Generic Picard Numbers

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of hypersurfaces in 3D weighted projective spaces

Ref:

(1). D. Cox. Picard Numbers of Surfaces in 3-Dimensional Weigted Projective Spaces

(2). A.J. de Jong. J.H.M. Steenbrink. Picard Numbers of Surfaces in 3-Dimensional Weigted Projective Spaces

(3). T. Shioda. The Hodge Conjecture for Fermat Varieties.

Grothendieck's theorem on $\mathrm{Pic}_{X/S}$

Our goal would be to review the general theory on generic Picard numbers here.

Recall that for any scheme X, its Picard group is by def. the group of isomorphism classes of invertible sheaves on X. If X is smooth projective / $k=\bar{k}$, $\mathrm{Pic}^0(X)$ is an abelian variety & the group $\mathrm{Pic}(X)/\mathrm{Pic}^0(X)$ is a finitely generated abelian group whose rank $\rho(X)$ is called the Picard number.

Let $\mathcal{X} \longrightarrow B$ be a smooth projective family.

Thm. (Grothendieck). The functor $\mathrm{Pic}_{\mathcal{X}/B}$ is represented by a countable disjoint union of projective schemes.
Furthermore, if for any $b \in B$, $H^2(\mathcal{X}_b, \mathcal{O}_{\mathcal{X}_b}) = 0$, then $\mathrm{Pic}_{\mathcal{X}/B}$ is smooth over B.

For the proof, see [BLR]. Néron Models, or Kleiman's notes, or SGA 6.

As an immediate application of the thm, we have a first result on Picard numbers of families of smooth surfaces of low degree in $\mathbb{P}^3$, namely they are constants:

Cor 1. Let $\mathcal{X} \longrightarrow B$ be the family of smooth degree d surfaces in $\mathbb{P}^3$ where $d \leq 3$, then the Picard rank is a constant.
Pf: $\forall b$, $H^2(\mathcal{O}_{\mathcal{X}_b}) \cong H^0(\omega_{\mathcal{X}_b})^\vee = H^0(\mathcal{O}_{\mathbb{P}^3}(-4+d)|_{\mathcal{X}_b})^\vee = 0$. $\square$

We can also explicitly compute ρ in these cases:

$d=1$, $\mathcal{X} \longrightarrow B$ is a family of $\mathbb{P}^2$, $\rho \equiv 1$.
$d=2$, $\mathcal{X} \longrightarrow B$ is a family of $\mathbb{P}^1 \times \mathbb{P}^1$, $\rho \equiv 2$.
$d=3$, $\mathcal{X} \longrightarrow B$ can be computed at the Fermat surface:

$$x_0^3 + x_1^3 + x_2^3 + x_3^3 = 0$$

$\rho \equiv \dim H^{1,1} = \dim R_2 + 1 = 7$ where R_2 denotes the degree 2 part of the Jacobian ring (to be defined later).

Second, it follows from this thm. that the image of each component of $Pic_{\mathcal{X}/B}$ in B is closed.

Assume we are now working over an uncountable, algebraically closed field k, and $\mathcal{X}, B$ are varieties over k. Then the above thm implies that, if the image of a particular component of $Pic_{\mathcal{X}/B}$ in B is not B, then it's a proper closed subset of B, which is "of measure zero" in B. So that these components will not help contributing to the generic Picard rank. Moreover, by the same theorem, if we are only worried about the generic Picard rank, we may safely remove all the components of $Pic_{\mathcal{X}/B}$ which do not surject onto B, since there are only countably many of them.

Now assume that the generic Picard rank of the family $\mathcal{X} \to B$ is ρ. Then we may pick components $P_0, \cdots, P_{\rho-1}$ of $Pic_{\mathcal{X}/B}$ accounting for these rank numbers (P_0 stands for the component where $\mathcal{O}_{\mathcal{X}/B}(1)$ lies).

Cor. 2. If the generic Picard rank is ρ, we can pick some components $P_0, \cdots, P_{\rho-1}$ of $\mathrm{Pic}_{X/B}$ whose c_1 in $R^2 f_*(\mathbb{Z})$ span the generic NS groups. $\square$

We will be interested in determining the generic Picard number ρ for the other families of surfaces in $\mathbb{P}^3$ of deg $d \geq 4$, and more generally families of surfaces in 3d weighted projective spaces. This is a purely geometric quantity. We will resort to Hodge theory, which turns geometry into algebra that we can deal with, to solve the problem.

IVHS of hypersurfaces in $\mathbb{P}^n$.

We recall the general theory of IVHS via the e.g. of hyper-surfaces in $\mathbb{P}^{n+1}/\mathbb{C}$.

Recall that the family of smooth hypersurface in $\mathbb{P}^{n+1}$ of deg d.

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d] & \mathbb{P}(\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))) \times \mathbb{P}^{n+1} \arrow[d, equal] \\ B \arrow[r, hook] & \mathbb{P}(\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))) \end{tikzcd}$$

where

$$B = \mathbb{P}(\Gamma(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(d))) \setminus (\text{Singular locus } \Delta).$$

We know that

$$(R^n f_*(\mathbb{Z}) = \mathbb{V}_{\mathbb{Z}}\ ,\ \mathcal{F}^p)$$

is a PVHS of wgt n, which leads to the global period map

$$B \xrightarrow{P} D : \text{ the period domain}$$

Recap: Period maps & IVHS

Note that given $F^\bullet$ a partial flag, we have

$$T_{F^\bullet}(\text{Partial flag variety}) = \left\{ (\alpha_p) \in \prod_p \mathrm{Hom}_{\mathbb{C}}(F^p, V_{\mathbb{C}}/F^p) \text{ s.t } \forall p \quad \begin{tikzcd} F^p \arrow[r, "\alpha_p"] & V_{\mathbb{C}}/F^p \\ F^{p+1} \arrow[u, phantom, "\cup\!\!\mid"] \arrow[r, "\alpha_{p+1}"] \arrow[ur, phantom, "\circlearrowleft"] & V_{\mathbb{C}}/F^{p+1} \arrow[u, two heads] \end{tikzcd} \right\}$$

By Griffiths transversality, an infinitesimal VHS lands in

$$\bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(F^p/F^{p+1},\ F^{p-1}/F^p)$$

For infinitesimal PVHS, it must land in the ψ-symmetric elements of the above subset:

$$\left\{ \begin{matrix} \psi\text{-symmetric} \\ \text{elements} \end{matrix} \right\} \subseteq \bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(F^p/F^{p+1},\ F^{p-1}/F^p)$$

Hence, we obtain, for any $b\in B$, which represents a hypersurface $X: F=0$ in $\mathbb{P}^{n+1}$,

$$\begin{array}{c} T_bB \xrightarrow{KS} T_X\mathcal{M} = H^1(X, TX) \xrightarrow{dP_F} \{\psi\text{-symmetric elts}\} \\ \hspace{8em}\cap\!\mid \\ \hspace{8em}\bigoplus_{p\in\mathbb{Z}} \mathrm{Hom}_{\mathbb{C}}(H^q(\Omega_X^{p+1}), H^{q+1}(\Omega_X^p)) \end{array}$$

Lemma 1. $\forall\ \theta \in H^1(X, T_X)$,

$$dP_F(\theta) = \oplus_p(\text{Cup product with } \theta : H^q(X, \Omega_X^p) \longrightarrow H^{q+1}(X, \Omega_X^{p-1})).$$

$\square$

From now on, we will focus on surfaces in $\mathbb{P}^3$, and assume we are in the case of generic rank ρ case.

By the cor, we have $c_1(P_i) \in \mathbb{V}_{\mathbb{Q}}$ and $c_1(P_i) \otimes_{\mathbb{C}} 1 \in \mathbb{V}_{\mathbb{C}}$ are $(1,1)$ classes, i.e.

$$0 = \mathcal{F}^3 \subseteq \underbrace{\mathcal{F}^2 \subseteq \mathcal{F}^1}_{\mathcal{H}^{1,1}} \subseteq \mathcal{F}^0 = \mathbb{V}_{\mathbb{C}}$$

Since these are flat sections of $\mathcal{H}^{1,1}$, $\nabla_{GM}(c_1(P_i) \otimes 1) = 0$, i.e. they are invariant under parallel transport along any direction of $H^1(X, TX)$. We thus obtain:

Criterion 1. (de Jong - Steenbrink)
If the generic Picard rank of the family of smooth surfaces in $\mathbb{P}^3$

$$f: \mathcal{X} \longrightarrow B$$

is ρ, then the map at any $X: F=0$:

$$\boxed{H^1(X, T_X) \otimes H^{1,1}(X, \mathbb{Q})_{prim} \longrightarrow H^2(X, \Omega_X^0)}$$

or equivalently:

$$\boxed{H^{1,1}(X, \mathbb{Q})_{prim} \longrightarrow \mathrm{Hom}(H^1(X, T_X), H^2(X, \Omega_X^0))}$$

has at least $\rho - 1$ dimensional (right) kernel. Here

$$H^{1,1}(X, \mathbb{Q})_{prim} \triangleq H^{1,1}(X, \mathbb{C})_{prim} \cap H^2(X, \mathbb{Q}),$$

the rational space of Lefschetz $(1,1)$ classes $\square$

The next lemma allows us to convert every quantity into pure algebra.

Lemma 2. (Griffiths) Let R be the Jacobian ring of X, defined by

$$R = \mathbb{C}[X_0, \cdots, X_{n+1}] / (\frac{\partial F}{\partial X_0}, \cdots, \frac{\partial F}{\partial X_{n+1}})$$

Then R is a finite dimensional graded algebra, and we have canonical isomorphisms:

$$H^1(X,TX) \cong R_d$$
$$H^q(X,\Omega_X^p)_{prim} \cong R_{(q+1)d-(n+2)} \quad (\text{where } p+q=n).$$

The cup product above translates into multiplication in the algebra:

$$R_d \otimes R_{n+2-(q+1)d} \longrightarrow R_{n+2-qd}$$

Sketch of proof.

The proof uses the residue map associated to F, we have identifications (for $n=3$)

$$\begin{cases} H^{2,0}(X) \cong R_{d-4}\,\Omega/F \\ H^{1,1}(X)_{prim} \cong R_{2d-4}\,\Omega/F^2 \\ H^{0,2}(X) \cong R_{3d-4}\,\Omega/F^3 \end{cases}$$

where $\Omega = dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3$.

That $H^1(X,TX) \cong R_d$ follows from the fact that

$$KS: T_bB \cong S_d \longrightarrow H^1(X,TX)$$

has as kernel J_d. □

Now after this algebraic translation, there are two ways of showing that the generic Picard number is 1

Approach I:

Thm. (Macauley). Let $f_0, \cdots, f_{n+1}$ be a regular sequence of degrees $d_0, \cdots, d_{n+1}$ resp. in $\mathbb{C}[x_0,\cdots,x_{n+1}]$ and $R = \mathbb{C}[x_0,\cdots,x_{n+1}]/(f_0,\cdots,f_{n+1})$. Then R is a finite dimensional graded algebra with top degree $\sigma = \sum(d_i - 1)$ and the multiplication:

$$R_a \otimes R_b \longrightarrow R_{a+b}$$

is non-degenerate for any $a+b \le \sigma$.

Approach II.

Find one particular hypersurface where

$$R_d \otimes R_{d-4} \twoheadrightarrow R_{2d-4}$$

Then calculate for the simplest case: Fermat surfaces. $(n=2)$

$$F = X_0^d + X_1^d + X_2^d + X_3^d$$

E.g. $d=4$. $R_{d-4} = R_0$: 1-dim'l $\quad R_d = R_{2d-4} = R_4$.

In general, use the Koszul resolution of R : $(S = \mathbb{C}[X_0, X_1, X_2, X_3])$

$$0 \longrightarrow S\{4(d-1)\} \longrightarrow S^{\oplus 4}\{3(d-1)\} \longrightarrow S^{\oplus 6}\{2(d-1)\} \longrightarrow \oplus_{i=0}^{3} S\{(d-1)\} \xrightarrow{X_i^{d-1}} S \longrightarrow R \longrightarrow 0$$

$$\Longrightarrow \dim R_n = \dim S_n - 4\dim S_{n-(d-1)} + 6\dim S_{n-2(d-1)} - 4\dim S_{n-3(d-1)} + \dim S_{n-4(d-1)} = \cdots$$

Rmks: Both approaches admit generalizations to wgted projective hypersurfaces, used by Cox, de Jong-Steenbrink resp.

In generalized approach I, one replaces Macauley's thm. by its analogue for wgted proj. hypersurfaces, namely Delorme's thm.

In approach II, one can identify the cohomology groups $H^{p,q}_{prim}$ with certain characters of the automorphism group (Katz, Shioda etc). This generalizes to weighted hypersurfaces (de Jong - Steenbrink) and can be used to deal with more cases that are not covered by approach I.

Weighted projective spaces

Let $q_0, q_1, \cdots, q_{n+1}$ be positive integers. Let $\mathbb{C}^*$ act on $\mathbb{C}^{n+2}\setminus\{0\}$ by

$$t\cdot(x_0, x_1, \cdots, x_{n+1}) = (t^{q_0}x_0, t^{q_1}x_1, \cdots, t^{q_{n+1}}x_{n+1})$$

The quotient space is the weighted projective space $\mathbb{P}(q_0, \cdots, q_{n+1})$, which can also be regarded as $\mathbb{P}^{n+1}/\mu_{q_0}\times\cdots\times\mu_{q_{n+1}}$.

Def. A weighted homogeneous polynomial f defines a hypersurface in $\mathbb{P}(q_0, \cdots, q_{n+1})$. It's called quasi-smooth if all partial derivatives $\partial_{x_0}f, \cdots, \partial_{x_{n+1}}f$ have no common zeros on X.

Prop. A quasi-smooth hypersurface is a V-variety, i.e. a variety which is locally the quotient of a smooth variety by a finite group. □

Just as for a smooth projective hypersurfaces in $\mathbb{P}^{n+1}$, the Hodge theory of a quasismooth hypersurface can be described in terms of the Jacobian ring

$$R = \mathbb{C}[x_0, \cdots, x_{n+1}]/(\partial_{x_0}f, \cdots, \partial_{x_i}f)$$

Thm. (Steenbrink). Let $H^{p,q}$ be the primitive (p,q)-cohomology of the quasismooth weighted hypersurface of degree d, and R_a the degree a part of the Jacobian ring. Then there are isomorphisms:

$$H^{p,q} \cong R_{(q+1)d-s} \qquad (s=\sum q_i,\ p+q=n)$$

and the first order deformation space:

$$H^1(T_X)_0 \cong R_d \quad (\text{image under } KS)$$ □

Cox's theorem

We apply the general theory developed so far on weighted projective surfaces.

Now let $X = V(F) \subseteq \mathbb{P}(q_0, q_1, q_2, q_3)$ be a wgted hypersurface of degree km, and let J be the Jacobian ideal of F, and let $R = S/J$, $S = \mathbb{C}[x_0, x_1, x_2, x_3]$. We further assume that $\gcd(q_i, q_j, q_k) = 1$, $\forall i \neq j \neq k$, which is not a restriction by Delorme. By Steenbrink's thm., we have:

$$\begin{cases} H^{2,0}(X) \cong R_{km-s} \\ H^{1,1}_{\text{prim}}(X) \cong R_{2km-s} \\ H^1(X, T_X) \cong R_{km} \end{cases}$$

Under these isomorphisms,

$$H^1(X, T_X) \otimes H^{2,0}(X) \longrightarrow H^{1,1}(X)$$

becomes ring multiplications.

$$R_{km} \otimes R_{km-s} \longrightarrow R_{2km-s}.$$

Thus if this map is always surjective, then the generic X will have $\rho(X) = 1$. But this will be true if we have

$$S_{km} \otimes S_{km-s} \twoheadrightarrow S_{2km-s}$$

This condition on weighted polynomial ring is studied by Delorme:

Thm. (Delorme). Given wgts $(q_0, \cdots, q_n)$, $m = \mathrm{lcm}(q_0, \cdots, q_n)$ and $s = q_0 + \cdots + q_n$. Let G be the Delorme constant:

$$G = -s + \frac{1}{n} \sum_{\nu=2}^{n+1} \frac{1}{\binom{n-1}{\nu-2}} \sum_{1 \le i_1 < \cdots < i_\nu \le n+1} \mathrm{lcm}(q_{i_1}, \cdots, q_{i_\nu}) \quad (\le -s + nm)$$

Then for any $k \geq 1$, the multiplication map

$$S_{km} \otimes S_\ell \longrightarrow S_{km+\ell}$$

is surjective whenever $\ell > G$. $\square$

In our case, $n=3$ and $G \le -s+3m$, and $\ell = km-s$. Thus if $k \ge 4$, $km-s > G$ and the generic X has $\rho(X)=1$.

When $k=3$, let $a_{ij} = a_{ji} = \gcd(q_i, q_j)$ for $i \ne j$. Thus $\gcd\{a_{ij}, a_{k\ell}\} = 1$ if $\{i,j\} \ne \{k,\ell\}$. We can write:

$$q_i = a_{ij}\, a_{i\ell}\, a_{ik}\, b_i \qquad \{j,k,\ell\} = \{0,1,2,3\} \setminus \{i\}.$$

Then $m = \prod_{0\le i<j\le 3} a_{ij} \prod_{0 \le i \le 3} b_i$. Furthermore, if $\{i,j,k,\ell\} = \{0,1,2,3\}$ then $m = \operatorname{lcm}(q_i, q_j)\, a_{k\ell}\, b_k b_\ell = \operatorname{lcm}(q_i, q_j, q_k)\, b_\ell$, so that

$$\begin{aligned} G &= -s + \Big(\frac{1}{3}\sum_{0\le i<j\le 3} \frac{1}{a_{ij} b_i b_j} + \frac{1}{6}\sum_{0\le i \le 3} \frac{1}{b_i} + \frac{1}{3}\Big)m \\ &\le -s + \Big(\frac{1}{3}\binom{4}{2} + \frac{1}{6}\cdot 4 + \frac{1}{3}\Big)m \\ &= -s + 3m \end{aligned}$$

"=" iff $a_{ij} = b_k = 1$, $\forall i,j,k$, which happens iff $q_0 = q_1 = q_2 = q_3 = 1$. Thus $G < -s+3m$ whenever $m > 1$. This shows that $\rho(X) = 1$ when $k=3$.

When $k=2$, Cox did a more careful argument and showed that

$$S_{2m} \otimes S_{2m-s} \longrightarrow S_{4m-s}$$

is surjective whenever we are in the case of either

(i). $\operatorname{lcm}(q_0, q_1) < m$ and $\operatorname{lcm}(q_2, q_3) < m$

(ii). $q_3 = m$ and all other $q_i > 1$

(iii). $q_0 = 1$, except $(1, q_1, q_2, q_3) = (1,1,1,2)$ or $(1,1,a,a)$ $a > 1$ up to permutation.

Then he proceeded to show that this covers all $k=2$ case (the exceptional case in (iii) has $P_g(X) = 0$).

Thus he obtained:

Thm (Cox) Let $\mathcal{F}$ be the family of quasismooth surfaces X in $\mathbb{P}(q_0, q_1, q_2, q_3)$ of deg. km. ($m = \mathrm{lcm}(q_0, q_1, q_2, q_3)$, $\gcd(q_i, q_j, q_k) = 1$)
If $m > 1$, $k > 1$, then either
(i) $P_g(X) = 0$, or
(ii). The generic member X of $\mathcal{F}$ has $\rho(X) = 1$.
Furthermore, (i) happens iff $k = 2$ and $(q_0, q_1, q_2, q_3) = (1, 1, 1, 2)$ or $(1, 1, a, a)$ $a > 1$ up to permutation. $\square$

Rmk: The same proof also works for $d = km + s$. However, Cox's proof breaks down for $k = 1$, by a counter e.g. of Dolgachev for X in $\mathbb{P}(1, 3, 3, 4)$

$$h^{2,0}(X) = 1, \quad h^{1,1}(X) = 16, \quad h(X, T_X)_0 = 14.$$

But using a different method, de Jong & Steenbrink showed that $\rho(X) = 1$ generically in this family and more generally for many more cases of $k = 1$.

In fact, via the earlier general theory, we can easily work out examples not covered by Delorme's thm, simply by checking the surjectivity of

$$R_{km} \otimes R_{km-s} \longrightarrow R_{2km-s}$$

at the Fermat surface!

E.g. $\mathbb{P}(1, 2, 2, 5) \supseteq X : x_0^{10} + x_1^5 + x_2^5 + x_3^2 = 0$
$m = 10$, $k = 1$, $s = 10$. $R = \mathbb{C}[x_0, x_1, x_2, x_3] / (x_0^9, x_1^4, x_2^4, x_3)$
$km = 10 = 2km - s$, $km - s = 0$ $\quad R_{km} = R_{2km-s}$, $R_{km-s} \cong \mathbb{C}$.

Hodge theory of Fermat surfaces

Let $X_d^n \triangleq \{x_0^d + \cdots + x_{n+1}^d = 0\} \subseteq \mathbb{P}^{n+1}$. μ_d acts on $\mathbb{P}^{n+1}$ coordinate-wise and preserves X_d^n. $\Longrightarrow$ $\mu_d^{\times n+1}$ acts on X_d^n, with the diagonal subgroup acting trivially. Define G_d^n

$$1 \longrightarrow \mu_d \xrightarrow{\Delta} \mu_d^{\times(n+1)} \longrightarrow G_d^n \longrightarrow 1$$

Then $\hat{G}_d^n$: $1 \longrightarrow \hat{G}_d^n \longrightarrow \hat{\mu}_d^{\times(n+1)} \longrightarrow \hat{\mu}_d \longrightarrow 1$. If we identify $\hat{\mu}_d \cong \mathbb{Z}/d$. Then

$$\hat{G}_d^n \cong \{\alpha = (a_1, \cdots, a_{n+1}) \mid a_i \in \mathbb{Z}/d,\ \Sigma a_i = 0\}.$$

G_d^n acts on X_d^n and thus induces an action on $H^i(X_d^n, \mathbb{Z})$ (resp. $\mathbb{Q}, \mathbb{R}, \mathbb{C}$). It decomposes $H^i(X_d^n, \mathbb{C})$ into isotypical components. Since $H^i(X_d^n, \mathbb{C})_{prim} = 0$ if $i \neq n$, we just need to consider the wgt n HS $H^n(X_d^n, \mathbb{C})_{prim}$ as a G_d^n-module

$\forall \alpha \in \hat{G}_d^n$, set $V(\alpha) \triangleq \{\xi \in H^n(X, \mathbb{C})_{prim} \mid g^*(\xi) = \alpha(g) \cdot \xi\}$.
Let $\amalg_d^n = \{\alpha = (a_0, \cdots, a_{n+1}) \in \hat{G}_d^n \mid a_i \neq 0, \forall i\}$, and $\forall \alpha \in \amalg_d^n$, define

$$|\alpha| = \left(\Sigma_{i=0}^{n+1} \langle a_i \rangle\right)/d$$

where $\langle a_i \rangle$ is the unique representative of a_i in $[0, d-1]$.

Thm. (Katz, Shioda) In this notation,

(i).

$$\dim V(\alpha) = \begin{cases} 1 & \alpha \in \amalg_d^n \\ 0 & \text{otherwise} \end{cases}$$

(ii).

$$H^{p,q} \cong \bigoplus_{|\alpha| = q+1} V(\alpha)$$

Sketch of proof.
The residue map gives us:

$$\begin{cases} H^{2,0}(X) \cong R_{d-4}\,\Omega/F \\ H^{1,1}(X)_{prim} \cong R_{2d-4}\,\Omega/F^2 \\ H^{0,2}(X) \cong R_{3d-4}\,\Omega/F^3 \end{cases}$$

Then given a monomial basis element of $H^{2-q,q}$,

$$x_0^{\ell_0} x_1^{\ell_1} x_2^{\ell_2} x_3^{\ell_3} \frac{dx_0 \wedge dx_1 \wedge dx_2 \wedge dx_3}{(x_0^d + x_1^d + x_2^d + x_3^d)^{q+1}} \quad \text{where} \begin{cases} \ell_0+\ell_1+\ell_2+\ell_3 = (q+1)d-4) \\ 0 \leq \ell_i \leq d-2 \end{cases}$$

$G_d^2 \ni g$, g acts with wgt $(\ell_0+1, \ell_1+1, \ell_2+1, \ell_3+1)$ mod d, which is in $\mathfrak{W}_d^2$. It also follows that each wgt space is 1-dim'l, and

$$|\alpha| = \sum \langle \alpha_i \rangle / d = \sum \langle \ell_i + 1 \rangle / d = q+1 \qquad \square$$

de Jong and Steenbrink's method

Their method is, as given in the previous example, to count dimensions for the weighted projective Fermat surfaces. With the aid of Katz-Shioda thm., we can further reduce counting dimensions of R to counting the finite number of group characters!

Identify the weighted Fermat surfaces as the quotient of the usual Fermat surface in $\mathbb{P}^3$

$$\begin{array}{lcc} \tilde{X} & \tilde{x}_0^d + \tilde{x}_1^d + \tilde{x}_2^d + \tilde{x}_3^d = 0 & \tilde{x}_i \\ & \Big\downarrow \; / \mu_{q_0} \times \mu_{q_1} \times \mu_{q_3} \times \mu_{q_4} & \Big\downarrow \\ X: & x_0^{\frac{d}{q_0}} + x_1^{\frac{d}{q_1}} + x_2^{\frac{d}{q_2}} + x_3^{\frac{d}{q_3}} = 0\,, & \tilde{x}_i^{q_i} \end{array}$$

we can describe $H^{p,q}(X)$ in terms of invariants of this group action. We set up some notation:

$$\hat{G} \triangleq \{(a_0, a_1, a_2, a_3) \in (\mathbb{Z}/d\mathbb{Z})^4 \mid a_i \equiv 0 \bmod q_i,\ \Sigma a_i = 0\}$$

i.e. these are $\prod \mu_{q_i}$ - invariant characters.

$$V(\alpha) = \{\xi \in H^2_{prim}(X, \mathbb{C}) \mid g^*(\xi) = \alpha(g)\xi\} \quad \alpha \in \hat{G}$$

$$\mathcal{A} = \{\alpha = (a_0, a_1, a_2, a_3) \in \hat{G} \mid a_i \neq 0 \text{ for all } i\}$$

Thm. (de Jong - Steenbrink).

(i) $\dim V(\alpha) = 1$ if $\alpha \in \mathcal{A}$ and else $\dim V(\alpha) = 0$.

(ii)

$$\begin{cases} H^{2,0}(X) = \sum_{|\alpha|=1} V(\alpha) \\ H^{1,1}_{prim}(X) = \sum_{|\alpha|=2} V(\alpha) \\ H^{0,2}(X) = \sum_{|\alpha|=3} V(\alpha) \end{cases}$$

(iii) Let $\mathcal{B} = \{\alpha \in \mathcal{A} \mid |t\alpha| = 2 \text{ for all } t \in (\mathbb{Z}/d\mathbb{Z})^*\}$. Then

$$H^{1,1}_{prim}(X, \mathbb{Q}) \otimes \mathbb{C} = \sum_{\alpha \in \mathcal{B}} V(\alpha).$$

Sketch of proof.

(i) & (ii) just follow from Katz-Shioda's thm by taking $\mu_{q_0} \times \cdots \times \mu_{q_3}$ invariants.

Part (iii) follows by considering the d-th cyclotomic Galois group $(\mathbb{Z}/d\mathbb{Z})^*$ action on the Fermat surfaces in $\mathbb{P}^3$, which commutes with the $\mu_{q_0} \times \cdots \times \mu_{q_3}$ action, inducing an action on

$$H^{1,1}_{prim}(X, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{C} \circlearrowleft (\mathbb{Z}/d\mathbb{Z})^*$$

$\square$

Combining this thm with Criterion 1 gives rise to the following purely (finite) numerical

Criterion 2. (de Jong - Steenbrink)
If $\forall \beta \in \mathcal{B}$, $\exists\, t \in (\mathbb{Z}/d\mathbb{Z})^*$ and $\alpha \in \mathcal{A}$ with $|\alpha| = 3$, s.t.

$$\langle t\beta_i \rangle \le \langle \alpha_i \rangle \qquad i = 0, \cdots, 3$$

then a generic weighted surface in the same family as the Fermat surface has Picard rank 1.

This criterion can be checked by computers and many more cases not covered by Delorme's thm are found!