / \prod_j g_j^{b_j})$.

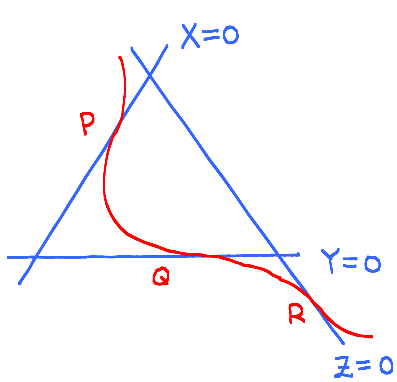
E.g. If X is non-singular, projective, $k = \bar{k}$, and $\mathrm{Pic}\, X \cong \mathbb{Z}$. Then $X \cong \mathbb{P}^1_k$.

Pf: Pick $x_1, x_2 \in X$, closed points, $\mathrm{Pic}(X) \cong \mathbb{Z} \Rightarrow [x_1] - [x_2] = \mathrm{div}(f)$. Consider $f: X \longrightarrow \mathbb{P}^1_k$, then it has degree 1 since $f^*([0]) = [x_1]$. Hence $k(X) \cong k(\mathbb{P}^1)$ (deg 1 extension). X regular, projective $\Rightarrow X \cong \mathbb{P}^1$.

Rmk: In general it's not true that $\mathrm{Pic}\, \mathbb{A}^1_R \cong \mathrm{Pic}\, \mathrm{Spec}\, R$ and $\mathrm{Pic}(\mathbb{P}^1_R) \cong \mathrm{Pic}(R) \times \mathbb{Z}$. It's only true for R "nice", for instance regular in codim 1. (c.f. Hartshorne).

Motivation: Why do we introduce cohomology?

Question: Are there any other curves than $\mathbb{P}^1$? (Yes)



Later we will see that, if $C \neq 0, -27, \infty$, $\operatorname{char} k \neq 3$, C is regular.

Claim: $C \not\cong \mathbb{P}^1_k$.

Otherwise, pick an isomorphism $\varphi: \mathbb{P}^1_k \longrightarrow C \subseteq \mathbb{P}^2_k$. Then f is given by an invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(d)$ and 3 global sections, $S_X, S_Y, S_Z \in \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(d))$. Furthermore, we have

(1). S_X, S_Y, S_Z have no common zeros on $\mathbb{P}^1$.

(2). $(S_X + S_Y + S_Z)^3 + C \cdot S_X S_Y S_Z = 0$ in $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(3d))$

(3). $Z(S_X) = P$, $Z(S_Y) = Q$, $Z(S_Z) = R$ are 3 distinct points (inflection points).

After a linear change of coordinates, we may assume that $P = [0, 1]$, $Q = [1, 0]$ and $R = (1, -1)$ in $\mathbb{P}^1$. It follows that:

$$S_X = \lambda X_0^d, \qquad S_Y = \mu X_1^d, \qquad S_Z = \nu (X_0 + X_1)^d \quad (\lambda, \mu, \nu \neq 0)$$

Plugging into the relation in (2), we have

$$(\lambda X_0^d + \mu X_1^d + \nu (X_0 + X_1)^d)^3 = -C \cdot \lambda \mu \nu X_0^d X_1^d (X_0 + X_1)^d$$

Obviously $d \neq 0$, otherwise $\operatorname{im}(\mathbb{P}^1) = pt$.

$\Rightarrow X_0 \mid$ r.h.s. $X_1 \mid$ r.h.s. $\Rightarrow X_0 \mid$ l.h.s, $X_1 \mid$ l.h.s.

$\Rightarrow \mu + \nu = 0$, $\lambda + \nu = 0$

$\Rightarrow C = 0$, $d = 1$, which we have excluded from the outset.

Imaging doing this case by case for curves! After enough cohomological machinary is developed, this will follow simply from calculating cohomological invariants:

$$\begin{cases} H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0 \\ H^1(C, \mathcal{O}_C) = 1 \end{cases}$$

Motivation / Question:

(1). X: a scheme, if X_{red} is affine, what can we say about X?

Thm. X_{red} is affine $\Leftrightarrow$ X is affine.

The proof is not easy and even in Noetherian case uses Serre's cohomological criterion of being affine.

(2). Over k. X_{red} proj $\overset{?}{\Longrightarrow}$ X proj (No)

No easy proof other than using cohomology.

§9. Cohomology

Let $\mathcal{A}$ be an abelian category.

Def: An object I of $\mathcal{A}$ is called injective iff $\forall$ diagram of solid arrows, the dotted arrow exists:

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr] & B \arrow[d, dotted] \\ & I \end{tikzcd}$$

Lemma. Given any s.e.s: $0 \to I \to A \to B \to 0$, with I injective
$A \cong I \oplus B$ (or $\mathrm{Ext}^1(B, I) = 0$). $\square$

Def's/ Lemmas.

(a). A complex $K^\bullet$ is $\cdots \to K^n \xrightarrow{d^n} K^{n+1} \to \cdots$ with $d^n \circ d^{n-1} = 0$, $\forall n \in \mathbb{Z}$.

(b). The n-th cohomology object is $H^n(K^\bullet) \triangleq \ker d^n / \mathrm{Im}\, d^{n-1}$.

(c). A morphism of complexes $\alpha: K^\bullet \to L^\bullet$ is a sequence of maps $\alpha^n: K^n \to L^n$ s.t.

$$\begin{tikzcd} K^n \arrow[r, "d"] \arrow[d, "\alpha^n"] & K^{n+1} \arrow[d, "\alpha^{n+1}"] \\ L^n \arrow[r, "d"] & L^{n+1} \end{tikzcd}$$

This induces maps $H^n(\alpha): H^n(K^\bullet) \to H^n(L^\bullet)$.

(d) Two morphisms of complexes $\alpha, \beta: K^\bullet \to L^\bullet$ are homotopic iff $\exists$ family of morphisms $h^i: K^i \to L^{i-1}$ s.t. $\alpha - \beta = hd + dh$. ($\alpha \sim \beta$ for short).

$$\begin{tikzcd} \cdots \arrow[r] & K^{i-1} \arrow[r] \arrow[d] \arrow[dl] & K^i \arrow[r] \arrow[d] \arrow[dl, "h^i"'] & K^{i+1} \arrow[r] \arrow[d] \arrow[dl, "h^{i+1}"'] & \cdots \\ \cdots \arrow[r] & L^{i-1} \arrow[r] & L^i \arrow[r] & L^{i+1} \arrow[r] & \cdots \end{tikzcd}$$

Lemma. (d) $\Rightarrow H^i(\alpha) = H^i(\beta)$, $\forall i \in \mathbb{Z}$. $\square$

Lemma. $\alpha\sim\beta \Rightarrow \delta\circ\alpha\circ\gamma \sim \delta\circ\beta\circ\gamma$, where $M^\bullet \xrightarrow{\gamma} K^\bullet \overset{\alpha}{\underset{\beta}{\rightrightarrows}} L^\bullet \xrightarrow{\delta} N^\bullet$. $\square$

(e). A morphism of complexes $\alpha: K^\bullet \to L^\bullet$ is a quasi-isomorphism (qis) iff $\forall n\in\mathbb{Z}$, $H^n(\alpha)$ is an isomorphism.

(f). We say $\mathcal{A}$ has enough injectives iff $\forall A$, object of $\mathcal{A}$, $\exists A \hookrightarrow I$, with I injective object.

(g). Given $A\in Ob(\mathcal{A})$, an injective resolution of A is a complex $I^\bullet$ and $A\hookrightarrow I^0$ s.t.

(1). $I^n = 0$, $\forall n<0$

(2). I^n injective, $\forall n$.

(3). $H^n(I^\bullet) = \begin{cases} 0 & n\neq 0 \\ A & n=0 \end{cases}$

Notation: $A[0] \triangleq$ the complex with A in degree 0, and everything else 0.
Then via this map:

$$\begin{array}{ccccccccc} \cdots & \longrightarrow & 0 & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & 0 & \longrightarrow & A & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

$A[0] \to I^\bullet$ is a qis. Or really, $A[0]\to I^\bullet$ is an injective resolution of the complex $A[0]$ of (h).

(h). Given a complex $K^\bullet$, an injective resolution $K^\bullet$ is a qis $\alpha: K^\bullet \to I^\bullet$ s.t.

(1). $I^n = 0$ $\forall n \ll 0$ (bounded below)

(2). Each I^n is injective.

Lemma. Assume $\mathcal{A}$ has enough injectives, then

(a). Every object and every complex $K^\bullet$ with all $H^n(K^\bullet)=0$ for all $n\ll 0$

has an injective resolution.

(a'). If $K^n = 0$, then we may pick $K^\bullet \xrightarrow{\alpha} I^\bullet$ termwise injective.

(b). Given a solid diagram:
$$\begin{tikzcd} K^\bullet \arrow[r, "\alpha"] \arrow[d] & L^\bullet \arrow[dl, dashed, "\beta"] \\ I^\bullet & \end{tikzcd}$$

with: (1). $I^\bullet$ a bounded-below complex of injectives.

(2). α is qis.

Then β exists making the diagram commute up to homotopy.

(b'). In (b), if α is termwise injective, then we can make the diagram commute.

(c). The β in (b) is unique up to homotopy.

Pf. (a) For objects it's easy. For complexes, we reduce it to (a') by introducing the truncation: pick $n \ll 0$:
$$\begin{tikzcd} \cdots \arrow[r] & K^{n-2} \arrow[r] \arrow[d] & K^{n-1} \arrow[r] \arrow[d] & K^n \arrow[r] \arrow[d] & K^{n+1} \arrow[r] \arrow[d] & \cdots & K^\bullet \\ \cdots \arrow[r] & 0 \arrow[r] & \operatorname{im} d^{n-1} \arrow[r] & K^n \arrow[r] & K^{n+1} \arrow[r] & \cdots & \tau_{\geq n} K^\bullet \end{tikzcd}$$

which is a qis of complexes and $\tau_{\geq n} K^\bullet$ is bounded from below.

To prove (b), we reduce it to (b'). Claim: Given any map of complexes, $K^\bullet \xrightarrow{\alpha} L^\bullet$, there exists a termwise injective map:
$$\begin{tikzcd} K^\bullet \arrow[r, "\tilde{\alpha}"] \arrow[rr, bend right, "\alpha"'] & \tilde{L}^\bullet \arrow[r, "\pi"] & L^\bullet \end{tikzcd}$$

where π has a section $L^\bullet \xrightarrow{s} \tilde{L}^\bullet$ s.t. $s \circ \pi$ is homotopic to the identity on $\tilde{L}^\bullet$. Then composing s with $\tilde{\beta}$ obtained from (b'), we get $\beta = \tilde{\beta} \circ s$.

Construction of $\tilde{L}^\bullet$ (mapping cylinder):
$$\begin{tikzcd} K^n \arrow[r] \arrow[d] & L^n \arrow[r, phantom, "\oplus"] \arrow[d, "d"'] & K^n \arrow[r, phantom, "\oplus"] \arrow[d, "d"'] & K^{n+1} \arrow[r] \arrow[d, "-d"] \arrow[dl, "1"'] \arrow[dll] & L^n \arrow[d] \\ K^{n+1} \arrow[r] & L^{n+1} \arrow[r, phantom, "\oplus"] & K^{n+1} \arrow[r, phantom, "\oplus"] & K^{n+2} \arrow[r] & L^{n+1} \end{tikzcd}$$

Now we only need to show (b'): Given

$$\begin{tikzcd}
K^\bullet \arrow[r, hook, "\alpha"] \arrow[d] & L^\bullet \arrow[dl, dashed, "\beta"] \\
I^\bullet &
\end{tikzcd}$$

with α termwise injective, qis. we need to define β. By induction, we may assume β is defined up to $\beta^{n-1}: L^{n-1} \longrightarrow I^{n-1}$.

$$\begin{tikzcd}
 & K^n \arrow[rr, hook] \arrow[dd] & & L^n \arrow[ddll, dashed, "?"] \\
K^{n-1} \arrow[ur] \arrow[rr, hook] \arrow[dd] & & L^{n-1} \arrow[ur] \arrow[ddll] & \\
 & I^n & & \\
I^{n-1} \arrow[ur] & & &
\end{tikzcd}$$

Note that ? is already defined on $d^{n-1}(L^{n-1})$, and agree with the map defined on $d^{n-1}K^{n-1}$. Since $K^\bullet \hookrightarrow L^\bullet$ is qis, we have $K^n \cap d^{n-1}L^{n-1} = d^{n-1}K^{n-1}$. Hence by injectivity, we can extend the map $K^n + d^{n-1}L^{n-1} \longrightarrow I^n$ to ?

$\square$

Upshot: Given any solid diagram:

$$\begin{tikzcd}
K_1^\bullet \arrow[r] \arrow[d, "\alpha_1"] & K_2^\bullet \arrow[d, "\alpha_2"] \\
I_1^\bullet \arrow[r, dashed] & I_2^\bullet
\end{tikzcd}$$

where α_1, α_2 are injective resolutions, $\exists$ dotted arrow making the diagram commute up to homotopy, and is unique up to homotopy.

Moreover, if $K_1^\bullet \longrightarrow K_2^\bullet$ is a qis, we have another dotted arrow in the other direction, which is inverse to the first one up to homotopy in both directions.

In the picture of categories, we have obtained:

$\mathcal{A}$	$\longrightarrow$ $\mathrm{Comp}^+(\mathcal{A})$	$\longrightarrow$ $K^+(\mathcal{A})$
Our original abelian category	Category of complexes, bdd below	Objects are the same, with morphism $\mathrm{Hom}_{K^+(\mathcal{A})}(K^\bullet, L^\bullet) = \mathrm{Hom}_{\mathrm{Comp}^+(\mathcal{A})}(K^\bullet, L^\bullet)/\sim$

Now with enough injectives in $\mathcal{A}$, we have

$$K^+(\mathcal{A}) \xrightarrow{j} \mathcal{D}^+(\mathcal{A})$$

where in $\mathcal{D}^+(\mathcal{A})$, objects are bdd below complexes of injective objects, morphisms as in $K^+(\mathcal{A})$. On objects, $j(K^\bullet) = I^\bullet$ where $I^\bullet$ is a chosen injective resolution. On morphisms,

$$\begin{tikzcd} K_1 \arrow[r, "\varphi"] \arrow[d] & K_2 \arrow[d] \\ I_1 \arrow[r, dashed, "j(\varphi)"] & I_2 \end{tikzcd}$$

and $j(\varphi)$ is unique up to homotopy. The functor j makes every qis an iso in $\mathcal{D}^+(\mathcal{A})$.

Def. A functor $F: \mathcal{A} \longrightarrow \mathcal{B}$ of abelian categories is called left exact iff F is additive for any s.e.s. $0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$ in $\mathcal{A}$, we have

$$0 \longrightarrow F(A) \longrightarrow F(B) \longrightarrow F(C)$$

is exact in $\mathcal{B}$.

E.g. (1). $\mathcal{A} = \mathcal{A}b(X)$ or $\mathcal{M}od(\mathcal{O}_X)$. $\mathcal{B} = \mathcal{A}b = \mathcal{A}b(*)$

Then $F = \Gamma(X, \cdot)$ is left exact.

(2). More generally, if $f: X \longrightarrow Y$ is a morphism of ringed spaces. $\mathcal{A} = \mathcal{A}b(X)$ or $\mathcal{M}od(\mathcal{O}_X)$, $\mathcal{B} = \mathcal{A}b(Y)$ or $\mathcal{M}od(\mathcal{O}_Y)$, then $F = f_*$ is left exact.

Recall that if $\mathcal{A}$ has enough injectives, we have

$$\mathcal{A} \longrightarrow Comp^+(\mathcal{A}) \longrightarrow K^+(\mathcal{A}) \xrightarrow{j} \mathcal{D}^+(\mathcal{A})$$

and the i-th homology functor $H^i: Comp^+(\mathcal{A}) \longrightarrow \mathcal{A}$ factors to give functors

$$H^i: K^+(\mathcal{A}) \longrightarrow \mathcal{A} \quad \text{and} \quad H^i: \mathcal{D}^+(\mathcal{A}) \longrightarrow \mathcal{A}$$

Thm. $F: \mathcal{A} \longrightarrow \mathcal{B}$, left exact, and $\mathcal{A}$ has enough injectives. Then there exists a functor $RF: \mathcal{D}^+(\mathcal{A}) \longrightarrow K^+(\mathcal{B})$ with the following properties:

$$\begin{tikzcd}
A \arrow[r, mapsto] & A[0] & & \\
\mathcal{A} \arrow[r] \arrow[rrrd, "RF"] \arrow[rrrdd, "R^iF"'] & \mathrm{Comp}^+(\mathcal{A}) \arrow[r] & K^+(\mathcal{A}) \arrow[r, "j"] & \mathbb{Q}^+(\mathcal{A}) \arrow[d, "RF"] \\
& & & K^+(\mathcal{B}) \arrow[d, "H^i"] \\
& & & \mathcal{B}
\end{tikzcd}$$

$(R^iF(\cdot) \triangleq H^i(RF(\cdot)))$

(0). $RF(I^\bullet) \triangleq F(I^\bullet) \in \mathrm{Ob}(K^+(\mathcal{B}))$, for any $I^\bullet$ in $\mathrm{Ob}(\mathbb{Q}^+(\mathcal{A}))$. (This makes sense since our def. of $\mathbb{Q}^+(\mathcal{A})$ is as bounded below complexes of injectives).

(1). For $A \in \mathrm{Ob}(\mathcal{A})$, we have

$$R^iF(A) = \begin{cases} 0 & i<0 \\ F(A) & i=0 \end{cases}$$

(2). If $I \in \mathrm{Ob}(\mathcal{A})$ is injective, then $R^iF(I) = 0$ if $i \neq 0$.

(3). Given any s.e.s. $0 \to A \to B \to C \to 0$ in $\mathcal{A}$, we get a l.e.s.

$$\begin{aligned} 0 &\to F(A) \to F(B) \to F(C) \\ &\to R^1F(A) \to R^1F(B) \to R^1F(C) \\ &\to R^2F(A) \to \cdots \end{aligned}$$

(4). If $0 \to K^\bullet \to L^\bullet \to M^\bullet \to 0$ is a s.e.s. in $\mathrm{Comp}^+(\mathcal{A})$, then we get a l.e.s:

$$\begin{aligned} \cdots &\to R^iF(K^\bullet) \to R^iF(L^\bullet) \to R^iF(M^\bullet) \\ &\to R^{i+1}F(K^\bullet) \to R^{i+1}F(L^\bullet) \to R^{i+1}F(M^\bullet) \\ &\to R^{i+2}F(K^\bullet) \to \cdots \end{aligned}$$

Pf: Recall that we defined $j: K^+(\mathcal{A}) \to \mathbb{Q}^+(\mathcal{A})$ by choosing for each $K^\bullet$ an injective resolution $K^\bullet \to j(K^\bullet)$. Note that if $A \to I^\bullet$ is an injective resolution, then $RF(A) = F(I^\bullet)$.

(1). The injective resolution $I^\bullet: 0 \to I^0 \to I^1 \to \cdots$ is an injective resolution of A ($A[0]$). Since F is left exact,

$$\left.\begin{aligned} &0 \to A \to I^0 \to Z \to 0 \\ &0 \to Z \to I^1 \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &0 \to F(A) \to F(I^0) \to F(Z) \\ &0 \to F(Z) \to F(I^1) \end{aligned}\right\} \text{exact}$$

$\Rightarrow \quad 0 \longrightarrow F(A) \longrightarrow F(I^0) \longrightarrow F(I^1)$ is exact.

(2). says that if I is injective, then $R^iF(I) = 0$, $\forall i > 0$. This is true since $I[0]: \cdots \longrightarrow 0 \longrightarrow I \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$ is an injective resolution of I $\Rightarrow RF(I) = F(I)[0]$ has no cohomology except at degree 0.

To prove (3), we need the following:

Lemma: Given any s.e.s. in $\mathcal{A}$:

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

there exists a diagram:

$$\begin{tikzcd} 0 \arrow[r] & A[0] \arrow[r] \arrow[d] & B[0] \arrow[r] \arrow[d] & C[0] \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I_A^\bullet \arrow[r] & I_B^\bullet \arrow[r] & I_C^\bullet \arrow[r] & 0 \end{tikzcd}$$

where the vertical arrows are injective resolutions, and the lower horizontal sequence is termwise split exact. More generally, the same statement holds with $0 \longrightarrow A[0] \longrightarrow B[0] \longrightarrow C[0] \longrightarrow 0$ replaced by a s.e.s. in $Comp^+(\mathcal{A})$.

$$0 \longrightarrow K^\bullet \longrightarrow L^\bullet \longrightarrow M^\bullet \longrightarrow 0$$

(Termwise split meaning $I_2^\bullet \cong I_1^\bullet \oplus I_3^\bullet$)

Pf omitted.

Rmk: (1). Any additive functor applied to a split s.e.s. gives a s.e.s. Hence we may compute $RF(A)$ as $F(I_1^\bullet)$, $RF(B)$ as $F(I_2^\bullet)$, etc. And for $\alpha: A \longrightarrow B$ we may compute $RF(\alpha)$ as $F(j(\alpha))$ etc. Then:

$$0 \longrightarrow F(I_A^\bullet) \longrightarrow F(I_B^\bullet) \longrightarrow F(I_C^\bullet) \longrightarrow 0$$

is a termwise split exact sequence in $Comp^+(\mathcal{B})$.

(2). Suppose $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ is a termwise split s.e.s. of complexes. and if we choose a splitting $B^n \cong A^n \oplus C^n$, then

$$d_{B^n} = \begin{bmatrix} d_{A^n} & \delta \\ 0 & d_{C^n} \end{bmatrix}$$

and $d_{B^{n+1}} \circ d_{B^n} = 0 \Rightarrow d_{A^{n+1}} \circ \delta + \delta \circ d_{C^n} = 0$. Thus $\delta : C^\bullet \longrightarrow A^\bullet[1]$ is a morphism of complexes where $(A^\bullet[1])^n \triangleq A^{n+1}$, and differential $-d_{A^{n+1}}$. (It follows $H^q(A^\bullet[1]) = H^{q+1}(A^\bullet)$.) By five lemma, the l.e.s. associated to $0 \longrightarrow A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \longrightarrow 0$ is given by:

$$\begin{tikzcd} \cdots \arrow[r] & H^i(A) \arrow[r] & H^i(B) \arrow[r] & H^i(C) \arrow[dll, "\delta"] \\ & H^{i+1}(A) \arrow[r] & \cdots \end{tikzcd}$$

The morphism δ gives:

$$A^\bullet \longrightarrow B^\bullet \longrightarrow C^\bullet \xrightarrow{\delta} A^\bullet[1]$$

which makes a distinguished triangle and $K^+(\mathcal{A})$ a triangulated category.

Now (3) and (4) follows directly from the lemma and remarks. $\square$

Leray's acyclicity lemma

Let $F : \mathcal{A} \longrightarrow \mathcal{B}$ be a left exact functor. If $K^\bullet$ is a bounded-below complex and $R^iF(K^n) = 0$, $\forall i > 0$, $\forall n$. Then:

$$RF(K^\bullet) = F(K^\bullet)$$

Rmk: This says that if K^n is acyclic for the functor F (i.e. all higher derived functors vanish on K^n), then if $A^\bullet \xrightarrow{qis} K^\bullet$, we have (in $D^+(\mathcal{B})$):

$$RF(A^\bullet) = RF(K^\bullet) = F(K^\bullet)$$

Upshot: If we have $A \in Ob(\mathcal{A})$ and a qis $A[0] \longrightarrow K^\bullet$, with $K^\bullet$ bounded below complex of acyclic's, then

$$RF(A[0]) \cong RF(K^\bullet) = F(K^\bullet)$$

and thus

$$R^iF(A) \cong H^i(K^\bullet)$$

Pf of Leray's lemma

Choose $K^\bullet \hookrightarrow I^\bullet$ injective resolution, and set $Q^\bullet = I^\bullet / K^\bullet$. So we have:

$$0 \longrightarrow K^\bullet \longrightarrow I^\bullet \longrightarrow Q^\bullet \longrightarrow 0$$

By the l.e.s. of cohomology groups, we have, for each $n \in \mathbb{Z}$

$$\begin{aligned} 0 \longrightarrow F(K^n) \longrightarrow F(I^n) \longrightarrow F(Q^n) \\ \longrightarrow \underset{\substack{\| \\ 0}}{R^1F(K^n)} \longrightarrow \underset{\substack{\| \\ 0}}{R^1F(I^n)} \longrightarrow R^1F(Q^n) \\ \longrightarrow \underset{\substack{\| \\ 0}}{R^2F(K^n)} \longrightarrow \underset{\substack{\| \\ 0}}{R^2F(I^n)} \longrightarrow \cdots \end{aligned}$$

$\Rightarrow Q^n$ is acyclic as well and

$$0 \longrightarrow F(K^n) \longrightarrow F(I^n) \longrightarrow F(Q^n) \longrightarrow 0$$

is exact. Hence

$$0 \longrightarrow F(K^\bullet) \longrightarrow F(I^\bullet) \longrightarrow F(Q^\bullet) \longrightarrow 0$$

is a s.e.s in $\mathrm{Comp}^+(\mathcal{B})$. Note that by the same argument as above without F, $Q^\bullet$ is an acyclic complex in $\mathrm{Comp}^+(\mathcal{A})$.

Since $F(I^\bullet)$ is a manifestation of $RF(K^\bullet)$, it suffices to show that $F(K^\bullet) \longrightarrow F(I^\bullet)$ is a qis, or $F(Q^\bullet)$ is acyclic.

Set $Z^i = \mathrm{Im}(Q^{i-1} \longrightarrow Q^i)$, then:

$$\begin{tikzcd}[column sep=small]
0 \arrow[r] & Q^n \arrow[rr] \arrow[dr] & & Q^{n+1} \arrow[rr] \arrow[dr] & & Q^{n+2} \arrow[r] \arrow[dr] & \cdots \\
& & Z^{n+1} \arrow[ur] \arrow[dr] & & Z^{n+2} \arrow[ur] \arrow[dr] & & Z^{n+3} \cdots \\
& 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \arrow[ur] & \cdots
\end{tikzcd} \qquad (*)$$

Look at the complex: $0 \longrightarrow Z^{i-1} \longrightarrow Q^i \longrightarrow Z^{i+1} \longrightarrow 0$

$i=n$, $Q^n \cong Z^{n+1} \Rightarrow Z^{n+1}$ is F-acyclic and $F(Q^n) \cong F(Z^n)$.

$i=n+1$, $0 \longrightarrow Z^{n+1} \longrightarrow Q^{n+1} \longrightarrow Z^{n+2} \longrightarrow 0 \Rightarrow Z^{n+2}$ is F-acyclic

and $0 \longrightarrow F(Z^{n+1}) \longrightarrow F(Q^{n+1}) \longrightarrow F(Z^{n+2}) \longrightarrow 0$ is exact.

$i=n+2$, $0 \longrightarrow Z^{n+2} \longrightarrow Q^{n+2} \longrightarrow Z^{n+3} \longrightarrow 0 \Rightarrow Z^{n+3}$ is F-acyclic

and $0 \longrightarrow F(Z^{n+2}) \longrightarrow F(Q^{n+2}) \longrightarrow F(Z^{n+2}) \longrightarrow 0$ is exact

etc. etc.

Thus applying F to $(*)$, we obtain a l.e.s. $0 \longrightarrow F(Q^\bullet)$. $\square$

Applications on ringed spaces

Thm. X: topological space. The category $\mathcal{A}b(X)$ of abelian sheaves on X has enough injectives. More generally, $(X, \mathcal{O}_X)$: ringed space, then the category $\mathcal{M}od(\mathcal{O}_X)$ has enough injectives.

Pf: Recall that $\mathcal{A}b(X) = \mathcal{M}od(\underline{\mathbb{Z}}_X)$. Hence it suffices to show the last statement.

Constructions: If for every $x \in X$, we are given an $\mathcal{O}_{X,x}$-module M_x, then the rule $U \longmapsto \prod_{x \in X} M_x$ is a sheaf of $\mathcal{O}_X$-modules. This sheaf is just

$$\prod_{x \in X} (j_x)_*(M_x)$$

a direct product of skyscraper sheaves.

Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules, then

$$\mathcal{F} \hookrightarrow \prod_{x \in X} (j_x)_*(\mathcal{F}_x) \hookrightarrow \prod_{x \in X} (j_x)_* I_x$$

is an injection of sheaves of $\mathcal{O}_X$-modules, where for each $x \in X$, we choose an injective $\mathcal{O}_{X,x}$-module I_x containing $\mathcal{F}_x$ as a submodule. The thm follows from the next two lemmas:

Lemma: A product of injectives is injective. $\square$

Lemma: If I_x is an injective $\mathcal{O}_{X,x}$-module, then $(j_x)_*(I_x)$ is an injective object in $\mathcal{M}od(\mathcal{O}_X)$.

Pf: $\forall\, \mathcal{G} \in \mathcal{M}od(\mathcal{O}_X)$. $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{G}, (j_x)_* I_x) = \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{G}_x, I_x)$. Since I_x is an injective object in $\mathrm{Mod}(\mathcal{O}_{X,x})$, $\mathrm{Hom}_{\mathcal{O}_{X,x}}(-, I_x)$ is exact. Moreover, taking stalks is an exact functor. Hence being the composition of two exact functors, $\mathrm{Hom}_{\mathcal{O}_{X,x}}((-)_x, I_x)$ is exact. $\square$

By previous results, we have defined:

Def: (1). X: topological space.

$$\Gamma(X,-),\ \Gamma(U,-): \mathcal{A}b(X) \longrightarrow \mathcal{A}b.$$

(2). $f: X \longrightarrow Y$ continuous map of topological spaces,

$$f_*: \mathcal{A}b(X) \longrightarrow \mathcal{A}b(Y)$$

(3). $(X, \mathcal{O}_X)$: ring space.

$$\Gamma(X,-): \mathcal{M}od(\mathcal{O}_X) \longrightarrow \mathrm{Mod}(\Gamma(X,\mathcal{O}_X))$$
$$\Gamma(U,-): \mathcal{M}od(\mathcal{O}_X) \longrightarrow \mathrm{Mod}(\Gamma(U,\mathcal{O}_X))$$

(4). $f: (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$ a morphism of ringed spaces.

$$f_*: \mathcal{M}od(\mathcal{O}_X) \longrightarrow \mathcal{M}od(\mathcal{O}_Y)$$

Notation:

(1). $H^i(X,\mathcal{F}) \triangleq R^i\Gamma(X,\mathcal{F})$, $H^i(U,\mathcal{F}) \triangleq R^i\Gamma(U,\mathcal{F})$

(2). If $\mathcal{F}^\bullet$ is a bounded below complex in $\mathcal{A}b(X)$ or $\mathcal{M}od(\mathcal{O}_X)$, then

$$\mathbb{H}^i(X,\mathcal{F}^\bullet) \triangleq R^i\Gamma(X,\mathcal{F}^\bullet) \quad \text{(hypercohomology)}$$

Rmk: By def. $\mathcal{F} \in \mathcal{M}od(\mathcal{O}_X)$, pick any injective resolution $\mathcal{F} \longrightarrow I^\bullet$, then

$$R\Gamma(X,\mathcal{F}) = \Gamma(X, I^\bullet)$$

is the complex:

$$0 \longrightarrow \Gamma(X, I^0) \longrightarrow \Gamma(X, I^1) \longrightarrow \Gamma(X, I^2) \longrightarrow \cdots$$

Then $H^i(X,\mathcal{F}) = H^i(\Gamma(X,I^\bullet))$, as $\Gamma(X,\mathcal{O}_X)$-modules. Similarly.

$$Rf_*(\mathcal{F}) = f_*(I^\bullet) \quad : \text{ a complex of } \mathcal{O}_Y\text{-modules}$$
$$R^if_*(\mathcal{F}) = H^i(f_*I^\bullet) \quad : \text{ a sheaf of } \mathcal{O}_Y\text{-modules}$$

Also note that by def.

$$H^0(X,\mathcal{F}) = \Gamma(X,\mathcal{F}) \quad \text{and} \quad H^i(X,\mathcal{F}) = 0 \ \text{ if } i<0$$
$$R^0f_*(\mathcal{F}) = f_*\mathcal{F} \quad \text{and} \quad R^if_*(\mathcal{F}) = 0 \ \text{ if } i<0.$$

Cor. Given a sheaf $\mathcal{F}$ and $[\xi] \in H^p(X,\mathcal{F})$, $p>0$, $\exists$ an open covering $X = \bigcup_{i\in I} U_i$ s.t. the image of $[\xi]$ in each $H^p(U_i,\mathcal{F})$ is 0.

Pf: $\xi \in \Gamma(X, I^p)$ and $0 = d\xi \in \Gamma(X, I^{p+1}) \Rightarrow (d\xi)_x = 0$ in I^{p+1}_x. But since $I^\bullet$ is acyclic away from 0, $\xi_x = d\eta_x$, for some $\eta_x \in I^{p-1}_x$. Hence in some open nhd U of x. $\xi|_U = d\eta$. $\square$

Čech complex

X: a topological space. $\mathcal{U}: U = \bigcup_{i\in I} U_i$ an open covering of $U \hookrightarrow X$, an open subset. $\mathcal{F} \in \mathcal{A}b(X)$.

Def. Let $\check{C}^p(\mathcal{U}, \mathcal{F}) \triangleq \prod_{i_0\cdots i_p} \mathcal{F}(U_{i_0\cdots i_p})$ where $(i_0, \cdots, i_p) \in I^{p+1}$. The Čech cplx is defined as

$$\check{C}^*(\mathcal{U}, \mathcal{F}) :\ 0 \longrightarrow \check{C}^0(\mathcal{U}, \mathcal{F}) \xrightarrow{d} \check{C}^1(\mathcal{U}, \mathcal{F}) \xrightarrow{d} \check{C}^2(\mathcal{U}, \mathcal{F}) \longrightarrow \cdots$$

with differential: $\forall\ f \in \check{C}^p(\mathcal{U}, \mathcal{F})$

$$(df)_{i_0\cdots i_{p+1}} = \sum_{j=0}^{p+1} (-1)^j f_{i_0\cdots \hat{i_j} \cdots i_{p+1}}\big|_{U_{i_0\cdots i_{p+1}}}$$

Lemma. $d^2 = 0$. $\square$

Def. $\check{H}^\cdot(\mathcal{U}, \mathcal{F}) \triangleq H^\cdot(\check{C}^*(\mathcal{U}, \mathcal{F}))$

In general, there is no long exact sequences of Čech cohomology of sheaves.

Lemma A. If I is an injective $\mathcal{O}_X$-module, then $\check{H}^i(\mathcal{U}, I) = 0$ for all $i > 0$ and any open covering $\mathcal{U}$ of $U \hookrightarrow X$.

Pf: Let $j: U \hookrightarrow X$ be the inclusion of an open set. Consider $j_!(\mathcal{O}_U)$, extension by 0 of $\mathcal{O}_U$ to a sheaf of $\mathcal{O}_X$-modules. It's characterized by $j_!\mathcal{O}_U \subseteq \mathcal{O}_X$, and $(j_!\mathcal{O}_U)_x = \mathcal{O}_{X,x}$ if $x \in U$, or 0 otherwise. Furthermore, we have the mapping property:

$$\mathrm{Hom}_{\mathcal{O}_X}(j_!(\mathcal{O}_U), \mathcal{F}) = \Gamma(U, \mathcal{F})$$

Denote $j_{i_0\cdots i_p}: U_{i_0\cdots i_p} \hookrightarrow X$ and consider for $\mathcal{U}: U = \bigcup_{i\in I} U_i$ the complex of $\mathcal{O}_X$-modules:

$$J^\bullet :\ \cdots \longrightarrow \oplus_{i_0 i_1} (j_{i_0 i_1})_!(\mathcal{O}_{U_{i_0 i_1}}) \longrightarrow \oplus_{i_0} (j_{i_0})_!(\mathcal{O}_{U_{i_0}}) \longrightarrow 0$$

Then:

(1). For any $\mathcal{F} \in \mathcal{M}od(\mathcal{O}_X)$. $\mathrm{Hom}(J^\cdot, \mathcal{F}) \cong \check{C}^\cdot(\mathcal{U}, \mathcal{F})$ canonically.

(2). The complex of sheaves $J^\cdot$ is exact except in degree 0, and in fact

$$\cdots \longrightarrow \oplus_{i_0 i_1} (j_{i_0 i_1})_! (\mathcal{O}_{U_{i_0 i_1}}) \longrightarrow \oplus_{i_0} (j_{i_0})_! (\mathcal{O}_{U_{i_0}}) \longrightarrow j_! \mathcal{O}_U \longrightarrow 0$$

is exact. Indeed, $\forall x \in X$, the stalk of this complex is the simplicial cplx of a single point with coefficient $\mathcal{O}_{X,x}$.

Since $\mathrm{Hom}_{\mathcal{O}_X}(-, I)$ is exact, the lemma follows by applying (1) and (2).

□

Lemma B. If $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ is a s.e.s. of $\mathcal{O}_X$-modules, and $\check{H}^1(\mathcal{U}, \mathcal{F}) = 0$ for all open coverings $\mathcal{U}$ of all opens, then $\forall U \hookrightarrow X$ open, the map $\mathcal{G}(U) \longrightarrow \mathcal{H}(U)$ is surjective. □

Thm. Suppose $\mathcal{F}$ is a sheaf of abelian groups or $\mathcal{O}_X$-modules s.t. $\check{H}^i(\mathcal{U}, \mathcal{F}) = 0$ for all $i > 0$ and all open coverings $\mathcal{U}$ of all opens. Then $H^i(U, \mathcal{F}) = 0$ for all $U \hookrightarrow X$ and $i > 0$.

Pf: Pick an embedding of $\mathcal{F} \longrightarrow I$ with I injective and set $Q = I/\mathcal{F}$. Then we have:

$$0 \longrightarrow \mathcal{F} \longrightarrow I \longrightarrow Q \longrightarrow 0$$

Form the Čech complexes:

$$0 \longrightarrow \check{C}^*(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^*(\mathcal{U}, I) \longrightarrow \check{C}^*(\mathcal{U}, Q) \longrightarrow 0$$

By lemma B, we have surjectivity on the r.h.s. Take the l.e.s. of Čech cohomology groups and by lemma A.

$$\begin{aligned}
0 &\longrightarrow \check{H}^0(\mathcal{U}, \mathcal{F}) \longrightarrow \check{H}^0(\mathcal{U}, I) \longrightarrow \check{H}^0(\mathcal{U}, Q) \\
&\longrightarrow \underset{\substack{\| \\ 0}}{\check{H}^1(\mathcal{U}, \mathcal{F})} \longrightarrow \underset{\substack{\| \\ 0}}{\check{H}^1(\mathcal{U}, I)} \longrightarrow \check{H}^1(\mathcal{U}, Q) \\
&\longrightarrow \underset{\substack{\| \\ 0}}{\check{H}^2(\mathcal{U}, \mathcal{F})} \longrightarrow \underset{\substack{\| \\ 0}}{\check{H}^2(\mathcal{U}, I)} \longrightarrow \check{H}^2(\mathcal{U}, Q) \\
&\longrightarrow \underset{\substack{\| \\ 0}}{\check{H}^3(\mathcal{U}, \mathcal{F})} \longrightarrow \cdots
\end{aligned}$$

$\Rightarrow$ Q is also among those sheaves with vanishing higher Čech cohomology for any open covering of any open. Take the l.e.s. of cohomology:

$$0 \longrightarrow H^0(\mathcal{U},\mathcal{F}) \longrightarrow H^0(\mathcal{U},I) \longrightarrow H^0(\mathcal{U},Q) \longrightarrow H^1(\mathcal{U},\mathcal{F}) \longrightarrow H^1(\mathcal{U},I) = 0$$

By lemma B again, $H^1(\mathcal{U},\mathcal{F}) = 0$. Hence $H^1(\mathcal{U},Q) = 0$. Continue the l.e.s.

$$\cdots \longrightarrow H^1(\mathcal{U},I) \longrightarrow \underset{\substack{\| \\ 0}}{H^1(\mathcal{U},Q)} \longrightarrow H^2(\mathcal{U},\mathcal{F}) \longrightarrow \underset{\substack{\| \\ 0}}{H^2(\mathcal{U},I)} \longrightarrow \cdots$$

$\Rightarrow H^2(\mathcal{U},\mathcal{F}) = 0$. Repeat the argument and we are done. $\square$

Cor. If $f: X \longrightarrow Y$ is a morphism of ringed spaces and I is an injective $\mathcal{O}_X$-module. Then f_*I is a sheaf of $\mathcal{O}_Y$-modules which satisfies the assumption of the thm.

Pf: Take $\mathcal{V}: V = \bigcup_{i\in I} V_i$ an open covering of an open set V in Y. Note that $(f_*I)(V_{i_0\cdots i_p}) = I(f^{-1}(V_{i_0\cdots i_p})) = I(f^{-1}(V_{i_0}) \cap \cdots \cap f^{-1}(V_{i_p}))$. It follows trivially that $\check{C}^*(\mathcal{V}, f_*I) \cong \check{C}^*(\mathcal{U}, I)$ where $\mathcal{U}: f^{-1}(V) = \bigcup_{i\in I} f^{-1}(V_i)$. The result follows from the lemma for injectives. $\square$

Cor. $(X,\mathcal{O}_X)$ ringed space. $\mathcal{F}$ an $\mathcal{O}_X$-module. Then

$$H^i_{\mathcal{M}od(\mathcal{O}_X)}(X,\mathcal{F}) \cong H^i_{\mathcal{A}b(X)}(X,\mathcal{F}).$$

as abelian groups.

Pf: Consider $f: (X,\mathcal{O}_X) \longrightarrow (X, \mathbb{Z}_X)$, where $f = id_X$ on X, as a morphism of ringed spaces. Choose an injective resolution $\mathcal{F} \longrightarrow I^\bullet$ in $\mathcal{M}od(\mathcal{O}_X)$.

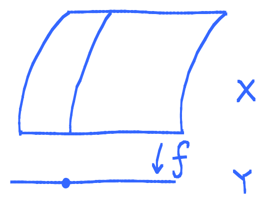
Note that $f_*: \mathcal{M}od(\mathcal{O}_X) \longrightarrow \mathcal{M}od(\mathbb{Z}_X) = \mathcal{A}b(X)$, $\mathcal{G} \mapsto f_*\mathcal{G}$ is just the forgetful functor (forgetting its $\mathcal{O}_X$-module structure), thus is exact $\Rightarrow f_*\mathcal{F} \longrightarrow f_*I^\bullet$ is a resolution.

The previous cor. $\Rightarrow f_*I^\bullet$ is acyclic for $\Gamma(X,-)$. Thus by Leray's acyclicity lemma. $\Gamma(X, f_*I^\bullet) \xrightarrow{qis} R\Gamma(X, f_*\mathcal{F})$. It follows that

$$\begin{aligned}
H^i_{\mathcal{M}od(\mathcal{O}_X)}(X,\mathcal{F}) &= H^i(\Gamma(X,I^\bullet)) \\
&= H^i(\Gamma(X,f_*I^\bullet)) \\
&= H^i(X, f_*\mathcal{F}) \text{ (by qis)} \\
&= H^i_{\mathcal{A}b(X)}(X,\mathcal{F}).
\end{aligned}$$

$\square$

In general, $f: X \longrightarrow Y$ a morphism of ringed spaces, it's not true that $f_*(\text{injectives}) = \text{injectives}$. Moreover, for an arbitrary morphism of schemes, the Cor will not be true unless f is rather trivial, for e.g. the forgetful map as in the cor. In general, Leray's spectral sequence says that



$$H^p(Y, R^q f_* \mathcal{F}) \Longrightarrow H^{p+q}(X, \mathcal{F})$$

By the same proof as in the thm, we have:

Thm (Variant). Let $(X, \mathcal{O}_X)$ be a ringed spaces. $\mathcal{F}$ an $\mathcal{O}_X$-module, $\mathcal{B}$: a basis of topology on X. Assume that $\forall\, U \in \mathcal{B}$, and any open covering $\mathcal{U}$: $U = \bigcup_{i \in I} U_i$, $\mathcal{U}$ can be refined into $\mathcal{V}$: $U = \bigcup_{j \in J} V_j$ s.t.

(a). $V_{j_0 \cdots j_p} \in \mathcal{B}$, $\forall\, p$

(b). $\check{H}^i(\mathcal{V}, \mathcal{F}) = 0$, $\forall\, i > 0$

Then $H^i(U, \mathcal{F}) = 0$, $\forall\, i > 0$ and $U \in \mathcal{B}$. $\square$

This doesn't imply that $H^i(X, \mathcal{F}) = 0$ for all $i > 0$ unless $X \in \mathcal{B}$.

Application on schemes

Lemma. X: scheme. $\mathcal{F}$: quasi-coherent (Q.C.) sheaf of $\mathcal{O}_X$-modules. Then for any affine open $U \subseteq X$ and any standard open covering $\mathcal{U}$: $U = \bigcup_{j=1}^n D(f_j)$, we have: $\forall\, i > 0$,

$$\check{H}^i(\mathcal{U}, \mathcal{F}) = 0$$

Pf: Say $U = \operatorname{Spec} A$ and $\mathcal{F}|_U = \tilde{M}$. We have to show that

$$0 \longrightarrow M \longrightarrow \oplus_{i_0} M_{f_{i_0}} \longrightarrow \oplus M_{f_{i_0} f_{i_1}} \longrightarrow \oplus M_{f_{i_0} f_{i_1} f_{i_2}} \longrightarrow \cdots$$

is exact. It's enough to show that the sequence, after localizing at all $\mathfrak{p} \subseteq A$ prime is exact. Since $D(f_i)$'s cover U, some f_i, say $f_i \notin \mathfrak{p}$. Then note that

$(M_{f_i})_\mathfrak{p} = M_\mathfrak{p}$ and $(M_{f_i f_{j_1}\cdots f_{j_r}})_\mathfrak{p} = (M_{f_{j_1}\cdots f_{j_r}})_\mathfrak{p}$. We can construct a homotopy operator

$$h: \oplus (M_{f_{i_0}\cdots f_{i_p}})_\mathfrak{p} \longrightarrow \oplus (M_{f_{i_0}\cdots i_{p-1}})_\mathfrak{p} \quad \text{by:}$$
$$m \mapsto (hm)_{i_0\cdots i_{p-1}} = (m_{i i_0 \cdots i_{p-1}})$$

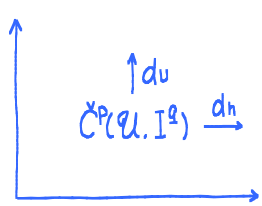
Then we compute:

$$\begin{aligned}(dh+hd)(m)_{i_0\cdots i_p} &= (d(hm))_{i_0\cdots i_p} + (h(dm))_{i_0\cdots i_p} \\ &= \textstyle\sum_{j=0}^{p}(-1)^j (hm)_{i_0\cdots \hat{i_j}\cdots i_p} + (dm)_{i i_0\cdots i_p} \\ &= \textstyle\sum_{j=0}^{p}(-1)^j m_{i i_0\cdots\hat{i_j}\cdots i_p} + \sum_{j=0}^{p}(-1)^{j+1} m_{i i_0\cdots\hat{i_j}\cdots i_p} + m_{i_0\cdots i_p} \\ &= m_{i_0\cdots i_p}\end{aligned}$$

$\Longrightarrow dh+hd = id$ on the complex and thus it's exact. $\square$

Cor. For any scheme X, and any $\mathcal{F}$: Q.C. sheaf of $\mathcal{O}_X$-modules, we have:
$$H^q(U, \mathcal{F}|_U) = 0$$
for any $q>0$ and U affine open in X. $\square$

Cor. Let X be a scheme, and $\mathcal{U}: X = \cup_{i\in I} U_i$ of an open covering by affines. s.t. each multi-intersection of U_i's is affine. (true for e.g. when X is seperated). Then $\forall$ Q.C. sheaf of $\mathcal{O}_X$-modules, we have
$$H^i(X, \mathcal{F}) = \check{H}^i(\mathcal{U}, \mathcal{F})$$
Pf: Pick an injective resolution $\mathcal{F} \longrightarrow I^\bullet$ and form the double complex
$$\check{C}^p(\mathcal{U}, I^q).$$



Note that fixing p, $(\check{C}^p(\mathcal{U}, I^q), d_v)$ is exact and $H^i(\check{C}^p(\mathcal{U}, I^q), d_v) = \prod_{i_0\cdots i_p} H^i(U_{i_0\cdots i_p}, I^q)$ $= 0$ if $i>0$ since $U_{i_0\cdots i_p}$ is affine, and equals $\prod_{i_0\cdots i_p}\Gamma(U_{i_0\cdots i_p}, \mathcal{F})$ for $i=0$ since the complex $\mathcal{F}|_{U_{i_0\cdots i_p}} \longrightarrow I^\bullet|_{U_{i_0\cdots i_p}}$ is an injective resolution.

On the other hand, fixing q, $(\check{C}^p(\mathcal{U}, I^q), d_h)$ is exact except at $p=0$, since I^q is injective, and has $H^0(\check{C}^p(\mathcal{U}, I^q), d_h) = \Gamma(X, I^q)$.

Hence the double complex computes both $\Gamma(X, I^\bullet)$ and $\check{C}^*(\mathcal{U}, \mathcal{F})$, and from the s.s. of the double complex, we have:

$$H^i(X, \mathcal{F}) = H^i(\Gamma(X, I^\bullet)) = H^i(C^\bullet(\mathcal{U}, I^\bullet)) = H^i(\check{C}^*(\mathcal{U}, \mathcal{F})) = \check{H}^i(\mathcal{U}, \mathcal{F}) \qquad \square$$

Alternating Čech cochains

$\mathcal{U}$: as before. $\check{C}^*_{alt}(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p})$, where we have chosen a total ordering on I.

d_{alt}: exactly as before: $\forall\, m \in \check{C}^{p-1}_{alt}(\mathcal{U}, \mathcal{F})$,

$$(dm)_{i_0 \cdots i_p} = \sum_{j=0}^{p} (-1)^j m_{i_0 \cdots \hat{i_j} \cdots i_p}$$

We have:

$$\check{C}^*_{alt}(\mathcal{U}, \mathcal{F}) \longrightarrow \check{C}^*(\mathcal{U}, \mathcal{F}) \xrightarrow{\text{proj.}} \check{C}^*_{alt}(\mathcal{U}, \mathcal{F})$$

$$m \mapsto (m_{i_0 \cdots i_p}) = \begin{cases} 0 & \text{if some } i_k = i_l \\ (-1)^{\operatorname{sgn}\sigma} m_{\sigma(i_0) \cdots \sigma(i_p)} & \text{if all } i_k \neq i_l \text{ and } \sigma(i_0) < \cdots < \sigma(i_p) \end{cases}$$

Fact: Actually this is a chain homotopy equivalence. (EGA. 0_{III}).

Cor. Let X be a scheme which has a covering $\mathcal{U}: X = U_1 \cup \cdots \cup U_n$ s.t. each $U_{i_0 \cdots i_p}$ is affine. Then $H^i(X, \mathcal{F}) = 0$, $\forall\, i \geq n$ and any Q.C. $\mathcal{O}_X$-module.

Pf: $H^i(X, \mathcal{F}) \cong \check{H}^i(\mathcal{U}, \mathcal{F}) \cong \check{H}^i_{alt}(\mathcal{U}, \mathcal{F})$. But $\check{C}^p_{alt}(\mathcal{U}, \mathcal{F}) = \prod_{i_0 < \cdots < i_p} \mathcal{F}(U_{i_0 \cdots i_p}) = 0$ for any $p > n$. $\square$

Typical application of the cor: X seperated and $X = U_1 \cup \cdots \cup U_n$, each U_i affine. For instance, $\mathbb{P}^{n-1}_R = D_+(X_0) \cup \cdots \cup D_+(X_n)$.

Mayer-Vietoris: Let $(X, \mathcal{O}_X)$ be a ringed space, and $X = U \cup V$ an open covering. For any $\mathcal{O}_X$-module $\mathcal{F}$, we have a l.e.s.

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(U, \mathcal{F}) \oplus H^0(V, \mathcal{F}) \longrightarrow H^0(U \cap V, \mathcal{F})$$
$$\longrightarrow H^1(X, \mathcal{F}) \longrightarrow \cdots$$

Pf: Choose an injective resolution $\mathcal{F} \longrightarrow I^\bullet$. Then:

$$0 \longrightarrow \underbrace{\Gamma(X, I^\bullet)}_{\substack{\text{This computes}\\ H^*(\mathcal{F})}} \longrightarrow \underbrace{\Gamma(U, I^\bullet)}_{\substack{\text{This computes}\\ H^*(\mathcal{F}|_U)}} \oplus \underbrace{\Gamma(V, I^\bullet)}_{\substack{\text{This computes}\\ H^*(\mathcal{F}|_V)}} \longrightarrow \underbrace{\Gamma(U \cap V, I^\bullet)}_{\substack{\text{This computes}\\ H^*(\mathcal{F}|_{U\cap V})}} \longrightarrow 0$$

The complex is short exact since:

$$0 \longrightarrow (j_{U\cap V})_!(\mathcal{O}_{U\cap V}) \longrightarrow (j_U)_!(\mathcal{O}_U) \oplus (j_V)_!(\mathcal{O}_V)$$

and $Hom(-, I^\bullet)$ is. ($j_!$ is left adjoint to j^*). Hence we have the l.e.s. $\square$

The correct generality of this result is:

There is a spectral sequence: $\check{H}^p_{\text{Čech}}(\mathcal{U}, \underline{H}^q(\mathcal{F})) \Longrightarrow \check{H}^{p+q}(\mathcal{U}, \mathcal{F})$, where $\mathcal{U}: U = \bigcup_{i\in I} U_i$ and $\underline{H}^q(\mathcal{F})$ is the presheaf on X which assigns U the group $H^q(U, \mathcal{F}|_U)$.

Higher direct images.

$f: (X, \mathcal{O}_X) \longrightarrow (Y, \mathcal{O}_Y)$, a morphism of ringed spaces. Recall that for an $\mathcal{O}_X$-module $\mathcal{F}$, $R^i f_* \mathcal{F} \triangleq H^i(f_* I^\bullet)$ where $\mathcal{F} \longrightarrow I^\bullet$ is an injective resolution, and $f_* I^\bullet$ is a complex of $\mathcal{O}_Y$-module.

Lemma. $R^i f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \longrightarrow H^i(V, \mathcal{F}|_V)$.

Pf: $H^i(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)}) = Ker(\Gamma(f^{-1}(V), I^i) \longrightarrow \Gamma(f^{-1}(V), I^{i+1}))/Im(\Gamma(f^{-1}(V), I^{i-1}) \longrightarrow \Gamma(f^{-1}(V), I^i))$.

$$= Ker(f_* I^i(V) \longrightarrow f_* I^{i+1}(V))/Im(f_* I^{i-1}(V) \longrightarrow f_* I^i(V)).$$
$$= (\text{presheaf cohomology of the complex } f_* I^\bullet)(V)$$

Sheafify it to get $H^i(f_* I^\bullet)$. $\square$

Lemma. (Mayer-Vietoris for higher direct images).

$f: X \longrightarrow Y$ as above, $X = U \cup V$ open covering, $\mathcal{F}$ an $\mathcal{O}_X$-module. Then there is a l.e.s.

$$0 \longrightarrow f_* \mathcal{F} \longrightarrow (f|_U)_*(\mathcal{F}|_U) \oplus (f|_V)_*(\mathcal{F}|_V) \longrightarrow (f|_{U\cap V})_*(\mathcal{F}|_{U\cap V})$$
$$\longrightarrow R^1 f_* \mathcal{F} \longrightarrow \cdots\cdots$$

Pf: The same proof as above using the previous lemma. $\square$

Lemma. Let $f: X \longrightarrow Y$ be a morphism of sheaves. Assume f is quasi-compact and quasi-seperated, then:

(a). For any QCoh $\mathcal{O}_X$-module $\mathcal{F}$, the sheaves $R^i f_* \mathcal{F}$ are QCoh $\mathcal{O}_Y$-module.

(b). If Y is quasi-compact. Then $\exists\, N \in \mathbb{N}$ s.t. $\forall\, i \geq N$, $R^i f_* \mathcal{F} = 0$, $\forall$ QCoh $\mathcal{O}_X$-module $\mathcal{F}$.

Pf: (a). The question is local, thus we may assume Y is affine.

Step 1: X is affine. We claim that, in this case, $R^i f_* \mathcal{F} = 0$, $i > 0$. Namely, $\forall$ standard opens, $D(g) \subseteq Y$, we have

$$H^i(f^{-1}(D(g)), \mathcal{F}) = H^i(D(f^{\#}(g)), \mathcal{F}) = 0, \quad \forall\, i > 0.$$

Sheafification $\Rightarrow$ it's 0.

Step 2. X seperated and q.c. Set $n(X) \triangleq$ smallest integer n s.t. we can cover X by n open affines, which is finite. Write $X = U \cup V$ with V affine and $n(U) \leq n-1$. The relative MV gives:

$$\begin{aligned} 0 &\longrightarrow f_* \mathcal{F} \longrightarrow (f|_U)_* (\mathcal{F}|_U) \oplus (f|_V)_* (\mathcal{F}|_V) \longrightarrow (f|_{U \cap V})_* (\mathcal{F}|_{U \cap V}) \\ &\longrightarrow R^1 f_* \mathcal{F} \longrightarrow \underbrace{(R^1 f|_U)_* (\mathcal{F}|_U) \oplus (R^1 f|_V)_* (\mathcal{F}|_V)}_{\text{Q.Coh by induction hypothesis}} \longrightarrow \underbrace{(R^1 f|_{U \cap V})_* (\mathcal{F}|_{U \cap V})}_{\text{Q.Coh. by induction hypothesis}} \\ &\longrightarrow R^2 f_* \mathcal{F} \longrightarrow \cdots\cdots \end{aligned}$$

$\Longrightarrow R^i f_* \mathcal{F}$ is Q.Coh.

Step 3. X is quasi-seperated and q.c. Induct on $n(X)$: $X = U_1 \cup \cdots \cup U_n$. Let $U' = U_1 \cup \cdots \cup U_{n-1}$, $V' = U_n$

$$\begin{aligned} 0 &\longrightarrow f_* \mathcal{F} \longrightarrow (f|_{U'})_* (\mathcal{F}|_{U'}) \oplus (f|_{V'})_* (\mathcal{F}|_{V'}) \longrightarrow (f|_{U' \cap V'})_* (\mathcal{F}|_{U' \cap V'}) \\ &\longrightarrow R^1 f_* \mathcal{F} \longrightarrow \underbrace{(R^1 f|_{U'})_* (\mathcal{F}|_{U'}) \oplus (R^1 f|_{V'})_* (\mathcal{F}|_{V'})}_{\text{Q.Coh by induction hypothesis}} \longrightarrow \underbrace{(R^1 f|_{U' \cap V'})_* (\mathcal{F}|_{U' \cap V'})}_{\substack{\text{Q.Coh. by step 2 since} \\ U' \cap V' \text{ is contained in an} \\ \text{affine open and thus seperated.}}} \\ &\longrightarrow R^2 f_* \mathcal{F} \longrightarrow \cdots\cdots \end{aligned}$$

(b). The same proof as above on the number of open affine covers of Y. $\square$

Rmk: The correct proof uses spectral sequence.

§10. Theorem of Coherence

Cohomology of $\mathbb{P}^n_R$.

R: any ring. $n \geq 1$. $\mathcal{U}$: $\mathbb{P}^n_R = \bigcup_{i=0}^n D_+(X_i)$, the standard affine cover of $\mathbb{P}^n_R$.
$\mathcal{F} = \mathcal{O}_{\mathbb{P}^n_R}(d)$. We will calculate $H^i(\mathbb{P}^n_R, \mathcal{F}) \cong \check{H}^i_{alt}(\mathcal{U}, \mathcal{F})$. Here $U_{i_0 \cdots i_p} = D_+(X_{i_0} \cdots X_{i_p})$, and $\check{C}_{alt}(\mathcal{U}, \mathcal{F})$:

$$0 \longrightarrow \oplus_{i_0=0}^n R[X_0, \cdots, X_n, X_{i_0}^{-1}]_d \longrightarrow \cdots \longrightarrow \oplus_{i_0 < \cdots < i_p} R[X_0, \cdots, X_n, (X_{i_0} \cdots X_{i_p})^{-1}]_d \longrightarrow \cdots$$
$$\longrightarrow R[X_0, \cdots, X_n, (X_0 \cdots X_n)^{-1}]_d \longrightarrow 0$$

From the previous section, we know that:

Lemma. For any QCoh sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on $\mathbb{P}^n_R$, we have:
(a). $H^j(\mathbb{P}^n_R, \mathcal{F}) = \check{H}^j(\mathcal{U}, \mathcal{F}) = \check{H}^j_{alt}(\mathcal{U}, \mathcal{F})$.
(b). $H^j(\mathbb{P}^n_R, \mathcal{F}) = 0, \ \forall j > n$. $\square$

Thm. We have:

$$H^j(\mathbb{P}^n_R, \mathcal{O}(d)) = \begin{cases} R[X_0, \cdots, X_n]_d & j = 0 \\ 0 & 0 < j < n \\ \left(R[X_0, \cdots, X_n, (X_0 \cdots X_n)^{-1}] / (\sum_{j=0}^n R[X_0, \cdots, X_n, (X_0 \cdots \hat{X_j} \cdots X_n)^{-1}])\right)_d & j = n \end{cases}$$

Pf: We use the alternating Čech complex above. For any $e = (e_0, \cdots, e_n) \in \mathbb{Z}^{n+1}$, with $\sum e_i = d$, set:

$$NEG(e) = \{0 \leq i \leq n \mid e_i < 0\}$$
$$POS(e) = \{0, \cdots, n\} \setminus NEG(e)$$

Then we have a subcomplex of $\check{C}_{alt}(\mathcal{U}, \mathcal{O}(d))$:

$$\check{C}(e): 0 \longrightarrow \oplus_{NEG(e) \subseteq \{i_0\}} R X^e \longrightarrow \oplus_{NEG(e) \subseteq \{i_0, i_1\}} R X^e \longrightarrow \cdots$$

and $\check{C}_{alt} = \oplus_{e \in \mathbb{Z}^{n+1}} \check{C}(e)$. $\check{C}(e)$ looks like, say $NEG(e) = \{j_0 < j_1 < \cdots < j_p\}$

$$\check{C}(e): \ 0 \longrightarrow \cdots \longrightarrow 0 \longrightarrow \underset{\deg p}{R X^e} \xrightarrow{\partial} \underset{\deg p+1}{\oplus_{i \in POS(e)} R X^e} \xrightarrow{\partial} \underset{\deg p+2}{\oplus_{i<j \in POS(e)} R X^e} \xrightarrow{\partial} \cdots$$

$$\cdots \xrightarrow{\partial} \underset{\deg n}{R X^e} \longrightarrow 0$$

with differential ∂ induced from the $\check{C}_{alt}$ as a subcomplex. Define M to be the degree $p+1$ part of $\check{C}(e)$: $M \triangleq \oplus_{i \in POS(e)} Rx^e$. A basis of M is given by $b_i = x^e$ in the summand corresponding to $i \in POS(e)$. Denote $m_0 \in M$ $m_0 \triangleq \partial x^e$, where x^e is the generator in deg p.

Claim: If $NEG(e) \neq \phi$ and $POS(e) \neq \phi$, then $\check{C}(e)$ is isomorphic to the Koszul complex:

$$0 \longrightarrow R \longrightarrow M \longrightarrow \Lambda^2 M \longrightarrow \cdots \longrightarrow \Lambda^{\# POS(e)} M \longrightarrow 0$$

with differential $\Lambda^i M \longrightarrow \Lambda^{i+1} M$: $m_1 \wedge \cdots \wedge m_i \longmapsto m_0 \wedge m_1 \wedge \cdots \wedge m_i$

E.g. $n=20$, $NEG(e) = \{0, \cdots, \hat{5}, \cdots, \hat{10}, \cdots 20\}$, $POS(e) = \{5, 10\}$.

$$0 \longrightarrow \underbrace{Rx^e} \xrightarrow{\partial} \underbrace{Rx^e} \oplus \underbrace{Rx^e} \xrightarrow{\partial} \underbrace{Rx^e} \longrightarrow 0$$

Summand index: $\underset{\xi}{\underset{\rotatebox{90}{$\in$}}{(0\cdots\hat{5}\cdots\hat{10}\cdots 20)}}$ $\underset{\eta}{\underset{\rotatebox{90}{$\in$}}{(0\cdots\hat{5}\cdots 20)}}$ $(0\cdots\hat{10}\cdots 20)$ $(0\cdots 20)$

The differentials are :

$$(\partial \xi)_{0\cdots\hat{5}\cdots 20} = (-1)^9 \xi_{0\cdots\hat{5}\cdots\hat{10}\cdots 20}$$

$$(\partial \xi)_{0\cdots\hat{10}\cdots 20} = (-1)^5 \xi_{0\cdots\hat{5}\cdots\hat{10}\cdots 20}$$

$$(\partial \eta)_{0\cdots 20} = (-1)^5 \xi_{0\cdots\hat{5}\cdots 20} + (-1)^{10} \xi_{0\cdots\hat{10}\cdots 20}$$

Thus the complex is isomorphic to:

$$0 \longrightarrow R \xrightarrow{\binom{-1}{-1}} R \oplus R \xrightarrow{\binom{-1}{1}} R \longrightarrow 0$$

which is the Koszul complex.

General Fact: M: free R-module, $m_0 \in M$, part of a basis of M. Then

$$0 \longrightarrow R \xrightarrow{\wedge m_0} M \xrightarrow{\wedge m_0} \Lambda^2 M \xrightarrow{\wedge m_0} \cdots \xrightarrow{\wedge m_0} \Lambda^{\mathrm{rank}} M \longrightarrow 0$$

is an acyclic complex of R-modules.

Hence if both $NEG(e) \neq \phi$ and $POS(e) \neq \phi$, the complex $\check{C}(e)$ contributes nothing to cohomology.

If $NEG(e) = \phi$, $POS(e) = \{0, \cdots, n\}$, this gives

$$\check{C}(e): 0 \longrightarrow \oplus_{i_0} Rx^e \longrightarrow \oplus_{i_0 < i_1} Rx^e \longrightarrow \cdots$$

contributing Rx^e to $H^0(\mathbb{P}^n_R, \mathcal{O}(d))$.

If $POS(e)=\phi$, $NEG(e)=\{0,\cdots,n\}$, this gives

$$0 \longrightarrow Rx^e \longrightarrow 0$$

contributing Rx^e to $H^n(\mathbb{P}^n_R, \mathcal{O}(d))$. $\square$

Cor. $H^j(\mathbb{P}^n_R, \mathcal{O}(d))$ is a finite free R-module $\square$

Our goal is to prove that:

If S is a locally Noetherian scheme and $f: X \longrightarrow S$ is a proper morphism. $\mathcal{F}$: a coherent $\mathcal{O}_X$-module. Then $R^if_*\mathcal{F}$ are all coherent. In particular, this says that if X is a proper variety$/k$ and $\mathcal{F} \in Coh(X)$, then $\dim_k H^i(X, \mathcal{F}) < \infty$, $\forall i$.

Def: Let S be a locally Noetherian scheme. A Q.Coh. $\mathcal{O}_X$-module $\mathcal{F}$ is called coherent iff $\forall$ affine open $U = \operatorname{Spec} R \subseteq S$, $\mathcal{F}|_U = \widetilde{M}$ with M a finite R-module.

Facts about coherent sheaves:

(1). It's enough to check the conditions in the def. just for an affine open cover of S.

(2). Kernels and cokernel of maps of coherent $\mathcal{O}_S$-modules are coherent.

(3). In a s.e.s. $0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$, if two out of three are coherent, so is the third.

Lemma. Let R be a ring, $n \geq 1$, $\mathcal{F}$ Q.Coh on $\mathbb{P}^n_R$. Then $\mathcal{F}$ is a quotient of $\bigoplus_{a\in A} \mathcal{O}(d_a)$ for some index set A and integers $d_a \in \mathbb{Z}$, $\forall a \in A$.

Pf: $\mathcal{F} = \widetilde{M}$ for some graded $S = R[x_0, \cdots, x_n]$-module. Pick a surjection $\bigoplus_{a\in A} S(d_a) \twoheadrightarrow M$ with $S(d_a)$ the shifted graded S-module. Applying the $\sim$ functor gives $\bigoplus_{a\in A} \mathcal{O}(d_a) \twoheadrightarrow \mathcal{F}$. $\square$

Lemma: Let R be a Noetherian ring, $n \geq 1$ and $\mathcal{F}$ a coherent sheaf on $\mathbb{P}^n_R$. Then $\exists\, t > 0$, $d_1, \cdots, d_t \in \mathbb{Z}$ with a surjection:

$$\oplus_{i=1}^{t} \mathcal{O}(d_i) \twoheadrightarrow \mathcal{F}.$$

Pf: This follows from the previous lemma and the next. □

Lemma. S: Noetherian scheme, $\mathcal{F}$: coherent $\mathcal{O}_S$-module and A any set and $\mathcal{G}_a$: Q.Coh $\mathcal{O}_S$-module, with

$$\oplus_{a \in A} \mathcal{G}_a \twoheadrightarrow \mathcal{F}$$

Then $\exists\, A' \subseteq A$ finite subset s.t.

$$\oplus_{a \in A'} \mathcal{G}_a \twoheadrightarrow \mathcal{F}$$ □

Cor. R: Noetherian, $n \geq 1$, $\mathcal{F}$: coherent on $\mathbb{P}^n_R$. Then $\forall\, i$, $H^i(\mathbb{P}^n_R, \mathcal{F})$ is a finite R-module.

Pf: Induction backwards. We know that $H^i(X, \mathcal{F}) = 0$ for $i > n$. Suppose the result true for any coherent sheaf for $i \geq k+1$. Then for any $\mathcal{F}$ coherent, pick a surjection: $\oplus_{i=0}^{t} \mathcal{O}(d_i) \twoheadrightarrow \mathcal{F}$. Then we have a s.e.s:

$$0 \longrightarrow \mathcal{K} \longrightarrow \oplus_{i=0}^{t} \mathcal{O}(d_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

and $\mathcal{K}$ is coherent as well. By the l.e.s.

$$\cdots \longrightarrow \underbrace{H^k(\mathbb{P}^n_R, \oplus_{i=0}^{t} \mathcal{O}(d_i))}_{\text{finite } R\text{-mod}} \longrightarrow H^k(\mathbb{P}^n_R, \mathcal{F}) \longrightarrow \underbrace{H^{k+1}(\mathbb{P}^n_R, \mathcal{K})}_{\text{finite } R\text{-mod}} \longrightarrow \cdots$$

$\Longrightarrow H^k(\mathbb{P}^n_R, \mathcal{F})$ is a finite R-module. □

Lemma. Let $f: X \longrightarrow Y$ be an affine morphism of schemes i.e. $\forall\, V \subseteq Y$ affine open in Y, $f^{-1}(V)$ is affine in X. (e.g. vector bundles over Y / closed immersions / finite morphisms). Then, for any Q. Coh. $\mathcal{O}_X$-module $\mathcal{F}$:

(a). $R^q f_* \mathcal{F} = 0$, $\forall\, q > 0$

(b). $H^p(X, \mathcal{F}) = H^p(Y, f_* \mathcal{F})$

Pf: (a) follows from the fact that $R^q f_* \mathcal{F}$ is the sheaf associated to the presheaf $V \longmapsto H^q(f^{-1}(V), \mathcal{F}|_{f^{-1}(V)})$, which is 0 for any affine open V

since Q.Coh. sheaves on affines have vanishing higher cohomology.
(b). follows from (a) and the following lemma. □

Lemma. $f: X \longrightarrow Y$: a morphism of ringed spaces. $\mathcal{F}$: an $\mathcal{O}_X$-module, with $R^q f_* \mathcal{F} = 0$, $\forall q > 0$, then:

$$H^p(X, \mathcal{F}) = H^p(Y, f_* \mathcal{F})$$

Pf: Pick an injective resolution $\mathcal{F} \longrightarrow I^\bullet$. By assumption, $f_* \mathcal{F}[0] \longrightarrow f_* I^\bullet$ is a qis. By previous results, each term in $f_* I^\bullet$ is $\Gamma(Y, -)$-acyclic. By Leray's acyclicity lemma, we can compute cohomology of $f_* \mathcal{F}$ by the complex $f_* I^\bullet$, since $\Gamma(Y, f_* I^\bullet)$ is qis to $R\Gamma(Y, f_* \mathcal{F})$. □

Rmk: The correct proof uses Leray's spectral sequence.

E.g. Let $C: F(X,Y,Z) = 0 \subseteq \mathbb{P}^2_k$, where $F(X,Y,Z)$ is homogeneous of degree d. $(d > 0)$. What's $H^i(C, \mathcal{O}_C)$?

By previous lemmas, $H^i(C, \mathcal{O}_C) = H^i(\mathbb{P}^2_k, i_* \mathcal{O}_C)$, where C is an effective Cartier divisor $\Rightarrow \mathcal{I}_C$ is an invertible sheaf.

Claim:
$$\begin{tikzcd}
\mathcal{O}_{\mathbb{P}^2_k} \arrow[r, "\cdot 1"] \arrow[dr, "\cdot F"'] & \mathcal{O}_{\mathbb{P}^2_k}(C) \cong \mathcal{H}om_{\mathcal{O}_{\mathbb{P}^2_k}}(\mathcal{I}_C, \mathcal{O}_{\mathbb{P}^2_k}) \arrow[d, "\alpha", "\wr"'] \\
& \mathcal{O}_{\mathbb{P}^2_k}(d)
\end{tikzcd}$$

α is an isomorphism of $\mathcal{O}_{\mathbb{P}^2_k}$-modules

Thus $\mathcal{I}_C \cong \mathcal{O}_{\mathbb{P}^2_k}(-d)$ and we have a s.e.s. of sheaves:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2_k}(-d) \longrightarrow \mathcal{O}_{\mathbb{P}^2_k} \longrightarrow \mathcal{O}_C \longrightarrow 0$$

$$\Rightarrow \quad \begin{array}{cccccccc}
0 \longrightarrow & 0 & \longrightarrow & k & \longrightarrow & H^0(C, \mathcal{O}_C) \\
\longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & H^1(C, \mathcal{O}_C) \\
\longrightarrow & H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) & \longrightarrow & 0 & \longrightarrow & H^2(C, \mathcal{O}_C) \\
\longrightarrow & 0 & & & &
\end{array}$$

By our previous calculation, $\dim H^2(\mathbb{P}^2_k, \mathcal{O}_{\mathbb{P}^2_k}(-d)) = \frac{1}{2}(d-1)(d-2)$. It follows that $\dim H^0(C, \mathcal{O}_C) = 1$ and $\dim H^1(C, \mathcal{O}_C) = \frac{1}{2}(d-1)(d-2)$.

As a corollary, we see that if C_1 and C_2 are two such curves having degrees $d_1 \neq d_2$, then $C_1 \ncong C_2$ unless both $d_1, d_2 \in \{1,2\}$.

Def. A morphism $f: X \longrightarrow S$ is called locally projective iff $\forall s \in S$, $\exists V \ni s$ open neighborhood s.t. $f|_{f^{-1}V}: f^{-1}(V) \longrightarrow V$ is H-projective. i.e.

$$\begin{tikzcd} X \supseteq f^{-1}(V) \arrow[rr, hook] \arrow[dr] & & \mathbb{P}^n_V \arrow[dl] \\ & V & \end{tikzcd} \quad (\text{closed immersion})$$

Lemma. Let $f: X \longrightarrow \operatorname{Spec} R$ be quasi-compact and quasi-seperated, and $\mathcal{F}$ Q. Coh on X. Then:

(i). $H^i(X, \mathcal{F}) = \Gamma(\operatorname{Spec} R, R^i f_* \mathcal{F})$

(ii). $R^i f_* \mathcal{F} = H^i(X, \mathcal{F})^\sim$.

Pf: From a previous thm we know that $R^i f_* \mathcal{F}$ is Q.Coh on $\operatorname{Spec} R$. Thus (i) $\Rightarrow$ (ii). For (i), note that since $R^i f_* \mathcal{F}$ is Q.Coh on $\operatorname{Spec} R$, and by affineness, we have $H^j(\operatorname{Spec} R, R^i f_* \mathcal{F}) = 0$, $\forall j > 0$. Thus (i) follows from:

Lemma. $f: X \longrightarrow Y$ morphism of ringed spaces. $\mathcal{F}$: an $\mathcal{O}_X$-module s.t. $H^j(Y, R^i f_* \mathcal{F}) = 0$ for all $j>0$ and all i. Then:

$$H^i(X, \mathcal{F}) = H^0(Y, R^i f_* \mathcal{F})$$

Pf: (This follows directly from Leray's spectral sequence:

$$H^p(Y, R^q f_* \mathcal{F}) \Longrightarrow H^{p+q}(X, \mathcal{F}). \quad)$$

Pick an injective resolution $\mathcal{F} \longrightarrow I^\bullet$ on X. Then since I^n is injective, we know that $f_* I^n$ has $H^j(Y, f_* I^n) = 0$ for all $j > 0$. Split the complex $f_* I^\bullet$ into short exact sequences:

$$\begin{tikzcd}[column sep=small] 0 \arrow[r] & f_* I^0 \arrow[r] \arrow[dr] & f_* I^1 \arrow[r] \arrow[dr] & f_* I^2 \arrow[r] \arrow[dr] & f_* I^3 \arrow[r] \arrow[dr] & \cdots \\ & & \mathcal{B}^1 \subseteq \mathcal{Z}^1 \arrow[u] & \mathcal{B}^2 \subseteq \mathcal{Z}^2 \arrow[u] & \mathcal{B}^3 \subseteq \mathcal{Z}^3 \arrow[u] & \mathcal{B}^4 \cdots \end{tikzcd}$$

we have:

$$
\begin{aligned}
&1)\quad 0 \longrightarrow f_*\mathcal{F} \longrightarrow f_*I^0 \longrightarrow \mathcal{B}^1 \longrightarrow 0\\
&2)\quad 0 \longrightarrow \mathcal{B}^1 \longrightarrow \mathcal{Z}^1 \longrightarrow R^1f_*\mathcal{F} \longrightarrow 0\\
&3)\quad 0 \longrightarrow \mathcal{Z}^1 \longrightarrow f_*I^1 \longrightarrow \mathcal{B}^2 \longrightarrow 0\\
&4)\quad 0 \longrightarrow \mathcal{B}^2 \longrightarrow \mathcal{Z}^2 \longrightarrow R^2f_*\mathcal{F} \longrightarrow 0\\
&\qquad \cdots
\end{aligned}
$$

Then 1) $\Rightarrow \mathcal{B}^1$ is acyclic. 2) $\Rightarrow \mathcal{Z}^1$ is acyclic. 3) $\Longrightarrow \mathcal{B}^2$ is acyclic $\cdots$
$\Longrightarrow$ all $\mathcal{B}^i, \mathcal{Z}^i$ are acyclic. Now taking global sections gives:

$$
\begin{aligned}
&0 \longrightarrow \underset{\Gamma(X,\mathcal{F})}{\underset{\parallel}{\Gamma(Y, f_*\mathcal{F})}} \longrightarrow \underset{\Gamma(X,I^0)}{\underset{\parallel}{\Gamma(Y, f_*I^0)}} \longrightarrow \Gamma(Y,\mathcal{B}^1)\\
&0 \longrightarrow \Gamma(Y,\mathcal{B}^1) \longrightarrow \Gamma(Y,\mathcal{Z}^1) \longrightarrow \Gamma(Y, R^1f_*\mathcal{F}) \longrightarrow 0\\
&0 \longrightarrow \Gamma(Y,\mathcal{Z}^1) \longrightarrow \Gamma(X,I^1) \longrightarrow \Gamma(Y,\mathcal{B}^2) \longrightarrow 0\\
&0 \longrightarrow \Gamma(Y,\mathcal{B}^2) \longrightarrow \Gamma(Y,\mathcal{Z}^2) \longrightarrow \Gamma(Y, R^2f_*\mathcal{F}) \longrightarrow 0\\
&\cdots\cdots
\end{aligned}
$$

which gives the l. e. s.

$$
\begin{tikzcd}
 & & & 0 \arrow[d] & & & & \\
0 \arrow[r] & \Gamma(X,\mathcal{F}) \arrow[r] & \Gamma(X,I^0) \arrow[r] \arrow[drr, "d^0"] & \Gamma(Y,\mathcal{B}^1) \arrow[d] & & 0 \arrow[d] & & \\
 & & 0 \arrow[r] & \Gamma(Y,\mathcal{Z}^1) \arrow[r] \arrow[d] & \Gamma(X,I^1) \arrow[r] \arrow[drr, "d^1"] & \Gamma(Y,\mathcal{B}^2) \arrow[r] \arrow[d] & 0 & \\
 & & & \Gamma(Y,R^1f_*\mathcal{F}) \arrow[d] & 0 \arrow[r] & \Gamma(Y,\mathcal{Z}^2) \arrow[r] \arrow[d] & \Gamma(X,I^2) & \cdots \\
 & & & 0 & & \Gamma(Y,R^2f_*\mathcal{F}) \arrow[d] & & \\
 & & & & & 0 & &
\end{tikzcd}
$$

$\Longrightarrow H^i(X,\mathcal{F}) = \ker d^i / \operatorname{Im} d^{i-1} = \Gamma(Y,\mathcal{Z}^i)/\Gamma(Y,\mathcal{B}^i) \cong \Gamma(Y, R^if_*\mathcal{F})$. $\square$

Thm. S: Noetherian. $f: X \longrightarrow S$ locally projective, $\mathcal{F}$ coherent on X. Then $\mathcal{F}$ coherent $\Rightarrow R^if_*\mathcal{F}$ is coherent.

Pf: The problem is local on S. Thus we may assume that $S = \operatorname{Spec} R$, with

R Noetherian and:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_R \arrow[d, "\pi"] \\ & \operatorname{Spec} R \end{tikzcd} \quad : \text{ closed immersion}$$

Step 2. $R^q i_* \mathcal{F} = 0 \;\; \forall q > 0$, since i is affine. Hence

$$H^q(X, \mathcal{F}) = H^q(\mathbb{P}^n_R, i_*\mathcal{F}),$$

by a previous lemma.

Step 3. Combining the previous lemma that $R^q f_* \mathcal{F} = H^q(X, \mathcal{F})^\sim$, with the finiteness of cohomology of coherent sheaves on $\mathbb{P}^n_R$, we have:

$$R^q f_* \mathcal{F} = H^q(X, \mathcal{F})^\sim = H^q(\mathbb{P}^n_R, i_*\mathcal{F})^\sim$$

we obtain the desired result. $\square$

Now to reach our goal:

$$f: X \longrightarrow S \text{ proper}, \quad S \text{ Noetherian}, \quad \mathcal{F} \in \operatorname{Coh}(X)$$
$$\Longrightarrow R^q f_* \mathcal{F} \in \operatorname{Coh}(X).$$

We need to use Chow's lemma: $\exists$ surjective, birational, proper morphism π:

$$\begin{tikzcd} X' \arrow[dr, "\pi"] \arrow[dd, "\text{H-proj.}"'] & \\ & X \arrow[dl] \\ S & \end{tikzcd}$$

We need to understand how cohomology changes under π. Let:

(*) S: Noetherian, $f: X \longrightarrow S$ be proper

(**): $\mathcal{F} \in \operatorname{Coh}(X) \Longrightarrow R^q f_* \mathcal{F} \in \operatorname{Coh}(S)$, $\forall q$.

Lemma. In situation (*), suppose $\mathcal{F}_i \in \operatorname{Coh}(X)$ and

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is s.e.. If (**) holds for any 2 out of 3, then it holds for the third.

Pf: We know that each $R^qf_*\mathcal{F}_i$ is QCoh. By the l.e.s.

$$\cdots \longrightarrow R^qf_*\mathcal{F}_1 \longrightarrow R^qf_*\mathcal{F}_2 \longrightarrow R^qf_*\mathcal{F}_3 \longrightarrow \cdots$$

the result follows. $\square$

Notation. In situation $(*)$, set $Coh(X)$ the category of coherent $\mathcal{O}_X$-modules on X, which is an abelian category.

Let $C_f \triangleq \{\mathcal{F} \in Coh(X) \mid (**) \text{ is true}\}$. We shall show that $C_f = Coh(X)$.

Lemma. Let $C \subseteq Coh(X)$ be a subclass of objects s.t.

(a). In any s.e.s. of $Coh(X)$, 2 out of 3 is in $C \Rightarrow$ so is the 3^{rd}.

(b). $\forall\ 0 \neq \mathcal{F} \in Coh(X)$, $\exists\ \alpha: \mathcal{F}' \longrightarrow \mathcal{F}$, with $\mathcal{F}' \in C$, and

$$\mathrm{supp}(\ker\alpha) \cup \mathrm{supp}(\mathrm{coker}\alpha) \subsetneq \mathrm{supp}\mathcal{F}$$

Then $C = Coh(X)$.

Pf: By Noetherian induction. Recall that supp. of a coherent sheaf is always closed. Consider $\mathcal{T} = \{Z \subseteq X \mid Z = \mathrm{Supp}\mathcal{F},\ \mathcal{F} \notin C\}$. We need to show that $\mathcal{T}$ is empty. Otherwise, we can take $Z \in \mathcal{T}$ minimal. Write $Z = \mathrm{Supp}\mathcal{F}$ with $\mathcal{F} \notin Coh(X)$. Apply (b). $\exists\ \alpha: \mathcal{F}' \longrightarrow \mathcal{F}$ with the condition on supports. By minimality of Z, $\ker\alpha$ and $\mathrm{coker}\alpha \in C$. Then

$$0 \longrightarrow \ker\alpha \longrightarrow \mathcal{F}' \longrightarrow \mathrm{im}\alpha \longrightarrow 0 \overset{(a)}{\Longrightarrow} \mathrm{Im}\alpha \in C.$$

$$0 \longrightarrow \mathrm{im}\alpha \longrightarrow \mathcal{F} \longrightarrow \mathrm{coker}\alpha \longrightarrow 0 \overset{(a)}{\Longrightarrow} \mathcal{F} \in C.$$

Contradiction. $\square$

A variant of the lemma:

Lemma': The same condition as before, with (b) replaced by:

(b'): For any $\mathcal{F} \in Coh(X)$, with $\mathrm{supp}\mathcal{F}$ irreducible with generic point ξ, and $\mathcal{F}_\xi \cong \kappa(\xi)$ as $\mathcal{O}_{X,\xi}$-modules, there exists an $\mathcal{F}' \in C$ with $\mathcal{F}' \xrightarrow{\alpha} \mathcal{F}$, and

$$\mathrm{supp}(\ker\alpha) \cup \mathrm{supp}(\mathrm{coker}\alpha) \subsetneq \mathrm{supp}\mathcal{F}.$$

Then the same condition holds as in the previous lemma.

Before proving this, we need a few lemmas.

Lemma. X: Noetherian scheme, $\mathscr{F} \in Coh(X)$. $U \subseteq X$ open and $\mathscr{G} \subseteq \mathscr{F}|_U$ is a quasi-coherent subsheaf (coherent then). Then $\exists$ coherent subsheaf $\mathscr{F}'$ of $\mathscr{F}$ s.t. $\mathscr{F}'|_U = \mathscr{G}$ as subsheaves of $\mathscr{F}|_U$.

Pf: $j: U \hookrightarrow X$ is q.c. and q.s. $\Rightarrow$ $j_*\mathscr{G}$ is Q.Coh on X and is a subsheaf of $j_*\mathscr{F}|_U = j_*j^*\mathscr{F}$. We have:

$$\begin{tikzcd} \mathscr{F}' \arrow[r, hook, dashed] \arrow[d] & \mathscr{F} \arrow[d] \\ j_*\mathscr{G} \arrow[r, hook] & j_*j^*\mathscr{F} \end{tikzcd}$$

Thus taking the fiber product ($\mathscr{F}' = \ker(\mathscr{F} \oplus j_*\mathscr{G} \longrightarrow j_*j^*\mathscr{F})$) in the (abelian) category $QCoh(X)$, $\mathscr{F}'$ is then a Q.Coh subsheaf of $\mathscr{F}$ (thus coherent) and $j^*\mathscr{F}' = \mathscr{G}$. $\square$

Rmk: This result generalizes to X quasi-compact and quasi-seperated, then we can obtain $\mathscr{F}'$ Q.Coh subsheaf of $\mathscr{F}$.

Recall the result from algebra:

R: Noetherian, M: finite R-module. Then there exists a filtration:

$$0 \subsetneq M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_n = M$$

s.t. $M_i / M_{i-1} \cong R/\mathfrak{p}_i$ for some prime $\mathfrak{p}_i$.

We shall globalize this result to Noetherian schemes.

Def. Let Z be an integral scheme with generic point $\xi \in Z$, and $\mathscr{F}$ be a Q.Coh $\mathcal{O}_Z$-module.

(1). $\mathscr{F}$ is torsion free on Z iff $\mathscr{F}$ corresponds to a torsion free module on every affine open of Z.

(2). $\operatorname{rank} \mathscr{F} \triangleq \dim_{\kappa(\xi)} \mathscr{F}_\xi$ (recall that $\mathcal{O}_{Z,\xi} = \kappa(\xi)$).

Lemma. X: Noetherian scheme. $\mathcal{F}$ coherent on X. $\xi \in \operatorname{Supp}\mathcal{F}$ is a generic point (of an irred. component of $\operatorname{supp}\mathcal{F}$). Then $\exists$ a s.e.s.

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}_2 \longrightarrow 0$$

s.t. $\mathcal{F}_{1,\xi} = 0$, $\mathcal{F}_{2,\xi} = \mathcal{F}_\xi$ and $\operatorname{supp}\mathcal{F}_2 = \overline{\{\xi\}}$.

Pf: Let $Z = \overline{\{\xi\}}$. $\mathcal{I}_Z \subseteq \mathcal{O}_X$ (with the reduced induced structure). Set $G_N \triangleq \mathcal{I}_Z^N \mathcal{F}$. Since $\xi \in Z$ is a generic point, and Z is an irreducible component of $\operatorname{Supp}\mathcal{F}$, $\mathcal{F}$ is of finite length over $\mathcal{O}_{X,\xi}$, and $\mathcal{I}_\xi = m_\xi \subseteq \mathcal{O}_{X,\xi} \Rightarrow \mathcal{I}_\xi^N \mathcal{F}_\xi = 0$ for $N \gg 0$. Set $\mathcal{F}_1 = G_N$ and $\mathcal{F}_2 = \mathcal{F}/G_N$. Then $\mathcal{I}^N \mathcal{F}_2 = 0 \Rightarrow \forall x \notin Z$, $(\mathcal{I}^N \mathcal{F}_2)_x = \mathcal{F}_{2x} = 0$. Thus $\operatorname{supp}\mathcal{F} = \overline{\{\xi\}}$ and $\mathcal{F}_{1,\xi} = 0$.

$\square$

Lemma. X: Noetherian scheme, $\mathcal{F} \in \operatorname{Coh}(X)$. Then there exists a filtration:

$$0 \subsetneq \mathcal{F}_0 \subsetneq \cdots \subsetneq \mathcal{F}_n \subsetneq \mathcal{F}$$

s.t. $\mathcal{F}_i / \mathcal{F}_{i-1} \cong (Z_i \hookrightarrow X)_* \mathcal{L}_i$, where Z_i is integral, closed and $\mathcal{L}_i$ a coherent rank 1 $\mathcal{O}_{Z_i}$-module. Furthermore, $\mathcal{L}_i$ can be taken to be torsion free $\mathcal{O}_{Z_i}$-module.

Question: Can we make $\mathcal{L}_i$ locally free? For quasi-projective it's true. In general, not known.

Pf: By Noetherian induction. Set $\mathcal{T} = \{Z \subseteq X \mid$ closed subset s.t. $\exists \mathcal{F}$ with $\operatorname{supp}\mathcal{F} = Z$ s.t. the lemma is false for $\mathcal{F}\}$. If $\mathcal{T} \neq \phi$, pick Z minimal such, with $\operatorname{supp}\mathcal{F} = Z$. Z minimal $\Rightarrow$ Z must be irreducible. Otherwise pick ξ_1 the generic point of an irred. component. By the previous lemma, $\exists \mathcal{F}_1, \mathcal{F}_2$ with strictly smaller support, then $\mathcal{F}_1, \mathcal{F}_2$ have the required filtrations $\Rightarrow$ so does $\mathcal{F}$, contradiction.

Pick $U \subseteq X$ affine open s.t. $Z \cap U \neq \phi$. By our algebraic lemma quoted above, $\exists$ filtration: $0 \subseteq G_1 \subseteq G_2 \subseteq \cdots \subseteq G_r = \mathcal{F}|_U$, with $G_i/G_{i-1} \cong \mathcal{O}_{T_i}$ for $T_i \subseteq U$ integral closed. By a previous lemma, $\exists$ filtration:

$$0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \cdots \subseteq \mathcal{F}_r = \mathcal{F}$$

with $\mathcal{F}_i|_U = G_i$. Thus $\operatorname{supp}(\mathcal{F}_i/\mathcal{F}_{i-1}) \cap U = T_i = \overline{\{\xi_i\}}$. Apply the previous lemma to the sheaves $\mathcal{F}_i/\mathcal{F}_{i-1}$ and the point ξ_i, we obtain:

$$0 \longrightarrow K_i \longrightarrow \mathcal{F}_i/\mathcal{F}_{i-1} \longrightarrow Q_i \longrightarrow 0$$

with $\operatorname{supp} Q_i = Z_i = \overline{\{\xi_i\}}$ and $K_{i,\xi_i} = 0$, $Q_{i,\xi_i} = (\mathcal{F}_i/\mathcal{F}_{i-1})_{\xi_i}$. Now if $\xi_i = \xi$, we leave Q_i unchanged, and K_i then has strictly smaller support. By induction, K_i has a required filtration and we may take the preimage of this filtration in $\mathcal{F}_i$ to obtain a filtration between $\mathcal{F}_{i-1}$ and $\mathcal{F}_i$ with required condition. If $\xi_i \neq \xi$, both K_i and Q_i have strictly smaller support and again we may enlarge the filtration between $\mathcal{F}_{i-1}$ and $\mathcal{F}_i$. Furthermore, for this enlarged filtration, we may further modify it so that $\mathcal{F}_i'/\mathcal{F}_{i-1}'$ is an $\mathcal{O}_{Z_i}$-torsion free module. (for instance take the preimage of $I^n_{Z_i}(\mathcal{F}_i'/\mathcal{F}_{i-1}')$). Now we obtain a filtration of $\mathcal{F}$ with the required conditions, contradiction to our choice of Z. $\square$

Now we can prove the variant lemma.

Lemma' Let C be a subclass of $Coh(X)$ s.t.

(a). In any s.e.s. of C, 2 out of 3 are in $C \Rightarrow$ so is the 3rd.

(b). For any $\mathcal{F} \in Coh(X)$, with $\mathcal{F} = (Z \hookrightarrow X)_* \mathcal{L}$, with $\mathcal{L}$ a rank 1 (torsion free) $\mathcal{O}_Z$-module, $\exists\, \alpha: \mathcal{F} \longrightarrow \mathcal{F}'$ s.t. $\mathcal{F}' \in C$ and

$$\operatorname{supp}(\ker\alpha) \cup \operatorname{supp}(\operatorname{coker}\alpha) \subsetneq Z$$

Then $C = Coh(X)$.

Pf: By Noetherian induction.

Filter $\mathcal{F} \in Coh(X)$:

$$0 \subsetneq \mathcal{F}_1 \subsetneq \mathcal{F}_2 \subsetneq \cdots \subsetneq \mathcal{F}_n = \mathcal{F}$$

with each $\mathcal{F}_i/\mathcal{F}_{i-1} \cong (Z_i \hookrightarrow X)_* \mathcal{L}_i$ for some rank 1 (torsion free) sheaf of $\mathcal{O}_{Z_i}$-module. By (b), $\exists\, \mathcal{F}' \in C$ with $\mathcal{F} \xrightarrow{\alpha} \mathcal{F}'$ and

$$\operatorname{supp}(\ker\alpha) \cup \operatorname{supp}(\operatorname{coker}\alpha) \subsetneq Z$$

Thus by induction, $\ker\alpha$, $\operatorname{coker}\alpha \in C$. Then $\operatorname{coker}\alpha \in C$, $\mathcal{F}' \in C$ with (a) $\Rightarrow$ $\mathcal{F}/\ker\alpha \in C$. With $\ker\alpha \in C$ and (a) again, $\mathcal{F} \in C$. $\square$

To reach our goal, we need to show that:

S: Noetherian affine, $f: X \longrightarrow S$ proper. Let:

$$\mathcal{C}_f \triangleq \{\mathcal{F} \in Coh(X) \mid R^q f_* \mathcal{F} \in Coh(S),\ \forall q\}$$

Then $\mathcal{C}_f = Coh(X)$. By a previous lemma, we have "2 out of 3" rule applies to $\mathcal{C}_f$. Thus it suffices to check condition (b) of lemma', i.e.

Given

$$\begin{tikzcd} Z \arrow[r, hook, "i"] \arrow[dr, "f|_Z"'] & X \arrow[d, "f"] \\ & S \end{tikzcd},$$

where Z is integral closed subscheme of X and $\mathcal{F} = i_* \mathcal{L}$, and $\mathcal{L} \in Coh(Z)$, of rank 1. Then we can find $\alpha: \mathcal{F} \longrightarrow \mathcal{F}'$ s.t.

$$(*) \begin{cases} (1). \ \mathcal{F}' \text{ has } R^q f_* \mathcal{F}' \in Coh(S) \text{ for all } q \\ (2). \ \mathrm{Supp}\ker\alpha \cup \mathrm{Supp}\,\mathrm{coker}\,\alpha \subsetneq Z \end{cases}$$

Apply Chow's lemma to $Z \longrightarrow S$, we get a diagram:

$$\begin{tikzcd} \mathbb{P}^n_S \arrow[ddrr, "g"] & Z' \arrow[l, hook', "i'"'] \arrow[dr, "\pi"] \arrow[ddr, "f'"] & & \\ & & Z \arrow[r, hook, "i"] \arrow[d, "f|_Z"] & X \arrow[dl, "f"] \\ & & S & \end{tikzcd}$$

where π is proper birational (H-projective), and i' is a closed immersion (f' is H-projective). Set $\mathcal{L}' = \pi^* \mathcal{L}$, and $\mathcal{O}_{Z'}(1) = i'^* \mathcal{O}_{\mathbb{P}^n_S}(1)$, the very ample invertible sheaf on Z'.

Claim: For some $d \gg 0$, we have:

(a). $R^q \pi_* \mathcal{L}'(d) = 0 \quad \forall q \geq 1$

(b). $\exists\, \beta: \mathcal{L}' \longrightarrow \mathcal{L}'(d)$, which is an isomorphism at the generic point of Z'.

(This is only done when S is affine).

The $(*)$ condition follows from this claim. Indeed, set $\mathcal{F}' = i_* \pi_* (\mathcal{L}'(d))$ we have:

$$\begin{tikzcd} \mathcal{F} = i_* \mathcal{L} \arrow[r] \arrow[rrr, bend right, "\alpha"'] & i_* \pi_* \pi^* \mathcal{L} = i_* \pi_* \mathcal{L}' \arrow[r, "i_* \pi_* (\beta)"] & i_* \pi_* \mathcal{L}'(d) & \triangleq \mathcal{F}' \end{tikzcd}$$

Conclusion $(*).(2)$ follows because π is birational and β is an isomorphism.
To show $(*).(1)$, we need to use the following result:

- $R^q i_* = 0 \ \forall q > 1$ (since i is a closed immersion) $\Longrightarrow$
$$R^p f_*(\underbrace{i_*\pi_*(\mathcal{L}'(d))}_{\mathcal{F}'}) = R(\underbrace{f\circ i}_{f|_Z})_*(\pi_*(\mathcal{L}'(d)))$$
- By (a). above $\Longrightarrow R^p(f|_Z)_*(\pi_*\mathcal{L}'(d)) = R^p(f')_*(\mathcal{L}'(d))$.

(These follow in general from spectral sequences, by in our case it's proved in the lemma below).

Thus $\forall q.$ $R^q f_*\mathcal{F}' = R^q(f|_Z)_*(\pi_*(\mathcal{L}'(d))) = R^q f'_*(\mathcal{L}'(d))$, which is coherent since f' is projective and $\mathcal{L}'(d)$ is coherent. $\square$

Lemma. $X \xrightarrow{f} Y \xrightarrow{g} Z$: morphism of ringed spaces. $\mathcal{F}$: an $\mathcal{O}_X$-module. Suppose $R^q f_*\mathcal{F} = 0$, $\forall q \geq 1$. Then:
$$R^p g_*(f_*\mathcal{F}) = R^p(g\circ f)_*(\mathcal{F})$$
Pf: Take an injective resolution $\mathcal{F} \longrightarrow I^\bullet$ in $\mathcal{M}od(\mathcal{O}_X)$. Then
$$0 = R^q f_*\mathcal{F} = h^q(f_*I^\bullet) \Longrightarrow f_*I^\bullet \text{ is a resolution of } f_*\mathcal{F}.$$
More over, since $R^q f_* I^\bullet$ is the sheaf associated with the presheaf
$$V \longmapsto H^q(g^{-1}(V), f_*I^\bullet) = 0$$
$f_*I^\bullet$ is g_*-acyclic. By Leray's acyclicity lemma, we can compute $R^p g_*(f_*\mathcal{F})$ as $h^p(g_*f_*I^\bullet) = h^p((g\circ f)_*I^\bullet) = R^p(g\circ f)_*(\mathcal{F})$. $\square$

Claims (a) and (b) follow from:
Lemma. Given
$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr, "f"'] & \mathbb{P}^n_S \arrow[d] \\ & S \end{tikzcd}$$
where i is a closed immersion and S is noetherian. Set $\mathcal{O}_X(1) = i^*\mathcal{O}_{\mathbb{P}^n_S}(1)$.
For any $\mathcal{F} \in Coh(X)$, $\exists d(\mathcal{F}) \gg 0$ s.t. $R^q f_*(\mathcal{F}(d)) = 0$, $\forall q \geq 1$, $d \geq d(\mathcal{F})$. If S is affine, then for all $d \geq d(\mathcal{F})$, the sheaf $\mathcal{F}(d)$ is generated by

global sections.

Rmk: The claims follow since, if

$$\begin{tikzcd} \mathbb{P}^n_S \arrow[dr] & Z' \arrow[l, "i"'] \arrow[r, "\pi"] \arrow[d] & Z \arrow[dl] \\ & S & \end{tikzcd}$$

with Z projective over S and taking the base change we get:

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[dr, "f"'] & \mathbb{P}^n_S \times_S Z = \mathbb{P}^n_Z \arrow[d, "\pi'"] \\ & Z \end{tikzcd}$$

where i' is a closed immersion since f is proper and π' is seperated. Then f is H-projective and $\mathcal{O}_{Z'}(1) = i'^*\mathcal{O}_{\mathbb{P}^n_Z}(1)$.

E.g. The second conclusion fails when S is not affine, in which case $\mathcal{O}_X(1)$ is only relatively ample. For instance, $S=\mathbb{P}^1_k$, $X=\mathbb{P}^1_k\times\mathbb{P}^2_k$ with i the identity:

$$\begin{tikzcd} X = \mathbb{P}^1_k \times \mathbb{P}^2_k \arrow[r, "id"] \arrow[dr, "f"'] & \mathbb{P}^1_k \times \mathbb{P}^2_k = \mathbb{P}^2_{\mathbb{P}^1_k} \arrow[d] \\ & \mathbb{P}^1_k \end{tikzcd}$$

It's known that $\mathrm{Pic}(X) \cong \mathbb{Z}\times\mathbb{Z}$. Pick $\mathcal{F} = \mathcal{O}_X(-2,0) = f^*\mathcal{O}_{\mathbb{P}^1}(-2)$. Then the relative ample sheaf "$\mathcal{O}_X(1)$" $= \mathcal{O}_X(0,1)$. Thus $\mathcal{F}(d) = \mathcal{O}_X(-2,d)$. Hence:

$$\begin{aligned} H^1(X, \mathcal{F}(d)) &= H^1(\mathbb{P}^1\times\mathbb{P}^2, \pi_1^*\mathcal{O}_{\mathbb{P}^1}(-2)\otimes\pi_2^*\mathcal{O}_{\mathbb{P}^2}(d)) \\ &= H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))\otimes H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \oplus H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2))\otimes H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(d)) \\ &\cong k \otimes (k[x,y,z])_{(d)} \oplus 0 \\ &\neq 0 \quad , \forall d>0. \end{aligned}$$

Pf of lemma.

It reduces easily to the case when S is affine. Since i is a closed immersion, $R^q i_*(\mathcal{F}) = 0$, $\forall q\geq 1$, $\mathcal{F}\in \mathrm{Coh}(X)$. Since $i_*\mathcal{F}\in\mathrm{Coh}(\mathbb{P}^n_S)$, we may well assume that $X=\mathbb{P}^n_S$.

If $\mathcal{F} = \bigoplus_{i=0}^m \mathcal{O}(d_i)$, then the result is OK whenever $d\geq -\min\{d_i\}$, by our explicit calculation for $\mathbb{P}^n_S$ case. In general, we know that $\mathcal{F}$ may be written as a quotient:

$$0 \longrightarrow \mathcal{G} \longrightarrow \oplus_{i=0}^{m} \mathcal{O}(d_i) \longrightarrow \mathcal{F} \longrightarrow 0$$

Thus taking $d \geq -\min\{d_i\}$, $\mathcal{F}(d)$ is globally generated. We prove by dimension shifting downwards that $H^i(\mathbb{P}^n_S, \mathcal{F}) = 0$, $\forall i > 0$. $i = n+1$ follows since $\mathbb{P}^n_S$ can by covered by $(n+1)$ open affines. Note that $\mathcal{G} \in Coh(\mathbb{P}^n_S)$ as well. By induction, for $d >> 0$, and $k \geq 1$,

$$\begin{tikzcd} \cdots \arrow[r] & H^k(\oplus_{i=0}^{m} \mathcal{O}(d+d_i)) \arrow[r] \arrow[d, equal] & H^k(\mathcal{F}(d)) \arrow[r] & H^{k+1}(\mathcal{G}(d)) \arrow[r] \arrow[d, equal, "\text{induction}"] & \cdots \\ & 0 & & 0 & \end{tikzcd}$$

$\Longrightarrow H^k(\mathcal{F}) = 0.$ $\square$

A partial converse to this lemma is proved in the next section.

Prop. $S = \mathrm{Spec} R$ R: Noetherian. $f: X \longrightarrow S$ proper. $\mathcal{L}$: an invertible sheaf on X. Suppose $\forall \mathcal{F}$ coherent, $\exists\, d(\mathcal{F}) \in \mathbb{Z}$ s.t. $\forall d \geq d(\mathcal{F})$,

$$H^1(X, \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{L}^d) = 0$$

Then there exists $t \in \mathbb{N}$ s.t.

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & \mathbb{P}^n_S \arrow[d] \\ & S \end{tikzcd}$$

where i is a closed immersion and $\mathcal{L}^t \cong i^* \mathcal{O}_{\mathbb{P}^n_S}(1)$.

Rmk: C.f. Hartshorne. Prop III 5.3. The same setting as above. Then $\mathcal{L}$: ample on $X \Longleftrightarrow \forall \mathcal{F} \in Coh(X)$ $\mathcal{F} \otimes \mathcal{L}^{\otimes d}$ is globally generated for all $d \geq d(\mathcal{F})$.

§11. Ample Invertible Sheaves

Def. (EGA). (1). $\mathcal{L}$: an invertible sheaf on X is called ample iff

(a). X is quasi-compact

(b). $\forall\, x \in X$, $\exists\, s \in \Gamma(X, \mathcal{L}^{\otimes n})$, $n \geq 1$ s.t.

(b)$_1$: $X_s = \{x \in X \mid s \text{ generates } \mathcal{L}_x \text{ as an } \mathcal{O}_{X,x}\text{-module}\}$

(b)$_2$: X_s is affine

(2). $f: X \longrightarrow S$ is called projective iff there exists

$$\begin{tikzcd} X \arrow[r, hook] \arrow[dr] & \mathbb{P}(\mathcal{E}) \arrow[d] \\ & S \end{tikzcd}$$

where $\mathcal{E}$ is a quasi-coherent $\mathcal{O}_S$-module of finite type, and

$$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}_S(\mathrm{Sym}^*\mathcal{E}).$$

Locally on S, $\mathcal{E}|_{\mathrm{Spec} R} = \tilde{M}$ for some finite type R-module, then

$$\mathbb{P}(\mathcal{E})|_{\mathrm{Spec} R} = \mathrm{Proj}(\mathrm{Sym}^*_R(M)).$$

(3). $\mathcal{L}$ is relatively ample on X/S iff for any $V \subseteq S$ open affine, we have $\mathcal{L}|_{f^{-1}(V)}$ is ample on $f^{-1}(V)$.

(4). $\mathcal{L}$ is relatively very ample on X/S iff $\exists$ an immersion i:

$$\begin{tikzcd} X \arrow[r, hook, "i"] \arrow[dr] & \mathbb{P}(\mathcal{E}) \arrow[d] \\ & S \end{tikzcd}$$

s.t. $\mathcal{L} \cong i^*\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ for some $\mathcal{E}$ quasi-coherent of finite type.

(5). $f: X \longrightarrow S$ is quasi-projective iff f is of finite type and $\exists$ an f relatively ample invertible $\mathcal{O}_X$-module

Hartshorne has different definitions:

Def. (Hartshorne) $\mathcal{L}$: invertible sheaf on X, Noetherian.

(1). $\mathcal{L}$ is (H-) ample iff $\forall$ coherent $\mathcal{F}$ on X, $\exists\, d_0(\mathcal{F})$ s.t. $\mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^{\otimes d}$ is globally generated, $\forall d \geq d_0(\mathcal{F})$.

(2). $f: X \longrightarrow S$ is (H-) projective if there is a closed immersion

$$\begin{tikzcd} X \arrow[rr, hook] \arrow[dr] & & \mathbb{P}^n_S \arrow[dl] \\ & S & \end{tikzcd}$$

(3). $f: X \longrightarrow S$ is quasi-projective iff f factors as

$$X \xhookrightarrow[\text{immersion}]{\text{open}} X' \xrightarrow{\text{H-proj}} S$$

Lemma. $S = \operatorname{Spec} R$, R: Noetherian. $f: X \longrightarrow S$ proper. $\mathcal{L}$: an invertible sheaf on X. Suppose $\forall \mathcal{F}$ coherent, $\exists\, d(\mathcal{F}) \in \mathbb{Z}$ s.t. $\forall d \geq d(\mathcal{F})$,

$$H^1(X, \mathcal{F}\otimes_{\mathcal{O}_X}\mathcal{L}^d) = 0$$

Then $\mathcal{L}$ is H-ample.

The converse is not true!

E.g. $\mathcal{O}_{\mathbb{A}^2_{\mathbb{C}}\setminus\{0\}}$ is H-ample, but $H^1(\mathbb{A}^2\setminus\{0\}, \mathcal{O}) \neq 0$.

Pf of lemma.

Pick $x \in X$ a closed point. Then we have:

$$0 \longrightarrow \mathcal{I}_x \longrightarrow \mathcal{O}_X \longrightarrow (j_x)_*\kappa(x) \longrightarrow 0$$

Twisting by $\mathcal{L}^{\otimes d}$, we get:

$$0 \longrightarrow \mathcal{I}_x \otimes \mathcal{L}^{\otimes d} \longrightarrow \mathcal{L}^{\otimes d} \longrightarrow \mathcal{L}^{\otimes d}\otimes\kappa(x) \longrightarrow 0,$$

and $\Gamma(X, \mathcal{L}^{\otimes d}\otimes\kappa(x)) \cong \kappa(x)$ (not canonically). Thus by assumption, we may pick $s_i \in \Gamma(X, \mathcal{L}^{\otimes i})$ for all $i \in d(\mathcal{I}_x), \cdots, 2d(\mathcal{I}_x)-1$ with $s_i(x) \neq 0$. Then set $U_x = \bigcap_{i=d(\mathcal{I}_x)}^{2d(\mathcal{I}_x)-1} X_{s_i}$. Then we see that $\forall x' \in U_x$, and $\forall d \geq d(\mathcal{I}_x)$, $\exists\, s \in \Gamma(X, \mathcal{L}^d)$, s.t. $s(x') \neq 0$. Indeed, we may just take $s = s_{d(\mathcal{I}_x)}^m s_{d-md(\mathcal{I}_x)}$, where $m = [\frac{d}{d(\mathcal{I}_x)} - 1]$.

Then since X is quasi-compact, $X = U_{x_1}\cup\cdots\cup U_{x_t}$ for some $x_1, \cdots, x_t$ closed. Set $d_0(\mathcal{O}_X) = \max_{i=1,\cdots,t}\{d(\mathcal{I}_{x_i})\}$. Then this shows that $\forall d \geq d(\mathcal{O}_X)$, $\mathcal{L}^{\otimes d}$ is globally generated.

For a general coherent sheaf, we can do the same argument for the s.e.s.

$$0 \longrightarrow \mathcal{I}_x\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}\otimes_{\mathcal{O}_X} k(x) \longrightarrow 0 \qquad \square$$

Lemma. X: Noetherian. $\mathcal{L}$: H-ample, then $\mathcal{L}$ is ample (EGA).
(This works in both direction).

Pf: Pick $x \in X$ and $U \subseteq X$ affine open nhd of x. Let $\mathcal{I}$ = the ideal sheaf of $X \setminus U$ in X, which is coherent since X is Noetherian. By assumption, $\exists d >> 0$ and $s \in \Gamma(X, \mathcal{I}\otimes\mathcal{L}^{\otimes d})$ with $s(x) \neq 0$. Then $X_s \subseteq U$ by construction, and it's easy to show that X_s is affine. (Or we could have assumed that $\mathcal{L}|_U \cong \mathcal{O}_U$, so that $X_s = D(f)$ where $s \longleftrightarrow f \in \Gamma(U, \mathcal{O}_U)$.). $\square$

Lemma. If X is Noetherian and $\mathcal{L}$ is ample, then

$$X \xhookrightarrow[\text{immersion}]{\text{open}} \mathbb{P}(\Gamma_*(X, \mathcal{L}))$$

where $\Gamma_*(X,\mathcal{L}) = \bigoplus_{d\geq 0} \Gamma(X, \mathcal{L}^{\otimes d})$.

Pf: $\forall s \in \Gamma(X, \mathcal{L}^{\otimes d})$, $D_+(s) = \operatorname{Spec}(\Gamma_*(X,\mathcal{L})_{(s)}) \subseteq \mathbb{P}(\Gamma_*(X,\mathcal{L}))$. We then wish to have:

$$\begin{tikzcd} X \arrow[r, hook] & \mathbb{P}(\Gamma_*(X,\mathcal{L})) \\ X_s \arrow[u, hook] \arrow[r, hook] & D_+(s) \arrow[u, hook] \end{tikzcd}$$

Pick an s s.t. X_s is affine and by assumption, these X_s's cover X. By construction, $\mathcal{O}_X(X_s) = \Gamma_*(X,\mathcal{L})_{(s)}$. Then just take $\psi_s : X_s \longrightarrow D_+(s)$ to agree with this identification. $\square$

Combing these lemmas, we obtain:

Prop. X: proper over $\operatorname{Spec} A$, with A Noetherian. $\mathcal{L}$: invertible sheaf on X s.t. for all coherent $\mathcal{F}$ on X, $\exists d(\mathcal{F})$ s.t. $\forall d \geq d(\mathcal{F})$, we have $H^1(X, \mathcal{F}\otimes\mathcal{L}^d) = 0$. Then $X \cong \operatorname{Proj}(\Gamma_*(X,\mathcal{L}))$. (For $d >> 0$, $\mathcal{L}^d = \psi^*\mathcal{O}(d)$).

Pf: By lemma 3, ψ is an open immersion. Since π is seperated, $\mathrm{im}\psi$ is

$$\begin{tikzcd} X \arrow[rr, hook, "\psi"] \arrow[dr, "\text{proper}"'] & & \mathrm{Proj}(\Gamma_*(X,\mathcal{L})) \arrow[dl, "\pi"] \\ & \mathrm{Spec}(A) & \end{tikzcd}$$

closed. Thus $\mathrm{Proj}(\Gamma_*(X,\mathcal{L})) = \psi(X) \amalg Y$. Then $Y=\phi$. Otherwise, $D_+(s) \subseteq Y$ for some $s \Rightarrow X_s = \phi$. $\Rightarrow s$ is nilpotent : $s^N = 0$. Then:

$$D_+(s) = D_+(s^N) = \phi$$

Contradiction. □

Rmk: We know that each $H^0(X, \mathcal{L}^d)$ is a finite A-module. But more importantly, the whole ring $\Gamma_*(X,\mathcal{L})$ is a finitely generated A-algebra. This doesn't follow directly from the first result. For instance, the algebra $\mathbb{C}[x,y]$ is finitely generated in each degree yet the subring $\mathbb{C}[x, xy, xy^2, xy^3, \cdots]$ is not finitely generated.

Up to now, we have shown that : $f: X \longrightarrow S = \mathrm{Spec}R$: proper and R Noetherian. $\mathcal{L}$ an invertible sheaf on X s.t. for every coherent sheaf $\mathcal{F}$ on X, we have statements:

(i). $H^1(X, \mathcal{F}\otimes\mathcal{L}^n) = 0$, $\forall n >> 0$.

(ii). $\mathcal{F}\otimes\mathcal{L}^n$ is globally generated for $n>>0$. (ampleness in Hartshorne)

(iii). $\forall x\in X$, $\exists s \in \Gamma(X, \mathcal{L}^n)$, $n \geq 1$ s.t. $x \in X_s$ and X_s is affine (ampleness in EGA).

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). (Actually in this case (iii) $\Rightarrow$ (i) as well but this requires some work). Moreover $X \cong \mathrm{Proj}(\Gamma_*(X,\mathcal{L}))$.

Prop. Under the assumption $f: X \longrightarrow S = \mathrm{Spec}R$: proper and R Noetherian and any of (i), (ii), (iii), $\Gamma_*(X,\mathcal{L})$ is a finitely generated R-algebra.

Rmk: If X is a quasi-compact scheme with $\mathcal{L}$ invertible on X satisfying

(iii) above. Then:

(a). $X \subseteq \mathrm{Proj}(\Gamma_*(X, \mathcal{L}))$ is an open immersion

(b). This is what EGA calls an ample invertible sheaf.

In this case, $\Gamma_*(X, \mathcal{L})$ need not be finitely generated.

E.g. k: a field. $X = \mathrm{Proj}(k[u, v, z_1, z_2, z_3, \cdots]/I)$, where $\deg u = \deg v = 1$ and $\deg z_i = i$. $I = (z_i^2 - u^{2i})$. Then it can be shown that

(1). $X = D_+(u) \cup D_+(v)$;

(2). $\mathcal{O}_X(1)$ is an invertible sheaf on X and $\mathcal{O}_X(n) \cong \mathcal{O}_X(1)^{\otimes n}$;

(3). $\Gamma(X, \mathcal{O}_X(n)) = (k[u, v, z_i]/I)_n$: degree n part. (This needs some calculation), and thus is finite dimensional.

However, A is not finitely generated ($X \longrightarrow \mathrm{Spec}\, k$ is not proper; it's not finite type).

Proof of prop.

As a first step, we shall try to find a closed immersion: $X \hookrightarrow \mathbb{P}^m_S$. To do this, choose $s_i \in \Gamma(X, \mathcal{L}^{d_i})$, $i = 0, \cdots, n$ s.t. $X = \bigcup_{i=0}^n X_{s_i}$. This can be done since X is q.c.. Next, since X/S is finite type, $A_i = R[a_{i1}, \cdots a_{in_i}]/I$. Recall that $A_i = \mathcal{O}_X(X_i) \cong \Gamma_*(X, \mathcal{L})_{(s_i)}$. Choose $s_{ij} \in \Gamma(X, \mathcal{L}^{e_{ij}d_i})$ s.t. $s_i^{e_{ij}} a_{ij}$ extends to be the global section s_{ij}, for $j = 1, \cdots, n_i$. Let $N = \mathrm{l.c.m.}(d_i, e_{ij}d_i)$, and consider $\varphi \triangleq \varphi_{\mathcal{L}^N}: X \longrightarrow \mathbb{P}^m_S$ defined by the sections $(s_0^{N/d_0}, \cdots, s_n^{N/d_n}, s_{ij} s_i^{\frac{N}{d_i} - e_{ij}}, \cdots)$, and $m = n + \sum_{i=0}^n n_i$. Since $X_{s_i} = X_{s_i^{N/d_i}}$, $i = 0, \cdots, n$ cover X the map is a morphism.

Claim: φ is a closed immersion.

Since X/S is proper, $\varphi(X)$ is closed. Furthermore, let $\mathbb{P}^m_S = \mathrm{Proj}(R[T_i, T_{ij}])$, and note that $\varphi^{-1}(D_+(T_i)) = X_{s_i}$, $i = 0, \cdots, n$ cover X. On the ring level,

$$R\left[\frac{T_0}{T_{i_0}}, \cdots, \frac{T_n}{T_{i_0}}, \frac{T_{ij}}{T_{i_0}}\right] \longrightarrow \mathcal{O}_X(X_{i_0})$$

$$T_{i_0 j}/T_{i_0} \longmapsto (s_{i_0 j} \cdot s_{i_0}^{\frac{N}{d_{i_0}} - e_{i_0 j}})/s_{i_0}^{\frac{N}{d_{i_0}}} = s_{i_0 j}/s_{i_0}^{e_{i_0 j}} = a_{i_0 j}$$

is surjective. Hence φ is closed in the open $\bigcup_{i=0}^n D_+(T_i)$. So it's an immersion.

- Conclusion: there is a closed immersion s.t. $i^*\mathcal{O}_{\mathbb{P}^m_S}(1) \cong \mathcal{L}^N$ for some $N>0$.

Now,

$$\begin{aligned}\Gamma_*(X,\mathcal{L}) &= \oplus_{n\geq 0}(\Gamma_*(X,\mathcal{L}^n)) \\ &= \oplus_{n\geq 0}(\oplus_{n_1=0}^{N-1}\Gamma(X,\mathcal{L}^{\otimes(n_1+nN)})) \\ &= \oplus_{n\geq 0}(\oplus_{n_1=0}^{N-1}\Gamma(\mathbb{P}^m_R, i_*\mathcal{L}^{\otimes(n_1+nN)})) \\ &= \oplus_{n\geq 0}(\oplus_{n_1=0}^{N-1}\Gamma(\mathbb{P}^m_R, i_*(\mathcal{L}^{n_1}\otimes i^*\mathcal{O}_{\mathbb{P}^m_R}(n)))) \\ &= \oplus_{n\geq 0}(\oplus_{n_1=0}^{N-1}\Gamma(\mathbb{P}^m_R, (i_*\mathcal{L}^{n_1})\otimes \mathcal{O}_{\mathbb{P}^m_R}(n))) \quad \text{(projection formula)} \\ &= \oplus_{n\geq 0}(\Gamma(\mathbb{P}^m_R, i_*(\oplus_{n_1=0}^{N-1}\mathcal{L}^{n_1})\otimes \mathcal{O}_{\mathbb{P}^m_R}(n)))\end{aligned}$$

Let $\mathcal{F} = i_*(\oplus_{n_1=0}^{N-1}\mathcal{L}^{n_1})$. Then this is a coherent sheaf on $\mathbb{P}^m_R$. Then

$$\Gamma_*(X,\mathcal{L}) = \Gamma_*(\mathbb{P}^m_R,\mathcal{F}).$$

We claim that $\Gamma_*(\mathbb{P}^m_R,\mathcal{F})$ is a finite $R[T_0,\cdots,T_m]$-module. Then it follows that $\Gamma_*(X,\mathcal{L})$ is a finitely generated R-algebra. □

Lemma. For any coherent sheaf $\mathcal{F}$ on $\mathbb{P}^m_R$, with R Noetherian. Then $\forall k\in\mathbb{Z}$, the module $\oplus_{n\geq k}\Gamma(\mathbb{P}^m_R,\mathcal{F}(n))$ is a finite $R[T_0,\cdots,T_m]$-module.

Pf: Choose a surjection: $\varphi: \oplus_{i=1}^{r}\mathcal{O}(d_i) \twoheadrightarrow \mathcal{F}$. Then for $k>>0$, the higher cohomologies of $\ker\varphi(n)$ vanish and

$$\oplus_{n\geq k}\Gamma(\mathbb{P}^m_R, \oplus_{i=1}^{r}\mathcal{O}(d_i+n)) \twoheadrightarrow \oplus_{n\geq k}\Gamma(\mathbb{P}^m_R,\mathcal{F}(n))$$

Since each individual $\Gamma(\mathbb{P}^m_R,\mathcal{F}(n))$ is finitely generated, we may ignore the starting terms. The lemma follows from our previous computation of $\mathcal{O}_{\mathbb{P}^m}(d)$.

□