Grothendieck - Riemann - Roch

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Ref: Le théorème de Riemann-Roch. A. Borel and J.P. Serre. Bulletin de la S.M.F.

§1. K-Groups On Algebraic Varieties

Let X be a projective variety over an algebraically closed field.

Def. $K(X) = \oplus \mathbb{Z}[\mathcal{F}]/\sim$, where $[\mathcal{F}]$ denotes the isomorphism classes of coherent sheaves on X, and "$\sim$" is generated by $[\mathcal{F}]-[\mathcal{F}']-[\mathcal{F}'']$ whenever

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

Def. $K_1(X) = \oplus \mathbb{Z}[\mathcal{E}]/\sim$, where $[\mathcal{E}]$ denotes the isomorphism classes of locally free sheaves on X, and "$\sim$ is generated by $[\mathcal{E}]-[\mathcal{E}']-[\mathcal{E}'']$ whenever

$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$

Thm 1. The canonical map $\varepsilon: K_1(X) \longrightarrow K(X)$ is an isomorphism.

Idea: Any $\mathcal{F} \in Coh(X)$ can be resolved by locally frees of length $\leq \dim X$.

$$0 \longrightarrow \mathcal{L}_n \longrightarrow \cdots \longrightarrow \mathcal{L}_0 \longrightarrow \mathcal{F} \longrightarrow 0$$

Then define an inverse γ_1 of ε by $\gamma_1([\mathcal{F}]) = \sum_{i=0}^{n}(-1)^i[\mathcal{L}_i]$. It can be checked that γ_1 is well-defined.

Operations On $K(X)$

(1). Ring structure on $K(X)$: $\mathcal{F}, \mathcal{G} \in Coh(X)$. Define

$$[\mathcal{F}], [\mathcal{G}] \longmapsto \sum_{i=0}^{n}(-1)^i[Tor^{\mathcal{O}_X}(\mathcal{F},\mathcal{G})]$$

Then it's additive in both $\mathcal{F}$ and $\mathcal{G}$ and thus define a product structure

$$K(X) \otimes K(X) \longrightarrow K(X).$$

Equivalently in $K_1(X)$, we may just take locally free resolutions of $\mathcal{F}$ and $\mathcal{G}$, and then form the tensor complex, and take the Euler characteristic.

By constructing the "associative Tor-functor" $Tor(\mathcal{F},\mathcal{G},\mathcal{H})$ and using the spectral sequences

$$E^2_{p,q} = Tor_p(\mathcal{F}, Tor_q(\mathcal{G},\mathcal{H})) \Longrightarrow Tor(\mathcal{F},\mathcal{G},\mathcal{H}) \Longleftarrow \widetilde{E}^2_{p,q} = Tor_p(Tor_q(\mathcal{F},\mathcal{G}),\mathcal{H})$$

and the fact that Euler characteristic is constant within a spectral sequence we have:

- The product on $K(X)$ is associative and commutative.

(In $K_1(X)$ the associativity is obvious; commutativity is obvious in both cases).

(2). λ-operations

Given a s.e.s. of locally free sheaves on X
$$0 \longrightarrow \mathcal{E}' \longrightarrow \mathcal{E} \longrightarrow \mathcal{E}'' \longrightarrow 0$$
we have a filtration

so that in $K_1(X)$, we have:
$$\lambda^p(\mathcal{E}) = \sum_{r+s=p} \lambda^r(\mathcal{E}')\lambda^s(\mathcal{E}'')$$
$$\Rightarrow \qquad \sum \lambda^p(\mathcal{E})t^p = (\sum \lambda^r(\mathcal{E}')t^r)(\sum \lambda^s(\mathcal{E}'')t^s)$$
$\Rightarrow \mathcal{E} \longmapsto \sum \lambda^p(\mathcal{E})t^p$ is an additive map from $K_1(X) \longrightarrow K_1(X)[t]$, with image in the multiplicative subgroup $\{1+a_1t+a_2t^2+\cdots \mid a_i \in K_1(X)\}$. This in turn induces

- $\Lambda: K(X) \longrightarrow K(X)[t]$

(3). $f^!$ and $f_!$

Let $f: Y \longrightarrow X$ be any morphism (not necessarily proper). $\forall \mathcal{E}$ locally free on X, $f^{-1}(\mathcal{E})$ is locally free on Y. This is an additive function, and thus induces

- $f^!: K(X) \cong K_1(X) \longrightarrow K_1(Y) \cong K(Y)$

We may also define this directly on $K(X)$ via the Tor-formula: given $\mathcal{F} \in Coh(X)$,
$$f^!(\mathcal{F}) \triangleq \sum_i (-1)^i [Tor_i^{\mathcal{O}_X}(\mathcal{O}_Y, \mathcal{F})]$$
which is additive on $Coh(X)$.

Next suppose $f: Y \longrightarrow X$ is proper. A fundamental result of Grothendieck states that:

- $\mathcal{F} \in Coh(Y) \Rightarrow R^q f(\mathcal{F}) \in Coh(X)$

Thus we may define:
$$f_!(\mathcal{F}) \triangleq \sum_i (-1)^i [R^q f(\mathcal{F})]$$
which is an additive functor on $Coh(Y) \longrightarrow K(X)$.

$f_!$ is not a ring homomorphism, but we have the projection formula:

$$\bullet\ f_!(y\cdot f^!(x)) = f_!(y)\cdot x$$

where $y\in K(Y)$, $x\in K(X)$. This follows from the projection formula for $\mathcal{F}\in Coh(Y)$, $\mathcal{E}$ locally free on X:

$$R^q f(\mathcal{F}\otimes_{\mathcal{O}_Y} f^{-1}\mathcal{E}) \cong R^q f(\mathcal{F})\otimes_{\mathcal{O}_X}\mathcal{E}$$

By definition, if $Z\xrightarrow{g} Y\xrightarrow{f} X$, then

$$\bullet\ (f\circ g)^! = g^!\cdot f^!.$$

If both f and g are proper, then

$$\bullet\ (f\circ g)_! = f_!\cdot g_!.$$

This follows from the spectral sequence: $\forall\,\mathcal{F}\in Coh(Z)$:

$$R^p f(R^q g(\mathcal{F})) \Longrightarrow R^{p+q} f\circ g(\mathcal{F}).$$

Chern Classes

Let $A(X)$ be the Chow ring of X. (Or we may use $H^*(X,\mathbb{Z})$ if $X/\mathbb{C}$). Chern classes can be extended to $K(X)\cong K_1(X)$ since the Chern polynomial

$$C(\mathcal{E})_t = 1 + C_1(\mathcal{E})t + C_2(\mathcal{E})t^2 + \cdots$$

is an additive function. Given $x\in K(X)$, let its Chern polynomial be $\prod_i(1+a_i t)$ (bearing in mind the splitting principal).

Def. (Todd Class). $Td(x) \triangleq \prod_i \dfrac{a_i}{1-e^{-a_i}}$ $(\in A(X)\otimes_{\mathbb{Z}}\mathbb{Q})$.

Def. (Chern Character) $ch(x) \triangleq rank(x) + \sum(e^{a_i}-1)$ $(\in A(X)\otimes_{\mathbb{Z}}\mathbb{Q})$.
(Note that rank is well-defined on $K_1(X)\cong K(X)$.)

We have, for $x,y\in K(X)$:

- $Td(x+y) = Td(x)\cdot Td(y)$
- $ch(x+y) = ch(x)+ch(y)$; $ch(x\cdot y) = ch(x)\,ch(y)$

§2. The Theorem

Let $f: Y \longrightarrow X$ be a proper morphism between non-singular quasi-projective varieties. Denote $T(X) \in A(X) \otimes \mathbb{Q}$ the Todd class of the tangent bundle of X.

Thm 2 (Grothendieck - Riemann - Roch)

$$\begin{array}{ccc} K(Y) & \xrightarrow{ch} & A(Y)\otimes\mathbb{Q} \\ \big\downarrow f_! & & \big\downarrow f_* \\ K(X) & \xrightarrow{ch} & A(X)\otimes\mathbb{Q} \end{array}$$

The lack of commutativity is measured by the formula

$$f_*(ch(y)\cdot Td(Y)) = ch(f_!(y))\cdot Td(X)$$

$X = \mathrm{Spec}\, k$. Then $A(X) \cong \mathbb{Z} \cong$ | /), and taking ch over X is just counting the dim over k. Furthermore, if $\mathcal{F} \in Coh(Y)$, then $f_!(\mathcal{F}) = \sum (-1)^i [H^i(Y, \mathcal{F})]$. Thus

Cor. (Hirzebruch - Riemann - Roch)

$$\sum (-1)^i h^i(Y, \mathcal{F}) = \int_Y ch(\mathcal{F})\cdot Td(Y)$$

Special cases

(1). $Y = \mathbb{P}^r$, $X = pt$. $\mathcal{F} = \mathcal{O}(n)$.

In this case, let $x = c_1(\mathcal{O}(1))$. Then we have $c_t(Y) = (1+tx)^{r+1}$, and thus $T(\mathbb{P}^r) = \frac{x^{r+1}}{(1-e^{-x})^{r+1}}$, $ch(\mathcal{O}(n)) = e^{nx}$. Hence R-R in this case reads:

$$\begin{aligned} \binom{n+r}{r} = \chi(\mathcal{O}(n)) &\overset{?}{=} \text{deg } r \text{ term in } \frac{e^{nx}\cdot x^{r+1}}{(1-e^{-x})^{r+1}} \\ &= \text{deg } (-1) \text{ term in } \frac{e^{nx}}{(1-e^{-x})^{r+1}} \\ &= \mathrm{Res}\left(\frac{e^{nx}}{(1-e^{-x})^{r+1}}dx\right) \\ &= \mathrm{Res}\, \frac{dy}{(1-y)^{n+1}\, y^{r+1}} \qquad (y = 1-e^{-x}) \\ &= \text{deg } r \text{ term in } (1-y)^{-(n+1)} \\ &= (-1)^r \binom{-n-1}{r} \\ &= \binom{n+r}{r}. \end{aligned}$$

(2). Y = Curve $X = \operatorname{Spec} k$, $y = \gamma(\mathcal{O}_Y(D))$ with D a divisor. Then:

$$h^0(D) - h^1(D) = (ch(\mathcal{O}_Y(D)) \cdot Td(Y))_1$$
$$= \left((1+D) \cdot \frac{(-K)}{1-e^{K}}\right)_1$$
$$= (1+D)\left(1-\frac{K}{2}\right)$$
$$= \deg(D) - \tfrac{1}{2}\deg K$$

Letting $D=0$, $g = h^1(\mathcal{O}_Y)$, we see that $\deg K = 2g-2$, and thus

$$h^0(D) - h^1(D) = \deg D - (g-1)$$

(3). Y = Surface $X = \operatorname{Spec} k$, $y = \gamma(\mathcal{O}_Y(D))$, with D a divisor

$$h^0(D) - h^1(D) + h^2(D) = \left[\left(1+D+\frac{D^2}{2}\right)\left(1+\tfrac{1}{2}C_1(Y) + \tfrac{1}{12}(C_1^2+C_2)\right)\right]_2 \quad (C_i = C_i(TY))$$
$$= \tfrac{1}{2}D\cdot(D+C_1(Y)) + \tfrac{1}{12}(C_1^2+C_2)$$
$$= \tfrac{1}{2}D\cdot(D-K) + \tfrac{1}{12}(K^2+C_2)$$

Taking $D=0$, $P_a = \frac{1}{12}(K^2+C_2)$, which is Noether's formula.

§. Proof of Theorem

We reduce the proof to two special cases: (1). projection. (2). injection.

First reductions.

Lemma 1. Let $Z \xrightarrow{g} Y \xrightarrow{f} X$ be proper morphisms. Let $z \in K(Z)$

(1). If GRR is true for (g, z) and $(f, g_!(z))$, then it's true for (fg, z).

(2). If GRR is true for (fg, z) and $(f, g_!(z))$, $f_*: A(Y) \longrightarrow A(X)$ is injective, then it's true for (g, z).

Pf: (1). By assumption, we have

$$g_*(ch(z)\cdot Td(Z)) = ch(g_!(z))\cdot Td(Y) \qquad (1)$$

$$f_*(ch(g_!(z))Td(Y)) = ch(f_!(g_!(z)))\cdot Td(X) \qquad (2)$$

$$\begin{aligned} \Rightarrow (fg)_*(ch(z)\cdot Td(Z)) &= f_*(g_*(ch(z)\cdot Td(Z))) \\ &= f_*((ch(g_!(z))\cdot Td(Y)) \\ &= ch(f_!(g_!(z)))\cdot Td(X) \\ &= ch((fg)_!(z))\cdot Td(X) \end{aligned}$$

(2). By assumption

$$\begin{array}{ccc} (fg)_*(ch(z)\cdot Td(Z)) & = & (ch((fg)_!(z))\cdot Td(X) \\ \| & & \| \\ f_*(g_*(ch(z)\cdot Td(Z))) & & ch(f_!(g_!(z)))\cdot Td(X) \end{array}$$

and

$$f_*(ch(g_!(z))\cdot Td(Y)) = ch(f_!(g_!(z)))\cdot Td(X)$$

$\Rightarrow f_*(g_*(ch(z)\cdot Td(Z))) = f_*(ch(g_!(z))\cdot Td(Y))$. Since f_* is injective, we have

$$g_*(ch(z)\cdot Td(Z)) = ch(g_!(z))\cdot Td(Y) \qquad \square$$

Lemma 2. Let $f: Y \longrightarrow X$, $f': Y' \longrightarrow X'$ be proper morphisms, $y \in K(Y)$, $y' \in K(Y')$. If GRR is true for (f, y), (f', y'), then it's true for $(f\times f', y\otimes y')$.

Pf: We shall use the following Künneth type formula:

$$(f\times f')_!(y\otimes y') = f_!(y)\otimes f'_!(y')$$

$$(f\times f')_*(z\otimes z') = f_*(z)\otimes f'_*(z') \quad \forall z \in A(Y), z' \in A(Y')$$

Furthermore, for Chern characters, we have

$$\mathrm{ch}(y \otimes y') = \mathrm{ch}(y) \otimes \mathrm{ch}(y') \qquad \square$$

Now given $f: Y \longrightarrow X$ proper morphism between projective varieties, we can factor it as

$$\begin{tikzcd} Y \arrow[r, hook, "i"] \arrow[dr, "f"'] & X \times \mathbb{P}^r \arrow[d, "p"] \\ & X \end{tikzcd}$$

where i is a closed immersion and p is the projection. By lemma 1 (a), we are reduced to show for the following two cases:

(i). $Y = X \times \mathbb{P}^r \longrightarrow X$ is the projection

(ii). $Y \longrightarrow X$ is a closed immersion.

Projection Case.

By lemma 2, we are further reduced to the case $\mathbb{P}^r \longrightarrow \mathrm{pt}$ if we have $K(X) \otimes K(\mathbb{P}^n) \longrightarrow K(X \times \mathbb{P}^n)$ is surjective (GRR is trivially true for $X \longrightarrow X$), which we have dealt with as an example before. We list some basic K theory properties:

(1). $X' \hookrightarrow X$ closed subscheme. $U = X \setminus X'$. Then

$$K(X') \longrightarrow K(X) \longrightarrow K(U) \longrightarrow 0$$

is exact.

(2). $p: X \times \mathbb{A}^1 \longrightarrow X$. Then $p^!: K(X) \xrightarrow{\cong} K(X \times \mathbb{A}^1)$. Inductively $K(X) \xrightarrow{\cong} K(X \times \mathbb{A}^r)$.

Lemma 3. X: projective variety. $K(X) \otimes K(\mathbb{P}^r) \longrightarrow K(X \times \mathbb{P}^r)$ is surjective.

Pf: By induction on r. $r = 0$ trivial. By (1), we have a commutative diagram:

$$\begin{tikzcd} K(X \times \mathbb{P}^{r-1}) \arrow[r] & K(X \times \mathbb{P}^r) \arrow[r] & K(X \times \mathbb{A}^r) \arrow[r] & 0 \\ K(X) \otimes K(\mathbb{P}^{r-1}) \arrow[r] \arrow[u, "\text{(by inducition)}"'] & K(X) \otimes K(\mathbb{P}^r) \arrow[r] \arrow[u, "\varepsilon"'] & K(X) \otimes K(\mathbb{A}^r) \arrow[r] \arrow[u, "\cong \text{ (by (2))}"'] & 0 \end{tikzcd}$$

$\Rightarrow \varepsilon$ is surjective by diagram chasing $\square$

Cor. GRR is true for the projection $X \times \mathbb{P}^r \longrightarrow X$. $\square$

A Special Case of Injection.

In case $Y \hookrightarrow X$ is a divisor, any $y \in K(Y)$ is of the form $y = i^!(x)$, $x \in K(X)$, we can check GRR as follows.

For an injection, we have $0 \longrightarrow I_Y/I_Y^2 \longrightarrow \Omega_X|_Y \longrightarrow \Omega_Y \longrightarrow 0$. Dualizing gives $0 \longrightarrow TY \longrightarrow TX|_Y \longrightarrow E \longrightarrow 0$ where E is the normal bundle. Then

$$i_*(ch(y) \cdot Td(Y)) \overset{?}{=} ch(i_!(y)) \cdot Td(X)$$

$\text{L.H.S.} = i_*(ch(y) \cdot Td(E)^{-1} \cdot Td(TX|_Y)) = i_*(ch(y) \cdot Td(E)^{-1} \cdot i^* Td(X)) = i_*(ch(y) \cdot Td(E)^{-1})\, Td(X)$

Thus it suffices to show that

$$ch(i_!(y)) = i_*(ch(y) \cdot Td(E)^{-1})$$

In the special case where Y is a smooth divisor, by adjunction formula, $E \cong \mathcal{O}_Y(Y)$. Let $y = i^!(x)$. We have:

$$\text{l.h.s} = ch(i_! i^!(x)) = ch(x \cdot i_!(1)) = ch(x) \cdot ch(i_!([\mathcal{O}_Y]))$$

But we have $0 \longrightarrow \mathcal{O}_X(-Y) \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_Y \longrightarrow 0 \Rightarrow i_!([\mathcal{O}_Y]) = [\mathcal{O}_X] - [\mathcal{O}_X(-Y)]$, and thus

$$\text{l.h.s} = ch(x) \cdot (1 - e^{-Y})$$

On the other hand,

$$\begin{aligned} \text{r.h.s} &= i_*(ch(i^!(x)) \cdot Td(\mathcal{O}_Y(Y))^{-1}) = i_*(i^* ch(x) \cdot Td(i^* \mathcal{O}_X(Y))^{-1}) = i_*(i^*(ch(x) \cdot Td(\mathcal{O}_X(Y))^{-1})) \\ &= ch(x) \cdot Td(\mathcal{O}_X(Y))^{-1} \cdot i_*(1) \\ &= ch(x) \frac{1-e^{-Y}}{Y} \cdot Y \\ &= ch(x) \cdot (1 - e^{-Y}) \\ &= \text{l.h.s.} \end{aligned}$$

Cor. GRR is true for $Y \hookrightarrow Y \times \mathbb{P}^r$, $a \mapsto (a, p_0)$, where p_0 is a fixed point of $\mathbb{P}$.

Pf: Since this morphism is the composition of $id_Y : Y \longrightarrow Y$ and $p_0 \hookrightarrow \mathbb{P}^r$, by lemma 2, it suffices to check for the latter case. Let $Y = pt$, $K(Y) \cong \mathbb{Z}$, and it suffices to show for $1 \in K(Y)$. But $1 \in i^!(K(\mathbb{P}))$. Thus if $r = 1$, the result follows from the divisorial case above. Now we induct on r.

Let H be a hyperplane in $\mathbb{P}^r$ containing p_0. $H \cong \mathbb{P}^{r-1}$

$$i: \{p_0\} \overset{u}{\hookrightarrow} H \overset{v}{\hookrightarrow} \mathbb{P}^r$$

By induction hypothesis GRR is true for $(u, 1)$. Since v is the inclusion of a divisor, if $u_!(1) \in v^!(K(\mathbb{P}^r))$, GRR will be true for $(v, u_!(1))$ by the divisorial case. It then follows from lemma 1 (1) that GRR is true for $(i, 1)$.

Now in w.l.o.g. assume $H \cong \mathbb{P}^{r-1} = \{X_r = 0\}$, we have a Koszul complex:

$$0 \longrightarrow \mathcal{O}_H(-r+1) \longrightarrow \overset{r-1}{\oplus}\mathcal{O}_H(-r+2) \longrightarrow \cdots \longrightarrow \oplus^{r-1}\mathcal{O}_H(-1) \longrightarrow \mathcal{O}_H \longrightarrow \mathcal{O}_{p_0} \longrightarrow 0$$

$\Rightarrow$ $u_!(1) = \Sigma(-1)^i$ [invertible sheaves on H]. But any invertible sheaf $\mathcal{O}_H(n)$ comes from $v^*\mathcal{O}_{\mathbb{P}^r}(n)$. The result follows.

$\square$

Cor. To prove GRR in general, it suffices to show for $Y \longrightarrow X$ with $\operatorname{codim} Y >> \dim Y$ (we just need $\operatorname{codim} Y \geq \dim Y + 2$).

Pf: Take the composition $Y \longrightarrow X \longrightarrow X \times \mathbb{P}^N$ for $N >> 0$. By assumption GRR is true for the composition and $X \longrightarrow X \times \mathbb{P}^N$. Moreover $i_*: A(X) \longrightarrow A(X \times \mathbb{P}^N)$ is injective. By lemma 1 (b), GRR is true for $Y \longrightarrow X$. $\square$

Rmk: The general case of injection will follow from the divisorial case by blowing up X along Y, so that the strict transform of Y is a divisor. By the cor above, we only need to consider Y of large enough codimension, which is needed for technical reasons.