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Lie Groups and Representation Theory I

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§1. Representation of Finite Groups over $\mathbb{C}$

1). Every rep is completely reducible

$\Leftrightarrow$ $\mathbb{C}[G]$ is a semi-simple ring

$\Leftrightarrow$ $\mathbb{C}[G] = \bigoplus_{i=1}^{m} \mathrm{Mat}(n_i, \mathbb{C})$

(Shur's Lemma) If L is an irrep (simple module) over a ring A, then $\mathrm{End}_A(L)$ is a division ring. In particular, $A = \mathbb{C}[G]$, L irrep. then $\mathrm{End}_A(L)$ ($\mathrm{Hom}_G(L, L)$) $= \mathbb{C}$

Notion: Intertwiners = homomorphisms of rep's.

If V is a G-rep. $\Rightarrow$ $\mathbb{C}[G] \longrightarrow \mathrm{End}(V) \cong \mathrm{Mat}(n, \mathbb{C})$, $n = \dim V$.

If V is irreducible $\Rightarrow$ this homo. is surjective

Since $\mathbb{C}[G]$ is semi-simple $\Rightarrow$ $\exists$ finitely many irrep. $V_1, \cdots, V_m$

$\Rightarrow$ $\varphi: \mathbb{C}[G] \longrightarrow \bigoplus_{i=1}^{m} \mathrm{Mat}(n_i, \mathbb{C})$ and φ is an isomorphism of rings.

Cor. The regular rep. of G contains each irrep of G with multiplicity equal to its dimension. In particular, $\sum_{i=1}^{m} n_i^2 = |G|$

In fact $\mathbb{C}[G] \cong \bigoplus_{i=1}^{m} \underset{\substack{\uparrow \\ \text{Left} \\ \mathbb{C}[G]\text{-mod}}}{V_i} \otimes \underset{\substack{\uparrow \\ \text{Right} \\ \mathbb{C}[G]\text{-mod}}}{V_i^*}$, and this is an isomorphism as a $\mathbb{C}[G]$-bimodule

- Tensor Representations

If V is a rep. of G and W is a rep. of H, $G \times H \curvearrowright V \otimes W$

$$g \times h : v \otimes w \mapsto gv \otimes hw$$

If V, W are rep's of G, $G \xrightarrow{\Delta} G \times G \curvearrowright V \otimes W$

Ex. $V \otimes W$ is irrep of $G \times H$ iff V and W are irrep's (Only $/\mathbb{C}$)

Pf by characters: $\chi_{V \otimes W}(g, h) = \chi_V(g)\chi_W(h)$

Thus $(\chi_{V\otimes W}, \chi_{V \otimes W}) = \frac{1}{|G||H|} \sum_{g,h} \chi_V(g)\chi_W(h)\chi_V(g^{-1})\chi_W(h^{-1})$

$$= \left(\frac{1}{|G|}\sum_{g} \chi_V(g)\chi_V(g^{-1})\right) \cdot \left(\frac{1}{|H|}\sum_{h} \chi_W(h)\chi_W(h^{-1})\right)$$

$$= (\chi_V, \chi_V)(\chi_W, \chi_W)$$

$\Rightarrow 1 = (\chi_{V\otimes W}, \chi_{V\otimes W}) \Leftrightarrow (\chi_V, \chi_V) = 1$ and $(\chi_W, \chi_W) = 1$ $\square$

$V_i \otimes V_j \cong \oplus V_k^{C_{ij}^k} \Rightarrow$ get a commutative ring $\mathrm{Rep}(G)$ with basis elts irrep's :
$[V_i] \in \mathrm{Rep}(G)$. $[V_i]\cdot[V_j] = \sum_k C_{ij}^k [V_k]$
As an abelian group, $\mathrm{Rep}(G)$ is free with basis $[V_1], \cdots, [V_m]$
$(V_i \otimes V_j) \otimes V_k \cong V_i \otimes (V_j \otimes V_k) \Rightarrow$ multiplication is associative
$V_i \otimes V_j \cong V_j \otimes V_i \Rightarrow$ multiplication is commutative
$\mathbb{C} \otimes V \cong V \Rightarrow \exists$ unit elt.
(For abitrary A-modules, no natural action on $V\otimes W$, no natural trivial repn)
Also define $[V]+[W] = [V \oplus W]$. If $V \cong \bigoplus_{i=1}^m V_i^{k_i}$, $[V] = \sum_{i=1}^m k_i [V_i]$
$\mathrm{Rep}(G) \supseteq \mathrm{Rep}(G)^+$ positive semi-ring with elements $\{[V]\}$

One dim'l rep $\Leftrightarrow$
$$\begin{tikzcd} G \arrow[rr] \arrow[dr] & & \mathbb{C}^* \\ & \frac{G}{[G,G]} \arrow[ur] & \end{tikzcd}$$
$\frac{G}{[G,G]} = H_1(G, \mathbb{Z}) \triangleq H_1(G)$

If H is abelian, $H^\vee = \mathrm{Hom}(H, \mathbb{Q}/\mathbb{Z}) = \mathrm{Hom}(H, \mathbb{C}^*)$: Pontryagin dual.
In general, $H^\vee \cong H$ (non-canonically) but $H^{\vee\vee} \cong H$ canonically.
Ex. Find the isomorphism $H^{\vee\vee} \cong H$.
In fact $H^\vee \times H \to \mathbb{C}^*$ $(\psi, h) \mapsto \psi(h)$ is a perfect pairing

If $\dim V = 1$, $W \mapsto V \otimes W$ is an invertible operation (inv. functor on the category of rep.)

- Matrix Coefficients of Irrep

Fact: Shur's lemma $\Rightarrow$ Any irrep of a finite abelian group is 1-dim'l (fails /$\mathbb{R}$)

V, W irrep of G

- $V \not\cong W$. $W \xrightarrow{f} V$ linear map.
  $f' \triangleq \sum_{g\in G} g \cdot f \cdot g^{-1} \Rightarrow \forall h \in G,\ hf' = f'h \Rightarrow f' = 0$
  Choose basis in V and W : $\{v_i\}$ & $\{w_j\}$, then g acts on V by matrix $A(g)$ on V, matrix $B(g)$ on W. f acts by F

$\Rightarrow \sum_g A(g) F B(g^{-1}) = 0$

Choose $F = E_{jk} \Rightarrow \sum_g a_{ij}(g) b_{k\ell}(g^{-1}) = 0$

- $V = W$, $f: V \longrightarrow V \Rightarrow \sum_g g f g^{-1} = \lambda \mathrm{Id}_V$

$\Rightarrow \lambda \cdot \dim V = tr(\lambda \mathrm{Id}_V) = |G| tr(f) \Rightarrow \lambda = \frac{|G|}{\dim V} tr(f)$

W.r.t. $\{v_i\}$, $F = E_{jk} \Rightarrow \sum_g a_{ij}(g) \cdot a_{k\ell}(g^{-1}) = \delta_{jk} \delta_{i\ell} \frac{|G|}{\dim V}$

Thus consider $a_{ij}: G \longrightarrow \mathbb{C} \in Fun(G)$. $\dim Fun(G) = |G|$

Introduce an inner product on $Fun(G)$: $\alpha, \beta \in Fun(G)$

$(\alpha, \beta) = \frac{1}{|G|} \sum_{g \in G} \alpha(g) \beta(g^{-1})$

Thus $\{a_{ij}\}$ over all irrep in $Fun(G)$ is dual to $\{a_{ji}\}$ itself.

If V is an irrep with a G-inv. inner product (G-inv. inner products are unique up to scalar on V. ex.) Choose an orthogonal basis of V

$\Rightarrow a_{ji}(g^{-1}) = \overline{a_{ij}(g)}$

$\Rightarrow \sum_g a_{ik}(g) \overline{a_{j\ell}(g)} = \delta_{ij} \delta_{k\ell} \frac{|G|}{\dim V}$

$\Rightarrow \{a_{ij}\}$ forms an orthogonal basis of $Fun(G)$

Ex. Work out the examples 1) $G = \mathbb{Z}/n$ 2) $G = S_3$

Fourier transform for finite groups is to write a function in this basis

- Characters $\chi_V(g) \triangleq \sum_{i=1}^{\dim V} a_{ii}(g)$

V_i: irrep, $\chi_i = \chi_{V_i} \Rightarrow (\chi_i, \chi_j) = \delta_{ij}$ where $(\chi_V, \chi_W) \triangleq \frac{1}{|G|} \sum_{g \in G} \chi_V(g) \chi_W(g^{-1})$

$V \cong \oplus V_i^{n_i}$, $W \cong \oplus V_j^{k_j} \Rightarrow (\chi_V, \chi_W) = \sum_{i=1}^m n_i k_i$

$(\chi_V, \chi_V) = 1$ iff V is irrep

Now $\mathbb{C}[G] \cong \bigoplus_{i=1}^m Mat(n_i, \mathbb{C})$

$e_i \longleftrightarrow 1$: e_i: primitive central idempotent

$e_i = \frac{\dim V_i}{|G|} \sum_g \chi_i(g^{-1}) g$

Ex. Show that $e_i e_j = \delta_{ij} e_i$, $e_i^2 = e_i$ using orthogonality of matrix coefficients

Pf: $e_i e_j = \frac{\dim V_i \cdot \dim V_j}{|G|^2} \sum_g \chi_i(g^{-1}) \sum_h \chi_j(h^{-1})\, gh$

$= \frac{\dim V_i \dim V_j}{|G|^2} \sum_k \sum_{gh=k} \chi_i(g^{-1}) \chi_j(k^{-1}g)\, gh$

$= \frac{\dim V_i \dim V_j}{|G|^2} \sum_k \sum_{gh=k} \sum_{i,j} a_{ii}(g^{-1})\, b_{jj}(k^{-1}g)\, gh$ (w.r.t. some orthonormal basis of V_i & V_j)

$= \frac{\dim V_i \dim V_j}{|G|^2} \sum_k \sum_{gh=k} \sum_{i,j,\ell} a_{ii}(g^{-1})\, b_{j\ell}(k^{-1})\, b_{\ell j}(g)\, gh$

If $V_i = V_j$, $e_i^2 = \frac{(\dim V_i)^2}{|G|} \sum_{i,j,\ell} \sum_k a_{j\ell}(k^{-1})k \sum_g a_{ii}(g^{-1})\, a_{\ell j}(g)$

$= \frac{(\dim V_i)^2}{|G|} \sum_{i,j,\ell} \sum_k a_{j\ell}(k^{-1})k\, \delta_{i\ell}\, \delta_{ij} \frac{1}{\dim V_i}$

$= \frac{\dim V_i}{|G|} \sum_k a_{jj}(k^{-1})k = e_i$

If $V_i \neq V_j$. $e_i e_j = \frac{\dim V_i \dim V_j}{|G|^2} \sum_{i,j,\ell} \sum_k b_{j\ell}(k^{-1})k \sum_g a_{ii}(g^{-1})\, b_{\ell j}(g) = 0$ $\square$

Thus: $Z\mathbb{C}[G]$ = ring of class functions $\cong Z(\oplus \mathrm{Mat}(n_i, \mathbb{C})) \cong \oplus\, \mathbb{C} e_i$.

- Character table

Recall the representation ring $\mathrm{Rep}(G)$ with basis $[V_0] \cong \mathbb{C}$, $[V_1], \cdots, [V_{m-1}]$

$[V_i]\cdot[V_j] = \sum a_{ij}^k [V_k]$ Applying χ: ($\chi_{V_i} = \chi_i$, $\chi_{V\otimes W} = \chi_V \cdot \chi_W$)

$\Rightarrow \chi_i \chi_j = \sum a_{ij}^k \chi_k$

E.g. Character table of S_3

S_3 has 3 irreducible representations.

The trivial rep	The sign rep	The 2-dim'l repn ($S_3 \cong D_6$)
$V_0 \cong \mathbb{C}$	V_1	V_2

There are 3 conjugacy classes in S_3

	$(1)_1$	$(12)_3$	$(123)_2$
χ_0	1	1	1
χ_1	1	-1	1
χ_2	2	0	-1

$V_2 \otimes V_1 \cong V_2$ $\quad V_1 \otimes V_1 \cong V_0$ $\quad V_2 \otimes V_2 \cong ?$

$\chi_2^2 = 4,\ 0,\ 1 = \chi_0 + \chi_1 + \chi_2$

How is V_2 defined?

$S_3 \curvearrowright \{1,2,3\} \Rightarrow S_3 \curvearrowright \mathbb{C}^3 = \mathbb{C}e_1 \oplus \mathbb{C}e_2 \oplus \mathbb{C}e_3$. Then there is a 1-dim'l trivial subrepresential spaned by $e_1+e_2+e_3$. $V_2 = \{(x_1, x_2, x_3) \mid \sum_{i=1}^3 x_i = 0\}$

Similarly $S_n \curvearrowright \{1, 2, \cdots, n\} \Rightarrow S_n \curvearrowright \mathbb{C}^n$, $\mathbb{C}^n = \mathbb{C} \oplus V^{n-1}$

Claim: V^{n-1} is irreducible.

Pf: It suffices to show that $\|\chi_{\mathbb{C}^n}\|^2 = 2$, since we know already that $\mathbb{C} \hookrightarrow \mathbb{C}^n$.

$$\|\chi\|^2 = \frac{1}{n!}\sum_g |\chi(g)|^2 = \frac{1}{n!}\sum_{k=0}^{n} \#\{g \mid g \text{ fixes } k \text{ elements of } \{1,\cdots,n\}\}\, k^2$$

But $\#\{g \mid g \text{ fixes } k \text{ elements of } \{1,\cdots,n\}\} = \#\{g \mid g \text{ fixes certain } k \text{ elements}\}$

$- \#\{g \mid g \text{ fixes certain } (k+1) \text{ elements}\} + \#\{g \text{ fixes certain } k+2 \text{ elements}\} - \cdots$

$$= \binom{n}{k}(n-k)! - \binom{n}{k+1}(n-k-1)! + \binom{n}{k+2}(n-k+2)! - \cdots$$

$$= \frac{n!}{k!} - \frac{n!}{(k+1)!} + \frac{n!}{(k+2)!} - \cdots$$

$$= n!\left(\sum_{\ell=k}^{n} \frac{(-1)^{\ell-k}}{k!}\right)$$

$$\Rightarrow \|\chi\|^2 = \sum_{k=0}^{n} k^2 \sum_{\ell=k}^{n} \frac{(-1)^{\ell-k}}{k!} = 2. \quad \text{(Mathematica)}$$ □

- More generally, whenever $G \curvearrowright X \Rightarrow G \curvearrowright \mathbb{C}\{X\}$. To consider the decomposition it suffices to assume $G \curvearrowright X$ transitively. Then $X \cong G/H$, $H = \mathrm{Stab}_x$. This is a special case of induced representations of H on G. (the trivial rep. of H).

$\chi_{\mathbb{C}\{X\}}(g) = \#\{x \mid gx = x\} \Rightarrow \chi_{V^{n-1}}(g) = \#\{x \mid gx = x\} - 1.$

Another point to observe is that: if V is irred, $V \otimes V \supseteq \mathbb{C}$ iff $V \cong V^*$.

Pf: "$\Leftarrow$" $V \otimes V^* \xrightarrow{\langle,\rangle} \mathbb{C}$

"$\Rightarrow$" $\mathrm{Hom}(V \otimes V, \mathbb{C}) = \mathrm{Hom}(V, V^*)$. Now $\mathrm{Hom}(V \otimes V, \mathbb{C})^G = \mathrm{Hom}_G(V \otimes V, \mathbb{C}) \neq 0$

$\Rightarrow \mathrm{Hom}_G(V, V^*) = \mathrm{Hom}(V, V^*)^G \neq 0 \Rightarrow V \cong V^*$. □

- Any irreducible rep of S_n is self-dual, since $\forall g \in S_n$, g is conjugate to g^{-1}.

$\Rightarrow \chi_V(g) = \chi_V(g^{-1}) = \overline{\chi_V(g)} \Rightarrow \chi_V(g)$'s are real valued. $\Rightarrow \langle \chi_{\mathbb{C}}, \chi_{V \otimes V}\rangle = \langle \chi_{\mathbb{C}}, \chi_V^2\rangle$

$= \frac{1}{n!}\sum 1 \cdot \chi_V(g)^2 > 0 \Rightarrow \mathbb{C} \hookrightarrow V \otimes V$, thus $V \cong V^*$.

- Induction and Restriction

A, B rings, $B \hookrightarrow A$ $\qquad$ $B\text{-modules} \underset{\text{Res}}{\overset{\text{Ind}}{\rightleftarrows}} A\text{-modules}$

$$V \longmapsto A \underset{B}{\otimes} V$$

$$_BW \longleftarrow W$$

Induction is left adjoint to restriction : $\mathrm{Hom}_A(A\otimes V, W) \stackrel{\sim}{=} \mathrm{Hom}_B(V, {}_BW)$

In case G is a finite group, and H a subgroup.

$\mathbb{C}[H] \hookrightarrow \mathbb{C}[G]$: $H\text{-mod} \underset{\text{Res}}{\overset{\text{Ind}}{\rightleftarrows}} G\text{-mod}$

$G = \coprod_{i=1}^{k} g_i H \quad k = [G:H] \Rightarrow \mathbb{C}[G]$ is a free right $\mathbb{C}[H]$ module with basis $\{g_i\}$.

$\mathrm{Ind}(V) = \bigoplus_{i=1}^{k} g_i \otimes V \Rightarrow \dim(\mathrm{Ind}\, V) = \dim V \cdot [G:H]$

The G-action on $\mathrm{Ind}\, V$ is given by $g \cdot g_i \otimes v = g_j h \otimes v = g_j \otimes hv$

If $V = \mathbb{C}[H]$ is the regular rep. $\mathrm{Ind}(V) = \mathbb{C}[G] \underset{\mathbb{C}[H]}{\otimes} \mathbb{C}[H] = \mathbb{C}[G]$

$\mathrm{Ind}(\mathbb{C}) = \mathbb{C}[G] \underset{\mathbb{C}[H]}{\otimes} \mathbb{C}$: a basis is given by cosets representations $\{g_i \otimes 1\}$

$g \cdot g_i \otimes 1 = g g_i \otimes 1 = g_j h \otimes 1 = g_j \otimes h \cdot 1 = g_j \otimes 1$; also action on cosets are given by left multiplication $\Rightarrow \mathrm{Ind}(\mathbb{C}) \cong \mathbb{C}\{G/H\}$

Frobenius reciprocity: $\mathrm{Hom}_G(\mathrm{Ind} V, W) \cong \mathrm{Hom}_H(V, {}_HW)$

If V, W are irreducible representations of H and G.

$\dim$ L.H.S. = multiplicity of W in $\mathrm{Ind}\, V$; $\dim$ R.H.S. = multiplicity of V in $_HW$

If W irred. V = trivial rep. of $\{1\} \subseteq G$

$\Rightarrow \mathrm{Ind}(V) = \mathbb{C}[G]$, and $\dim$ L.H.S. = mult. of W in $\mathbb{C}[G]$ and $\dim$ R.H.S. $= \dim_{\{1\}}(\mathbb{C}, W) = \dim W$. Once again: multiplicity of W in $\mathbb{C}[G]$ is $\dim W$.

If H abelian $\mathrm{Ind}\,\mathbb{C}[H] = \mathbb{C}[G]$, which contains all irrep of G.

Moreover $\mathbb{C}[H] = \bigoplus_{i=1}^{|H|} V_i$ and $\mathrm{Ind}\,\mathbb{C}[H] = \bigoplus_{i=1}^{|H|} \mathrm{Ind}(V_i)$

W irrep of $G \Rightarrow W$ is contained in at least one $\mathrm{Ind}(V_i) \Rightarrow \dim W \leq [G:H]$.

Cor. Any irrep of D_{2n} or D_{2n}^* is at most 2-dim'l.

Pf: Both D_{2n} and D_{2n}^* contain index 2 (normal) subgroups.

$$\begin{tikzcd}
 & 1 & 1 & & \\
1 \arrow[r] & \mathbb{Z}/n \arrow[r] \arrow[u] & D_{2n} \arrow[r] \arrow[u] & \mathbb{Z}/2 \arrow[r] & 1 \\
1 \arrow[r] & \mathbb{Z}/2n \arrow[r] \arrow[u] & D_{2n}^* \arrow[r] \arrow[u] & \mathbb{Z}/2 \arrow[r] & 1 \\
 & \mathbb{Z}/2 \arrow[u] \arrow[r, equal] & \mathbb{Z}/2 \arrow[u] & & \\
 & 1 \arrow[u] & 1 \arrow[u] & &
\end{tikzcd}$$

□

If $1 \to A \to B \to C \to 1$ is a short exact sequence of groups

- B is called an extension of C by A
- An extension is split if $1 \longrightarrow A \xrightarrow{\alpha} B \underset{\gamma}{\overset{\beta}{\rightleftarrows}} C \longrightarrow 1$ $\exists \gamma$ s.t $\beta \circ \gamma = Id_C$

Then B is called the semi-direct product of A & C: $B = C \ltimes A$. It is determined by $C \to Aut(A)$, $\gamma(c) \cdot a = a' \gamma(c)$.

- D_n is a split extension, since D_n contains many order 2 elements (reflections) not in the kernel $\mathbb{Z}/n$ (rotations)
- D_n^* is not a split extension, since $D^*(n) \subseteq SU(2)$ doesn't contain any order two element other than $-I$, which is in $\mathbb{Z}/2n$.
- If n is odd $1 \to SO(n) \to O(n) \to \mathbb{Z}/2 \to 1$

$$-I \longleftarrow 1$$

and $-I \in Z(O(n)) \Rightarrow O(n) = SO(n) \times \mathbb{Z}/2$.

- Symmetric and Exterior Powers

V vector space over k, $\mathrm{char}\, k = 0$.

$V^{\otimes n}$: $S_n \curvearrowright V^{\otimes n}$: $\sigma(v_1 \otimes \cdots \otimes v_n) = v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}$

$S^n V$: the subspace of S_n-invariants

$\Lambda^n V$: the subspace of signed S_n-invariants

In particular, $V^{\otimes 2} = S^2 V \oplus \Lambda^2 V$, and if $\{v_1, \cdots, v_k\}$ is a basis of V, then $\{v_i \wedge v_j\}_{i<j}$ is a basis of $\Lambda^2 V$ and $\{v_i v_j\}_{i \leq j}$ is a basis of $S^2 V$.

But if $n > 2$, $V^{\otimes n} = S^n V \oplus \Lambda^n V \oplus$ other terms.

If $H \curvearrowright W \Rightarrow W = \oplus W_i^{n_i}$: isotypical decomposition. $W_i^{n_i}$ sits canonically inside W:

$$\begin{aligned} W_i \otimes Hom(W_i, W) &\longrightarrow W \\ (w\ ,\ \ f\ ) &\mapsto f(w) \end{aligned}$$

This is a homomorphism of rep's and the image is $W_i^{n_i}$.

Note that, however, if $n_i \geq 2$, the choices of $W_i^{n_i} = W_i \oplus W_i \oplus \cdots$ is not canonical.

E.g. $H = S_3 \curvearrowright V \Rightarrow V^{\otimes 3} = S^3 V \oplus \Lambda^3 V \oplus V'$ and $S^3 V$ consists of trivial rep's and $\Lambda^3 V$ consists of sign rep's. V' consists of the other rep V^2 of S_3.

In general, if $v_1, \cdots, v_k$ is a basis of V. $\{v_{i_1} \wedge \cdots \wedge v_{i_n}\}_{1 \leq i_1 < \cdots < i_n \leq k}$ is a basis for $\Lambda^n V$ ($n \leq k$), and $\{v_{i_1} \cdots v_{i_n}\}_{1 \leq i_1 \leq \cdots \leq i_n \leq k}$ is a basis of $S^n V$.

In particular, $\Lambda^k V$ is 1-dim'l, spanned by $v_1 \wedge \cdots \wedge v_k$.

Now $G \curvearrowright V \Rightarrow G \curvearrowright V^{\otimes n}$ $\quad g(v_1 \otimes \cdots \otimes v_n) = gv_1 \otimes \cdots \otimes gv_n$.

This action commutes with the S_n action $\Rightarrow G \curvearrowright S^n V$ and $\Lambda^n V$ respectively.

How to compute characters of $\Lambda^n V$ and $S^n V$?

$n=2$, choose a basis s.t. $g = \begin{pmatrix} \lambda_1 & & * \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. In the basis $\{v_i \wedge v_j\}_{i<j}$ $\quad g(v_i \wedge v_j)$
$= \lambda v_i \wedge v_j + \cdots$. $\Rightarrow Tr_{\Lambda^2 V}(g) = \sum_{i<j} \lambda_i \lambda_j$. Moreover $g^2 = \begin{pmatrix} \lambda_1^2 & & * \\ & \ddots & \\ 0 & & \lambda_n^2 \end{pmatrix} \Rightarrow tr_V(g^2) = \Sigma \lambda_i^2$
$\Rightarrow tr_{\Lambda^2 V}(g) = \sum_{i<j} \lambda_i \lambda_j = \frac{1}{2}((\Sigma \lambda_i)^2 - \Sigma \lambda_i^2) = \frac{1}{2}((tr_V(g))^2 - tr_V(g^2))$.

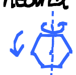
Similarly $tr_{S^2 V}(g) = \frac{1}{2}((tr_V(g))^2 + tr_V(g^2))$.

- Finite Subgroups of $SU(2)$

$SU(2) = \{U \in M_2(\mathbb{C}) \mid UU^* = I,\ \det U = 1\} = \left\{\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \middle| \ |a|^2 + |b|^2 = 1\right\}$

$\Rightarrow \exists \quad 1 \to \mathbb{Z}_2 \to SU(2) \xrightarrow[2:1]{\varphi} SO(3) \to 1$

Finite subgroups of $SO(3)$ are:

$\mathbb{Z}/n$	D_n	A_4	S_4	A_5
cyclic groups 	dihedral groups 	sym. group of 	sym. group of 	sym group of dotrahedron

Starting with $H \subseteq SO(3)$, we may construct $\varphi^{-1}(H) \subseteq SU(2)$. Note that D_n, A_4, S_4, A_5 have lots of elements of order 2, while $SU(2)$ has only 1 order 2 element, namely $-I$, thus $\varphi^{-1}(H)$ ($H = D_n, A_4, S_4, A_5$) are not direct products. Similarly $\varphi^{-1}(\mathbb{Z}/2n)$ cannot be $\mathbb{Z}/2n \times \mathbb{Z}/2$, otherwise there would be more than 1 order 2 elements

Thus we obtain a classification of finite subgroups of $SU(2)$

cyclic: $\mathbb{Z}/n = \left\{\begin{pmatrix} \zeta & 0 \\ 0 & \zeta^{-1} \end{pmatrix} \middle| \ \zeta = e^{\frac{2\pi i}{n}}\right\}$

binary dihedral group: D_{2n}^* (order $= 4n$) $= \left\langle \begin{pmatrix} \zeta^k & 0 \\ 0 & \zeta^{-k} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},\ \zeta = e^{\frac{\pi i}{n}} \right\rangle$

and 3 exceptional groups: $A_4^* = \varphi^{-1}(A_4)$, $S_4^* = \varphi^{-1}(S_4)$, and $S_5^* = \varphi^{-1}(S_5)$.

An interesting fact: $SU(2)/A_5^*$ is the Poincaré homology 3-sphere.

$H_1(SU(2)/A_5^*) = 0$ since $[A_5^*, A_5^*] = A_5^*$ (note that $[A_5^*, A_5^*]$ is a normal subgroup of A_5^* of index at most 2 since $\varphi([A_5^*, A_5^*]) = [\varphi(A_5^*), \varphi(A_5^*)] = [A_5, A_5] = A_5$, and the sequence $1 \to \mathbb{Z}/2 \to A_5^* \to A_5 \to 1$ cannot split.)

$\Rightarrow \pi_1(SU(2)/A_5^*) = A_5^* \Rightarrow H_1 = \pi_1/[\pi_1, \pi_1] = \{0\}$. Moreover it's oriented since $SU(2)$ is connected and each group element action is homotopic to identity. $\Rightarrow H_1 \cong H_2 = \{0\}$, $H_3 = H_1 = \mathbb{Z}$.

• McKay Correspondence

$G \subseteq SU(2)$ a finite subgroup, then G can only be:

$\mathbb{Z}/n$, D_n^* , A_4^* , S_4^* , A_5^*

Given G, we may construct a graph $\Gamma = \Gamma(G)$ called the McKay graph.

Vertices: irrep's

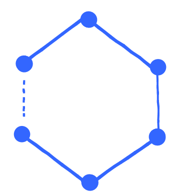
Edges: if $V_i \subseteq V_j \otimes V$, where V is the standard rep of $SU(2)$ on $\mathbb{C}^2$ restricted to G, then connect V_i with V_j with an edge.

$(V_i \subseteq V_j \otimes V \Rightarrow 0 \neq \mathrm{Hom}_G(V_i, V_j \otimes V) = \mathrm{Hom}_G(V_i \otimes V^*, V_j) = \mathrm{Hom}_G(V_i \otimes V, V_j))$

The last step holds since for $SU(2)$ (and its subgroups) $V \cong V^*$ $\ulcorner$ $V \otimes V = S^2V \oplus \Lambda^2 V$ but $\Lambda^2 V = \mathbb{C}$; or can be seen as follows: $\forall g \in SU(2)$, $g \sim \begin{pmatrix} \lambda & \\ & \lambda^{-1} \end{pmatrix}$ $\lambda^{|G|} = 1 \Rightarrow \lambda^{-1} = \lambda^*$. $\Rightarrow \chi_V(g) = \lambda + \lambda^*$. $\chi_{V^*}(g) = \overline{\lambda + \lambda^*} = \lambda^* + \lambda = \chi_V(g)$ $\lrcorner$

V is irreducible iff G is nonabelian (not $\mathbb{Z}/n$)

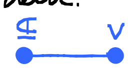
If G is abelian, the McKay graph is like:



Prop. The graph G is connected.

Pf: First of all, we may assume G is non-abelian. since the only abelian cases have connected graphs as above.

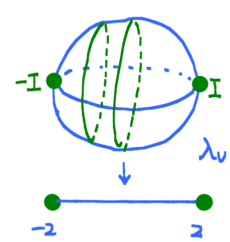
Thus V is an irrep and $\mathbb{C}$ and V are connected.



If $V_i \in$ this component $\Rightarrow V_i - V_j - V_k \cdots - V$ and $V_i \subseteq V_j \otimes V \subseteq V_k \otimes V \otimes V \subseteq \cdots \subseteq V \otimes \cdots \otimes V = V^m$ for some m. $\Rightarrow (\chi_{V_i}, \chi_V^m) \neq 0$

If $V_i \notin$ this component, $(\chi_{V_i}, \chi_V^m) = 0$, $\forall m$. Otherwise, for some m, $(\chi_{V_i}, \chi^m) \neq 0$ $V_i \subseteq V_j \otimes V$ for some $V_j \subseteq V^{m-1}$; similarly, $V_j \subseteq V_k \otimes V$ for some $V_k \subseteq V^{m-2}, \dots,$ $V_s \subseteq V_t \otimes V$ for some $V_t \subseteq V$ and $V_t \subseteq \mathbb{C} \otimes V \Rightarrow V_i - V_j - V_k - \cdots - V_s - V_t - \mathbb{C}$ contradiction.

Now $\chi_V(g) = \lambda + \lambda^{-1} \in [-2, 2]$, where $\chi_V : G \longrightarrow [-2, 2]$.



$0 = (\chi_i, \chi_V) = \frac{1}{|G|} \sum_g \chi_i(g) \chi_V(g)^m = \frac{1}{|G|} \sum_g \chi_i(g) (\lambda_g + \lambda_g^{-1})$

$\Rightarrow 0 = \sum_{g \in G} \chi_i(g) \left(\frac{\lambda_g + \lambda_g^{-1}}{2}\right)^m = 1 + (\pm 1)(-1)^m + \sum_{g \neq \pm I} \chi_i(g) \left(\frac{\lambda_g + \lambda_g^{-1}}{2}\right)^m$

But $g \neq \pm I$, $\left(\frac{\lambda_g + \lambda_g^{-1}}{2}\right)^m \to 0 \Rightarrow 0 = 1 \pm (-1)^m + o(1)$, $\forall m$, contradiction. $\square$

Now if $(V_i \hookrightarrow V_j \otimes V)$ has multiplicity ≥ 2 (which must be equal to the multiplicity of

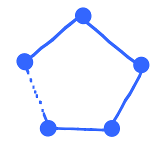
$(V_j \hookrightarrow V_i \otimes V)$, since $\mathrm{Hom}(V_i, V_j \otimes V) \cong \mathrm{Hom}(V_i \otimes V^*, V_j) \cong \mathrm{Hom}(V_i \otimes V, V_j)$).

$\Rightarrow$ (assuming $\dim V_i \geq \dim V_j$) $2\dim V_i \leq 2\dim V_j \Rightarrow \dim V_i = \dim V_j$ and $V_i \otimes V \cong V_j \oplus V_j$

$V_j \otimes V \cong V_i \oplus V_i$, i.e. only i and j are connected $\Rightarrow V_i, V_j$ are the only rep's of G

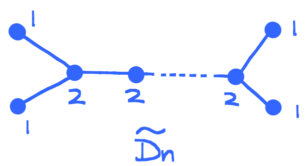
$\Rightarrow G = \{1\}$ or $\mathbb{Z}/2$, whose McKay graph is like:

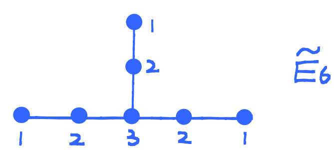
and can be realized as extreme cases of $\widetilde{A}_n$. $\leftrightarrow \mathbb{Z}/n$.

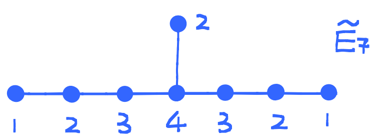


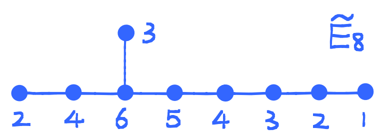
$\widetilde{A}_n$

There are other graphs:







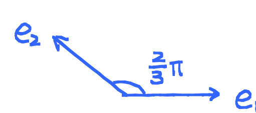


Thm. The only graphs Γ with weight numbers d_i assigned to vertex i satisfying $2d_i = \sum_{i-j} d_j$ are the graphs listed above

Pf: To each graph satisfying the equation we assign a vector space $\mathbb{R}^\Gamma$ with basis $\{e_i\}$ where i stands for vertices, and inner product $(e_i, e_j) = \begin{cases} 2 & i=j \\ -1 & i-j \\ 0 & \text{otherwise} \end{cases}$

For instance $\Gamma = \bullet$, $\mathbb{R}^\Gamma = \mathbb{R}e_1$, $(e_1, e_1) = 2$

$\Gamma = \bullet\!\!-\!\!\bullet$, $\mathbb{R}^\Gamma = \mathbb{R}e_1 \oplus \mathbb{R}e_2$



Lemma: The inner product for a McKay graph is positive semi-definite with null space spanned by $w_0 = \sum_i d_i e_i$.

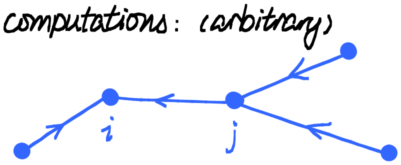
Indeed, $(w_0, e_j) = \sum_i (d_i e_i, e_j) = \sum_{i-j} d_i (e_i, e_j) + d_j (e_j, e_j) = 2d_j - \sum_{i-j} d_i = 0$

$\Rightarrow \forall v \in \mathbb{R}^\Gamma, (w_0, e_j) = 0$.

Now assign each edge an orientation to keep track of computations: (arbitrary)

Take $w = \sum x_i e_i$, then

$0 \leq \sum_{i \to j} d_i d_j \left(\frac{x_i}{d_i} - \frac{x_j}{d_j}\right)^2$



$$= \sum_{i\to j} d_i d_j \left(\frac{x_i^2}{d_i^2} - 2\frac{x_i x_j}{d_i d_j} + \frac{x_j^2}{d_j^2}\right)$$
$$= \sum_{i\to j} \left(\frac{d_j}{d_i} x_i^2 - 2x_i x_j + \frac{d_i}{d_j} x_j^2\right)$$
$$= \sum_{i\to j} \frac{d_j}{d_i} x_i^2 - 2\sum_{i\to j} x_i x_j + \sum_{j\to i} \frac{d_i}{d_j} x_j^2$$
$$= 2\sum_{i\to j} \frac{d_j}{d_i} x_i^2 - 2\sum_{i\to j} x_i x_j$$
$$= 2\sum_i x_i^2 - 2\sum_{i\to j} x_i x_j = (\omega, \omega)$$

$\Rightarrow (\omega, \omega) \geq 0$ and "=" iff $\frac{x_i}{d_i} = \frac{x_j}{d_j}$ i.e. $\omega = \lambda \omega_0$ $\square$ of lemma.

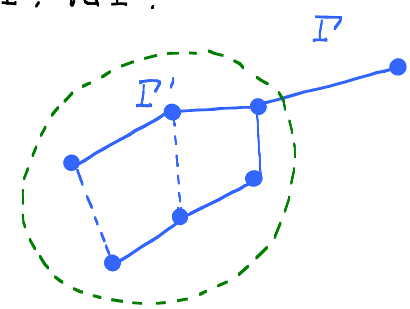
Lemma: If the connected Γ contains a proper subgraph Γ' which is a McKay graph, then the inner product on $\mathbb{R}^\Gamma$ is indefinite.

Indeed, there are two cases to consider:

(1) If Γ contains a vertex not in Γ', say $i \in \Gamma$, $i \notin \Gamma'$.

$\omega_0 \triangleq \sum_{j\in\Gamma'} d_j e_j$ and we have

$(\omega_0, \omega_0) \leq 0$ (there might be edges omitted between vertices of Γ' inside Γ, which contribute to multiples of (-1)'s.)



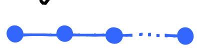
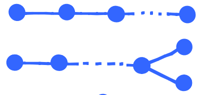
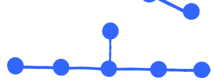
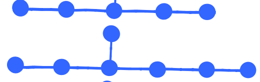
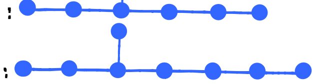
Take $\omega = \omega_0 + \varepsilon e_i$, $\varepsilon > 0$, then

$$(\omega, \omega) = \underset{(\leq 0)}{(\omega_0, \omega_0)} + 2\underset{(<0)}{(\omega_0, e_i)} \cdot \varepsilon + 2\varepsilon^2$$

$(\omega, \omega) < 0$ when $\varepsilon \ll 1$.

(2). All vertices are in Γ', but some edges are omitted. In this case, $(\omega_0, \omega_0) < 0$, since the omitted edges give back multiples of (-1)'s. $\square$ of lemma.

Thm. Any (simply-laced) connected graph is among:

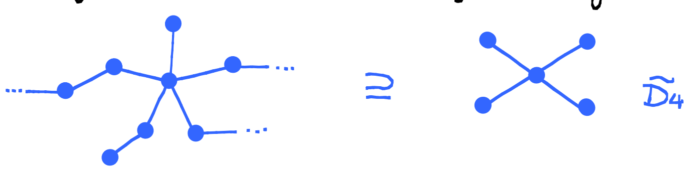
Dykin	Affine (McKay)	Indefinite
A_n : 	$\tilde{A}_n$	
D_n : 	$\tilde{D}_n$	($\mathbb{R}^\Gamma$, (,) indefinite)
E_6 : 	$\tilde{E}_6$	
E_7 : 	$\tilde{E}_7$	
E_8 : 	$\tilde{E}_8$	

For Dykin graphs, $(\mathbb{R}^{\Gamma}, (\ ,\ ))$ is positive definite; for affine graphs, $(\mathbb{R}^{\Gamma}, (\ ,\ ))$ is positive semi-definite. (Dykin graphs are obtained from affine ones by removing one of their weight one vertices)

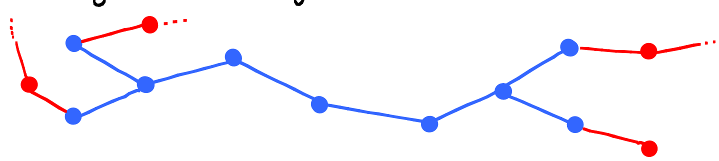
Pf: By its definition. Dykin graphs assign positive definite inner products on $\mathbb{R}^{\Gamma}$ since $\mathbb{R}^{\Gamma}$ is a subspace of a positive semi-definite space transversal to ω_0 we constructed. We will show that if any graph Γ which is neither Dykin nor McKay, then Γ contains properly a McKay graph Γ'. Then by the previous lemma, the inner product associated with Γ is indefinite.

(i). If Γ contains a cycle, then it contains $\tilde{A}_n$ properly.

(ii). If Γ contains a vertex of velancy ≥ 4, then it contains $\tilde{D}_4$ properly.

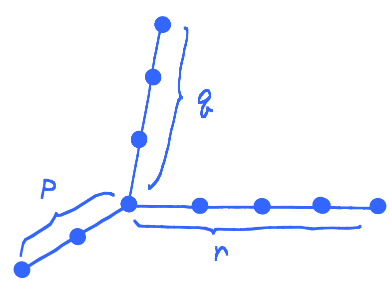


(iii). If Γ contains two velancy 3 vertices, we may find a path connecting them, and any such path gives rise to a $\tilde{D}_n$.



(iv). If Γ contains only 1 vertex of velancy 3

Let the number of vertices on each edge be p, q, r and without loss of generality, assume that $p \leq q \leq r$



p	q	r	Results
2	2	≥ 2	it's D_n of Dykin
2	2	3, 4, 5	it's E_6, E_7, or E_8 of Dykin
2	3	6	it's $\tilde{E}_8$,
2	3	≥ 7	it contains $\tilde{E}_8$ properly
2	4	4	it's $\tilde{E}_7$
2	4	≥ 5	it contains $\tilde{E}_7$ properly
2	≥ 5	≥ 5	it contains $\tilde{E}_7$ properly
3	3	3	it's $\tilde{E}_6$

Note the fact here:

$$\begin{cases} \frac{1}{p} + \frac{1}{q} + \frac{1}{r} > 1 & \text{Dykin} \\ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1 & \text{McKay} \\ \frac{1}{p} + \frac{1}{q} + \frac{1}{r} < 1 & \text{Indefinite} \end{cases}$$

≥ 3	≥ 3	≥ 3	it contains $\widetilde{E}_6$ properly.

$\square$ of thm.

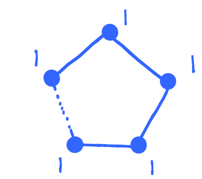
In summary, we have done:

A finite subgroup of $SU(2)$ $\longrightarrow$ A McKay graph labelling rep's $\longrightarrow$ positive semi-definite inner product on $\mathbb{R}^{\Gamma}$.

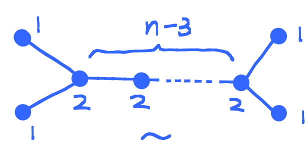
And for arbitrary graphs we have classified them by their associated inner product on $\mathbb{R}^{\Gamma}$.

$$\begin{cases} \text{Dykin: } A_n,\ D_n,\ E_6,\ E_7,\ E_8, \text{ with associated inner product positive definite} \\ \text{McKay: } \widetilde{A}_n,\ \widetilde{D}_n,\ \widetilde{E}_6,\ \widetilde{E}_7,\ \widetilde{E}_8, \text{ with associated inner product positive semi-definite.} \\ \text{Others: with associated inner product indefinite.} \end{cases}$$

Thus we conclude that to each finite subgroup of $SU(2)$, the associated graphs are precisely the MaKay graphs. Counting the order of group and use the relation that $|G| = \sum_{i=1}^{m} d_i^2$, we see the correspondence is like:

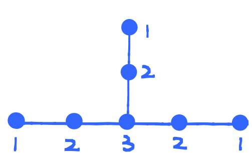


$\widetilde{A}_n$ $\quad C_n$ $(n \geq 1)$

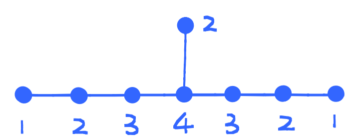


$\widetilde{D}_n$

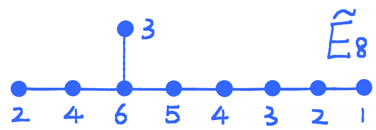
$\widetilde{D}_n$ $\quad D^*_{n-2}$ $(n \geq 4)$



$\widetilde{E}_6$ $\quad A^*_4$



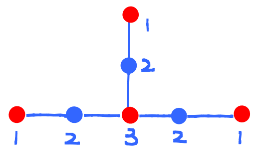
$\widetilde{E}_7$ $\quad S^*_4$



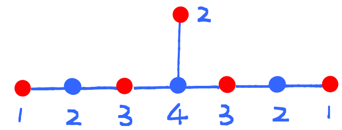
$\widetilde{E}_8$ $\quad A^*_5$

$\square$

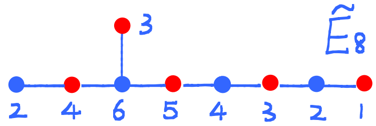
Now, if $z = -I_{2\times 2} \in G$, then on each irrep of G, z acts as the scalar ± 1 by Shur's lemma. Thus we can partition all irrep's of G into 2 classes, those which z acts as 1 or those as -1. Moreover if $i - j$, then $V_i \subseteq V_j \otimes V$. Hence if z acts as 1 on V_j, then it acts as -1 on V_i since $z = -\mathrm{id}$ on V, and if z acts as -1 on V_j, it acts as 1 on V_i. Those irrep's which z acts as 1 descend to irrep's of $SU(2)/\{1, z\} \cong SO(3)$



$\tilde{E}_6$ $\quad A_4^*$



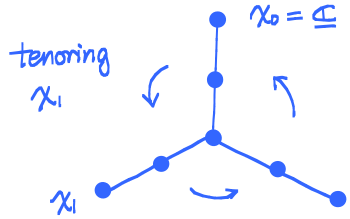
$\tilde{E}_7$ $\quad S_4^*$



$\tilde{E}_8$ $\quad A_5^*$

● : irrep's of A_4, S_4, A_5.

Moreover, tensoring with 1-dim'l irrep's gives automorphisms of the McKay graphs. For instance, tensoring with one of the non-trivial 1-dim'l irrep's of A_4^* gives a rotation of the graph:



- Dykin Diagrams and Weyl Groups

E.g. $\Gamma = A_n =$  . Let $\mathbb{R}^\Gamma \subseteq \mathbb{R}^{n+1} = \text{Span}\{\varepsilon_1, \cdots, \varepsilon_{n+1}\}$ $\quad \langle \varepsilon_i, \varepsilon_j \rangle = \delta_{ij}$

Let $e_i = \varepsilon_i - \varepsilon_{i+1}$ $\quad \Rightarrow \quad \langle e_i, e_i \rangle = 2 \quad \langle e_i, e_{i+1} \rangle = -1 \quad \mathbb{R}^\Gamma = \text{Span}\{e_1, \cdots, e_n\}$

Then $\mathbb{R}^\Gamma \subseteq \mathbb{R}^{n+1}$ is a hyperplane of codimension 1 $= \{ v = \sum v_i \varepsilon_i \mid \sum v_i = 0 \}$

Weyl group $W(\Gamma) \triangleq$ the group generated by reflections about the planes perpendicular to e_i $\quad i = 1, \cdots, n$.

These reflections extend to reflections of $\mathbb{R}^{n+1}$ (fixing $\sum \varepsilon_i$)

Let S_i be the reflection about the plane perpendicular to e_i :

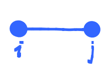
$S_i(x_1, \cdots, x_{n+1}) = \vec{x} - \langle \vec{x}, e_i \rangle e_i = (x_1, \cdots, x_{i+1}, x_i, \cdots, x_n)$

$\Rightarrow W(\Gamma) \cong S_{n+1}$ and the generators satisfy $S_i^2 = 1$, $S_i S_{i+1} S_i = S_{i+1} S_i S_{i+1}$,

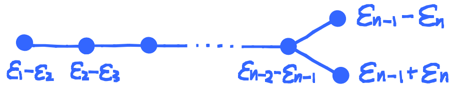
$S_i S_j = S_j S_i \quad (j \neq i-1, i+1)$

More generally, $\Gamma \rightsquigarrow W(\Gamma)$ has defining relations:

$S_i^2 = 1 \qquad S_i S_j S_i = S_j S_i S_j \qquad S_i S_j = S_j S_i$



E.g. D_n : $\mathbb{R}^{D_n} \triangleq \mathbb{R}^n = \mathrm{Span}\{\varepsilon_1, \cdots, \varepsilon_n\}$



Surely $S_n = W(A_{n-1}) \hookrightarrow W(D_n)$, corresponding to

The reflections by $\varepsilon_i - \varepsilon_{i+1}$ are given by:

$(x_1, \cdots, x_n) \longmapsto (x_1, \cdots, x_{i+1}, x_i, \cdots x_n)$, they generate S_n.

The reflection by $\varepsilon_{n-1} + \varepsilon_n$ is given by:

$(x_1, \cdots, x_n) \longmapsto (x_1, \cdots, x_{n-2}, -x_n, -x_{n-1})$

The composition of reflections by $(\varepsilon_{n-1} - \varepsilon_n)$ and $(\varepsilon_{n-1} + \varepsilon_n)$

$(x_1, \cdots, x_n) \longmapsto (x_1, \cdots, x_{n-2}, -x_{n-1}, -x_n)$

It can then be shown that $W(D_n)$ is the group of permutations of n letters and even number of sign changes. $\Rightarrow |W(D_n)| = 2^{n-1} \cdot n!$

We have the exact sequence $1 \to (\mathbb{Z}/2)^{n-1} \to W(D_n) \to S_n \to 1$ (forgetting about the sign changes!) The sequence splits since it contains a subgroup $W(A_{n-1}) \cong S_n$.

(Latter we will study a slightly better group G: $1 \to (\mathbb{Z}/2)^n \to G \to S_n \to 1$, which is the Weyl group of the B_n diagram:  : B_n Dykin diagram.)

Prop. $W(\Gamma)$ is finite.

Pf: Let $\mathbb{R}^\Gamma$ be the inner product space associated with Γ, and $\Lambda = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ the integral lattice spanned by $e_1, \cdots, e_n$. Then Λ is preserved by the actions of W:

$e_j \mapsto e_j - (e_i\ e_j) e_i$, $(e_i, e_j) \in \mathbb{Z}$.

Moreover, W is a group of isometries, thus preserves vectors of lenth^2 $2 = (e_i, e_i)$.

However, there are only finitely many length^2 2 vectors in the lattice.

Now, $W \curvearrowright \{w(e_1, e_2, \cdots, e_n)\}$ transitively, with stablizer $\{e\}$ $\Rightarrow |W| < \infty$. $\square$

- Real and Quaternionic Representations.

Recall that dim of an irrep divides $|G|$ (over $\mathbb{C}$, not true for $\mathbb{R}$)

If V is an irrep of $G/\mathbb{R}$, $\mathrm{End}_G(\mathbb{R})$ is a division algebra over $\mathbb{R}$

Thm. $\mathbb{R}, \mathbb{C}, \mathbb{H}$ are the only finite dimensional division algebras over $\mathbb{R}$.

$$\mathrm{End}_G(V) = \begin{cases} \mathbb{R} & : \text{real rep} \\ \mathbb{C} & : \text{complex rep} \\ \mathbb{H} & : \text{quaterion rep} \end{cases}$$

E.g.

(1). real rep's : trivial rep's of any G ; any rep's of S_n.

(2). complex rep's : $\mathbb{Z}/n \curvearrowright \mathbb{R}^2$ by rotation

(3). quaternion rep's : $G \subseteq SU(2) \hookrightarrow \mathbb{H}^* \curvearrowright \mathbb{H} \cong \mathbb{R}^4$, acting by left multiplication. It commutes with right multiplication by elements of $\mathbb{H}$. If G large, say, containing $Q_8 = \{1, \pm i, \pm j, \pm k\}$, then the rep must be quaternionic.

§2. Lie Groups

- Definitions

G is called a topological group if G is a group as well as a topological space, and these structures are compatible, i.e.

$$G\times G \longrightarrow G \quad (g,h)\mapsto gh \quad ; \quad G\longrightarrow G \quad g\mapsto g^{-1}$$

are continuous maps.

Ex. G is a topological group iff $G\times G\longrightarrow G$, $(g,h)\mapsto gh^{-1}$ is continuous.

Pf: "$\Leftarrow$" $G\longrightarrow G\times G$ ($g\mapsto (e,g)$) is continuous, thus the composition map $G\longrightarrow G\times G\longrightarrow G$ ($g\mapsto (e,g)\mapsto g^{-1}$ is continuous) Furthermore $G\times G\longrightarrow G\times G\longrightarrow G$ $(g,h)\mapsto (g,h^{-1})\mapsto g(h^{-1})^{-1}=gh$ is continuous.

"$\Rightarrow$" $G\times G\longrightarrow G\times G\longrightarrow G$: $(g,h)\mapsto (g,h^{-1})\mapsto g\cdot h^{-1}$ is a composition of continuous maps. $\square$

$L_g: G\longrightarrow G$ $h\mapsto gh$; $R_g: G\longrightarrow G$: $h\mapsto hg$ are homeomorphisms.

Moreover the conjugation map $G\times G\longrightarrow G$ $(g,h)\mapsto ghg^{-1}$ is a continuous map.

E.g.

1). G is discrete with discrete topology (every set is open)

2). G indiscrete. $\overline{\{1\}}\subseteq G$ is then a closed normal subgroup of G, and $G/\overline{\{1\}}$ is Hausdorff. More generally, H a subgroup of $G\Rightarrow \overline{H}$ is a closed subgroup of G $\overline{H}=H\Rightarrow G/H$ is a Hausdorff space.

Prop: $H\subseteq G$ locally closed, then H is closed.

Pf: "locally closed" means $\forall\, h\in H$. $\exists\, U\subseteq G$ open and $U\cap H$ is closed in U.

Take $\overline{H}\subseteq G$ $\Rightarrow \overline{H}=\coprod_i Hg_i$ $\quad g_i\in\overline{H}$. Furthermore $Hg_i\cong H$ homeomorphic by R_{g_i}

$\Rightarrow R_{g_i}(\overline{H})=\overline{H}\cdot g_i=\overline{Hg_i}$ But since R_{g_i} is also a homeomorphism of $\overline{H}$, thus $\overline{H}\cdot g_i=\overline{H}$ and Hg_i is dense in $\overline{H}$.

Next, since H is locally closed, H is open in $\overline{H}$, similarly for Hg_i. It follows that $H\cap Hg_i\neq\phi \Rightarrow Hg_i=H$, $\forall\, g_i \Rightarrow \overline{H}=H$. $\square$

For $\mathbb{R}$, closed subgroups are $\{0\}$, $\mathbb{R}$ or $\mathbb{Z}\cdot a$, $a>0$.

More examples are supplied by Lie groups. (e.g. $GL_n(\mathbb{R})$, $GL_n(\mathbb{C})$, $O_n(\mathbb{R})$, $\cdots$)

Profinite completions.

$G_1 \leftarrow G_2 \leftarrow G_3 \leftarrow \cdots$ $\quad \varphi_i : G_{i+1} \longrightarrow G_i$. We may assume φ_i's are surjective

Then $\varprojlim G_i$ is a topological group with profinite topology, the neighborhood of identity is given by sets of the form $\{1, 1, \cdots, 1, *, *, \cdots\}$

Ex. 1) G is homeomorphic to the Cantor set.

2). G is a topological group, and totally disconnected.

Examples of profinite groups: $Gal(\overline{F}/F)$ where F a field, with Krull topology.

$Gal(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \hat{\mathbb{Z}}$ $\quad$ ($G \to \hat{G}$. completion of a group with respect to finite quotients)

X: an algebraic variety, $\pi_1(X)$ is then profinite.

- Lie groups

– Topological group with smooth manifold structure, s.t

$G \times G \longrightarrow G$, $\ g, h \mapsto gh$; $\quad G \longrightarrow G \quad g \mapsto g^{-1}$ are smooth.

Prop. G: a Lie group. G_0: connected component of G containing 1. Then $G_0 \triangleleft G$ and G/G_0 is discrete and $1 \to G_0 \to G \to G/G_0 \to 1$. G is the disjoint union: $G = \coprod g_i G_0$ $\quad g_i \in$ the i-th component of G.

Pf: G_0 is normal since $\forall g \in G$, gG_0g^{-1} is connected, diffeomorphic to G_0 and contains $1 \Rightarrow gG_0g^{-1} = G_0$. The other statements follow. $\square$

3 Lie groups diffeomorphic to $S^1 \amalg S^1$

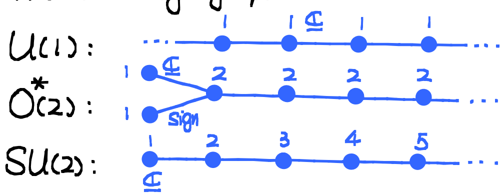
$SO(2) \times \mathbb{Z}/2 \qquad O(2) \quad$ and $\quad O^*(2)$ (These are the only 3 possibilities)

$$O^*(2): \quad \begin{array}{ccc} SU(2) & \subseteq & O^*(2) \\ \downarrow & & \downarrow \\ SO(3) & \subseteq & O(2) \end{array} \qquad \text{and} \quad O^*(2) = \left\{ \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \right\} \amalg \left\{ \begin{pmatrix} 0 & e^{i\varphi} \\ -e^{-i\varphi} & 0 \end{pmatrix} \right\}$$

Note that while $1 \to SO(2) \longrightarrow O(2) \longrightarrow \mathbb{Z}/2 \to 1$ is split ($-1 \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$)

the sequence $\quad 1 \longrightarrow U(1) \longrightarrow O^*(2) \longrightarrow \mathbb{Z}/2 \to 1$ is not. (there is only 1 order 2 element in $SU(2)$, namely $-I_{2\times 2}$, but lies in $U(1)$)

The McKay graph can be extended to infinite closed subgroups of $SU(2)$



Each irrep of the form $z \longmapsto z^n \in GL(1,\mathbb{C})$

1 irrep in each positive dim: $V \otimes S^n V \cong S^{n+1}V \oplus S^{n-1}V$

$S^n V$ irreducible of dim $= n+1$

Prop: The universal cover $\widetilde{G_0}$ of a connected Lie group G_0 has a natural Lie group structure. $\square$

E.g. $\widetilde{SO(2)} = \mathbb{R}$, $\widetilde{SO(3)} = SU(2)$. In general, $\pi_1(SO(n)) = \mathbb{Z}/2$ $(n \geq 3)$ and the univeral double cover of $SO(n)$ is $Spin(n)$. $Spin(n)$ has faithful $2^{[\frac{n}{2}]}$ representations.

$G \xrightarrow{L_g} G$: left multiplication by an element has no fixed point, by Lefschetz fixed point theorem, if we take $g \sim 1$.

Prop: G compact. non-discrete $\Rightarrow \chi(G) = 0$ $\square$

Def: $H \subseteq G$ is a Lie subgroup if it is both a subgroup and a submanifold.

$$\begin{tikzcd} H \times H \arrow[r, "m"] \arrow[d, hook] & H \arrow[d, hook] \\ G \times G \arrow[r, "m"] & G \end{tikzcd}$$

$\Rightarrow$ m smooth into H. In general, if A, B, C are smooth manifolds and $f(A) \subseteq B$, B a submanifold of C implies $A \longrightarrow B$ is smooth.

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr] & B \arrow[d, hook] \\ & C \end{tikzcd}$$

Rmk: Any closed subgroup of a Lie group is a Lie subgroup.

$t \mapsto (e^{2\pi i t}, e^{2\pi i \alpha t})$ α irrational is a subgroup of T^2 which is not closed.

E.g.

1). $GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$ are Lie groups

2). G, H Lie groups $\Rightarrow G \times H$ is a Lie group

3). Classical groups. $GL(n)$, $SL(n)$, $O(n)$, $U(n)$, $SU(n)$, $Sp(n)$ etc.

Prop. A normal discrete subgroup of a connected Lie group is central.

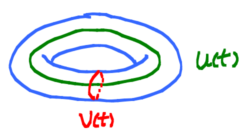
Pf: $H \trianglelefteq G$ discrete, normal. $\Rightarrow \forall g \sim 1$ (say in some small neighborhood $\mathcal{U}$ of 1)
$\Rightarrow ghg^{-1} \in H$. $ghg^{-1} \sim h$. $\Rightarrow ghg^{-1} = h$. This is true for all g, since $\bigcup_{n\in\mathbb{Z}} \mathcal{U}^n = G$
$\Rightarrow h \in Z(G) \Rightarrow H \subseteq Z(G)$. $\square$

$GL_n(\mathbb{Z}) \subseteq GL_n(\mathbb{R})$ discrete but not normal.

Cor. $\pi_1(G)$ is abelian (G connected).

Pf: Take the universal cover $\widetilde{G}$ of G $\Rightarrow 1 \to \pi_1(G) \to \widetilde{G} \to G \to 1$
$\Rightarrow \pi_1(G) \subseteq Z(\widehat{G})$ since it's normal $\Rightarrow \pi_1(G)$ abelian.

Another proof: Take $\alpha(t), \beta(t) \in \pi_1(G) \Rightarrow p(t,s) = \alpha(t)\beta(s) : T^2 \to G$
$\Rightarrow \alpha(t) = p_*(u(t))$ $\beta(t) = p_*(v(t))$ and $\pi_1(T^2)$ is abelian.

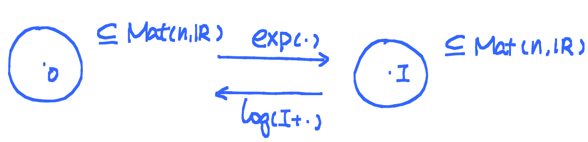


$\square$

Ex. $\pi_2(G) = 0$
$\pi_3(G) \cong \mathbb{Z}^n$ (torsion free)

$A \in Mat(n,\mathbb{R})$ (or $\mathbb{C}$), $\exp A \triangleq \sum_{n=0}^{\infty} \frac{A^n}{n!}$ (the sum converges uniformly and absolutely)

- $\exp(BAB^{-1}) = \sum_{n=0}^{\infty} \frac{(BAB^{-1})^n}{n!} = B \sum_{n=0}^{\infty} \frac{A^n}{n!} B^{-1} = B \exp A B^{-1}$
- $\exp(A+B) \neq \exp A \exp B$ unless $AB = BA$.
- $\exp A^t = (\exp A)^t$, $\exp(-A) = (\exp A)^{-1}$
- $\exp : Mat(n,\mathbb{R}) \to Mat(n,\mathbb{R})$ is real analytic (smooth) and $0 \mapsto I$
  $d\exp_0 = Id_{Mat(n;\mathbb{R})} \Rightarrow \exp$ is a diffeomorphism from a neighborhood of 0
  to a neighborhood of I. (Inverse: $\log(I+B) = \sum_{k\geq 1} \frac{(-1)^{k-1}}{k} B^k$)



$SL(n,\mathbb{R})$, $SO(n,\mathbb{R})$, $SL(n,\mathbb{C})$

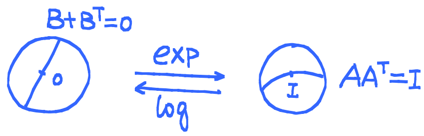
- $\det e^A = e^{\operatorname{tr} A}$. In particular if $\operatorname{tr} A = 0$, $\det e^A = 1$ and vice versa.

For $SO(n,\mathbb{R})$

$$AA^T = I \Rightarrow A = I + tB + O(t^2) \Rightarrow I = AA^T = I + t(B+B^T) + O(t^2)$$

$$\Rightarrow \frac{d}{dt}\Big|_{t=0} : \quad 0 = B + B^T$$

Conversely $B = -B^T \Rightarrow (\exp B)^T \exp B = \exp B^T \exp B = \exp(-B)\exp B = I$.



$U(n)$, $SU(n)$ (taking log gives hermitian and traceless hermitian matrices)

$Sp(n) \subseteq GL(n,\mathbb{H})$ $A\bar{A}^t = I$ ($\overline{a+bi+cj+dk} = a - bi - cj - dk$)

taking log gives $\{ B \mid B + \bar{B}^t = 0$, quaternionic matrices $\}$

In summary:

Group :	$GL(n,\mathbb{R})$	$SL(n,\mathbb{R})$	$O(n)/SO(n)$	$U(n)$	$SU(n)$	$Sp(n)$
Dimension :	n^2	n^2-1	$\frac{n(n-1)}{2}$	n^2	n^2-1	$2n^2+n$

Prop: All groups in the above table are Lie groups of that dimension. $\square$

Prop: $O(n)$, $SO(n)$, $U(n)$, $SU(n)$, $Sp(n)$ are compact Lie groups.

Pf: $AA^* = I \Rightarrow \sum |a_{ij}|^2 = 1 \Rightarrow |a_{ij}| \le 1 \Rightarrow$ These are bounded subsets of $\mathrm{Mat}(n)$.

Furthermore, they are closed since they are defined by 0's of polynomials. $\Rightarrow$ Compactness. $\square$

Thm. Any compact Lie group has a finite cover $\tilde{G}$ s.t. $\tilde{G} = T^n \times \prod_i G_i$, G_i among the list $\{ Spin(n)\ (n \ge 3), SU(n), Sp(n), E_6, E_7, E_8, F_4, G_2 \}$. $\square$

Here $O(n)$ ($Spin(n)$) are symmetries of $\mathbb{R}^n$, $\langle x, y\rangle = \sum x_i y_i$ $x_i, y_i \in \mathbb{R}$

$SU(n)$ ($U(n)$) are symmetrices of $\mathbb{C}^n$, $\langle x,y\rangle = \sum x_i \bar{y}_i$ $x_i, y_i \in \mathbb{C}$

$Sp(n)$ are symmetries of $\mathbb{H}^n$, $\langle x,y\rangle = \sum x_i \bar{y}_i$ $x_i, y_i \in \mathbb{H}$

Exceptional Lie groups are symmetries associated with $\mathbb{O}$: octonions.

- Vector fields on manifolds and Lie algebras

A smooth vector field on a manifold M is given locally in a coordinate chart U by $\zeta = \sum_{i=1}^{n} a_i(x) \frac{\partial}{\partial x^i}$, where $U\ (\cong \mathbb{R}^n) \subseteq M$.

We adopt Einstein's convention.

$$a^i(x)\frac{\partial}{\partial x^i} = a^i(x)\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} = b^j(y)\frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} \Rightarrow b^j(y) = \frac{\partial y^j}{\partial x^i} a^i(x(y))$$

Write $\vec{a} = \begin{pmatrix} a^1 \\ a^2 \\ \vdots \\ a^n \end{pmatrix}$ $\vec{b} = \begin{pmatrix} b^1 \\ b^2 \\ \vdots \\ b^n \end{pmatrix}$ $\Rightarrow \vec{b} = Jac \cdot \vec{a}$, where $Jac = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial y^n}{\partial x^1} & \cdots & \frac{\partial y^n}{\partial x^n} \end{pmatrix}$

Write $\frac{\partial}{\partial \vec{x}} = (\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n})$, $\frac{\partial}{\partial \vec{y}} = (\frac{\partial}{\partial y^1}, \cdots, \frac{\partial}{\partial y^n})$. Then $\frac{\partial}{\partial \vec{x}} = \frac{\partial}{\partial \vec{y}} \cdot Jac$.

Vector fields act on smooth functions $f \in C^\infty(M)$

$\zeta(f) \triangleq a^i(x)\frac{\partial f}{\partial x^i}(x)$, well-defined and independent of coordinate systems.

$\zeta: C^\infty(M) \longrightarrow C^\infty(M)$

1). ($\mathbb{R}$-linearity) $\zeta(af+bg) = a\zeta(f) + b\zeta(g)$, $a,b \in \mathbb{R}$, $f,g \in C^\infty(M)$

2). (Leibnitz rule) $\zeta(fg) = g\zeta(f) + f\zeta(g)$

i.e. ζ is a derivation of the algebra $C^\infty(M)$

For any algebra A over $\mathbb{k}$, we can define $Der(A) = \{\mathbb{k}\text{-linear derivations}\}$

$d \in Der(A)$. $d: A \longrightarrow A$ and $d(ab) = da \cdot b + a \cdot db$, $\forall a, b \in A$.

$Der(A)$ is a $\mathbb{k}$-vector space and if $d_1, d_2 \in Der(A) \Rightarrow [d_1, d_2] \in Der(A)$

Pf: $[d_1\ d_2](ab) = d_1 d_2(ab) - d_2 d_1(ab)$

$$\begin{aligned} &= d_1(d_2 a\cdot b + a\cdot d_2 b) - d_2(d_1 a\cdot b - a d_1 b) \\ &= d_1 d_2 a\cdot b + d_2 a\cdot d_1 b + d_1 a\cdot d_2 b + a\cdot d_1 d_2 b \\ &\quad - d_2 d_1 a\cdot b - d_1 a\cdot d_2 b - d_2 a\cdot d_1 b - a d_2 d_1 b \\ &= [d_1, d_2] a\cdot b - a\cdot [d_1, d_2] b \end{aligned}$$

Fact: Any derivation of $C^\infty(M)$ comes from a vector field.

i.e. $D \in Der(C^\infty(M)) \Leftrightarrow D(f) = \zeta(f)$ for a unique ζ

ξ, ζ vector fields, $[\xi, \eta](f) \triangleq \xi\eta(f) - \eta\xi(f)$ is then a derivation.

In a coordinate chart $\xi = a^i\partial_i$, $\zeta = b^j\partial_j \Rightarrow [\xi, \zeta] = (a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i})\partial_j$

In particular, $[\partial_i, \partial_j] = 0$

$Vect(M)$ = all smooth vector fields on M, an $\mathbb{R}$-vector space.

$[\ ,\ ] : Vect(M) \times Vect(M) \longrightarrow Vect(M)$

Def. A Lie algebra $\mathfrak{g}$ over a field $\mathbb{k}$ is a $\mathbb{k}$-vector space with a bilinear map $[\ ,\ ]$ (Lie bracket) : $\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$ which satisfies

1). skew-symmetric : $[a, b] = -[b, a]$, $\forall a, b \in \mathfrak{g}$ (char $\mathbb{k} \neq 2$)

2). Jacobi identity : $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0$

A Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ is a vector subspace closed under $[\ ,\ ]$.

Examples

0). $Vect(M)$. (needs to check Jacobi's identity)

1). A: any associative algebra over $\mathbb{k}$. Then A can be made into a Lie algebra A^L by defining $[a, b] \triangleq ab - ba$. Jacobi's identity is verified by a direct calculation.

For instance, $A = Mat(n; \mathbb{k})$, $A^L \triangleq \mathfrak{gl}(n, \mathbb{k}) = Mat(n, \mathbb{k})$. $\forall a, b \in \mathfrak{gl}(n, \mathbb{k})$, $tr([a, b]) = 0$. i.e. $[a, b] \in \mathfrak{sl}(n, \mathbb{k}) = \{$traceless matrices in $Mat(n, \mathbb{k})\}$. Thus $\mathfrak{sl}(n, \mathbb{k})$ is a Lie subalgebra of $\mathfrak{gl}(n, \mathbb{k})$

If char $\mathbb{k} \nmid n$, $\mathfrak{gl}(n, \mathbb{k}) = \mathfrak{sl}(n, \mathbb{k}) \oplus \mathbb{k}I$. But if char $\mathbb{k} \mid n$ this is not true, since $tr(a \cdot I) = n \cdot a = 0, \forall a \in \mathbb{k}$, since char $\mathbb{k} \mid n$.

2). A: any algebra over $\mathbb{k} \Rightarrow Der A$ is a Lie algebra

Thus $A \rightsquigarrow Der(A) \hookrightarrow End_{\mathbb{k}}(A) = \mathfrak{gl}(A)$. We don't need A to be associative or commutative, but only requires a product structure $A \times A \longrightarrow A$.

For instance $A = \mathbb{O}$ octonions, $A \cong \mathbb{R}^8$. $Der(A) = G_2$: an exceptional Lie algebra.

3). Classical Lie groups

Group :	$SL(n)$	$O(n)/SO(n)$	$U(n)$	$SU(n)$	$Sp(n)$
Lie(G) :	$\mathfrak{sl}(n)$	$\mathfrak{so}(n)$	$\mathfrak{u}(n)$	$\mathfrak{su}(n)$	$\mathfrak{sp}(n)$
	traceless $n \times n$ matrices	traceless antisymmetric matrices	antihermitian matrices	traceless anti-hermitian matrices	quaternionic anti hermitian matrices

Derivations / Vector fields are infinitesimal symmetries in the following sense

A: a finite dimensional algebra over $\mathbb{R}$, ($\mathbb{C}$), and $D \in \mathrm{Der}(A)$

Then $\exp D$ is a well-defined invertible linear map with inverse $\exp(-D)$.

$\exp D \in \mathrm{Aut}(A)$, in the sense that $\exp D\cdot(ab) = (\exp D\cdot a)b + a(\exp D\cdot b)$.

Pf:
$$\begin{aligned}\exp(D)(ab) &= \sum_{n=0}^{\infty}\frac{1}{n!}D^n(ab) = \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}(D^k a)\cdot(D^{n-k}b)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{1}{k!}D^k a\,\frac{1}{(n-k)!}D^{n-k}b\\ &= \sum_{k=0}^{\infty}\frac{1}{k!}D^k a\sum_{m=0}^{\infty}\frac{1}{m!}D^m b\\ &= \exp(D)(a)\cdot\exp(D)(b).\end{aligned}$$
□

In this case $D \rightsquigarrow$ a 1-parameter group of automorphisms of A: $\exp(tD)$, since $tD \in \mathrm{Der}(A)$; and since $[t_1D, t_2D] = 0 \Rightarrow \exp(t_1D)\exp(t_2D) = \exp((t_1+t_2)D)$. Hence $\mathbb{R} \longrightarrow \mathrm{Aut}(A)$, $t \longmapsto \exp(tD)$ is a group homomorphism.

In general, the concept of exp runs into trouble over finite fields $\mathbb{k}$ or $\dim A = \infty$. thus it doesn't make sense to apply this definition.

E.g. $a \in \mathrm{Mat}(n,\mathbb{R})$, then we can associate with a a derivation $D_a: \mathrm{End}(\mathrm{Mat}(n,\mathbb{R})) \cong \mathrm{End}(n^2,\mathbb{R})$: $D_a(b) \triangleq [a,b]$ (It's a derivation since $[a,[b,c]] = [[a,b],c] + [b,[a,c]]$, $\forall\, b,c \in \mathrm{Mat}(n,\mathbb{R})$, by Jacobi's identity).

Then $(\exp D_a) \in GL(n^2,\mathbb{R})$, and
$$\begin{aligned}(\exp D_a)(b) &= \sum_{n=0}^{\infty}\frac{1}{n!}D_a^n(b) = b + [a,b] + \tfrac{1}{2}[a,[a,b]] + \cdots\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}a^k b a^{n-k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(-1)^{n-k}}{k!(n-k)!}a^k b a^{n-k}\\ &= \sum_{k=0}^{\infty}\sum_{n=k}^{\infty}\frac{1}{k!}a^k\cdot b\cdot\frac{(-1)^{n-k}}{(n-k)!}a^{n-k}\\ &= \Big(\sum_{k=0}^{\infty}\frac{1}{k!}a^k\Big)b\Big(\sum_{\ell=0}^{\infty}\frac{(-a)^{\ell}}{\ell!}\Big)\\ &= \exp(a)\,b\,\exp(-a)\end{aligned}$$

Usually $D_a(b) \triangleq \mathrm{ad}_a(b)$, $\exp(a)b\exp(-a) = \mathrm{Ad}(\exp(a))(b) \Rightarrow e^{\mathrm{ad}a} = \mathrm{Ad}\,e^a$.

Tangent vector v.s. vector fields

A tangent vector to M at p is a linear map $\alpha: C^\infty(M) \longrightarrow \mathbb{R}$ s.t.
$$\alpha(fg) = f(p)\alpha(g) + g(p)\alpha(f)$$

$T_pM \triangleq \{\text{tangent vectors at } p\}$ is a vector space of $\dim = \dim M$.

$M \xrightarrow{\nu} N$ smooth map, $T_pM \xrightarrow{d\nu_p} T_pN$ by the commutative diagram:

$$\begin{tikzcd} \mathbb{R} & \\ C^\infty(M) \arrow[u, "\alpha"] & C^\infty(N) \arrow[l, "\nu^*"'] \arrow[ul, dashed, "d\nu_p(\alpha)"'] \end{tikzcd}$$

If $\nu: M \to N$ is a diffeomorphism, then one can transfer anything on M to N, including vector fields. $\xi \in \mathrm{Vect}(M) \Rightarrow \nu_*(\xi) \in \mathrm{Vect}(N)$.

G: Lie group. $\mathfrak{g} = T_1G$. If $\alpha \in T_1G$, we can define a vector field on G
$L_g: G \longrightarrow G$ left translation $\Rightarrow \xi_\alpha(g) = L_{g*}(\alpha)$. ξ_α is then left invariant.
Conversely, every left invariant vector field is defined by ξ_α for some $\alpha \in \mathfrak{g}$.
Moreover, if ξ_α, ξ_β are left invariant, then so is $[\xi_\alpha, \xi_\beta]$. Thus $[\xi_\alpha, \xi_\beta] = \xi_\gamma$ for some $\gamma \in \mathfrak{g}$. Define $[\alpha, \beta] \triangleq \gamma$. Then $\mathfrak{g}$ becomes a Lie algebra and $\mathfrak{g} \subseteq \mathrm{Vect}(G)$.
Now since $\mathfrak{g} = T_1G$ depends only on a neighborhood of 1 in G. (since multiplication is continuous, if U is a neighborhood of 1, then $\exists\, V$ a neighborhood of 1, $V \subseteq U$, $V = V^{-1}$ and $V^2 \subseteq U$) Thus if $\varphi: G_0 \longrightarrow G$ is a homomorphism of Lie groups as well as a covering map $\Rightarrow T_1G_0 \cong T_1G$.
Thus if G is a Lie group, G_0 its connected component containg 1, $\widetilde{G_0}$ the universal cover of G_0. $T_1G = T_1G_0 \cong T_1\widetilde{G_0}$ $(\cong \mathfrak{g})$

E.g. $G = GL(n, \mathbb{R}) \subseteq \mathrm{Mat}(n, \mathbb{R}) \cong \mathbb{R}^{n^2}$. $T_1G = \mathfrak{gl}(n, \mathbb{R}) \cong \mathrm{Mat}(n, \mathbb{R})$
We will see that the definition of $[A, B] = AB - BA$ agrees with that of left invariant vector field.
$A \in \mathfrak{gl}(n, \mathbb{R}) = \mathrm{Mat}(n, \mathbb{R}) \Longrightarrow \zeta_A(X) = X \cdot A$, $X \in GL(n, A)$
More precisely, $\zeta_A(X) = \sum_{i,j} X_{ik}A_{kj} \frac{\partial}{\partial X_{ij}}$

$$\begin{aligned} \Rightarrow \quad [\zeta_A, \zeta_B] &= [X_{ik}A_{kj}\partial_{ij},\ X_{uw}B_{wv}\partial_{uv}] \\ &= A_{kj}B_{wv}\big((\partial_{ij}X_{uw})X_{ik}\partial_{uw} - \partial_{uw}(X_{ik})X_{uw}\partial_{ij}\big) \\ &= A_{kj}B_{wv}(\delta_{iu}\delta_{jw}X_{ik}\partial_{uw} - \delta_{ui}\delta_{uk}X_{uw}\partial_{ij}) \\ &= A_{kj}B_{ju}X_{ik}\partial_{iv} - A_{kj}B_{wk}X_{iw}\partial_{ij} \\ &= A_{ku}B_{uj}X_{ik}\partial_{ij} - A_{uj}B_{ku}X_{ik}\partial_{ij} \end{aligned}$$

$$= X_{ik} [A, B]_{kj} \, \partial_{ij}$$

$$= \zeta_{[A,B]} \qquad \square$$

§3. Lie Algebras

The theory of Lie algebra works for more general fields than $\mathbb{R}$ or $\mathbb{C}$.

L: Lie algebra over a field $\mathbb{k}$, $\operatorname{char}\mathbb{k} = 0$.

Def: $L_1 \subseteq L$ is called a subalgebra if L_1 is a subspace and $[L_1, L_1] \subseteq L$

$I \subseteq L$ is called an ideal if $[I, L] \subseteq I$.

If I is an ideal, L/I is a Lie algebra : $[a+I, b+I] \triangleq [a,b]+I$.

We have a short exact sequence: $0 \to I \to L \to L/I \to 0$

E.g.

1). L is called abelian if $[L,L]=0$, and any subspace $I \subseteq L$ is an ideal.

2). $a \in L$, $\mathbb{k}a \subseteq L$ is an abelian subalgebra.

3). $\dim L = 2$. Choose x, y a basis of L.

$[x,y]=0 \Rightarrow L$ abelian

$[x,y] \neq 0 \Rightarrow [x,y] = ax+by$. (W.L.O.G assume $b \neq 0$)

$\Rightarrow [\frac{x}{b}, (ax+by)] = ax+by$, let $y' = ax+by$, $x' = \frac{x}{b}$.

$\Rightarrow [x', y'] = y'$

This is a Lie algebra (by checking the Jacobi's identity):

$[x,[x,y]] + [x,[y,x]] + [y,[x,x]] = [x,y] + [x,-y] + 0 = 0$

$[x,[y,y]] + [y[y,x]] + [y,[x,y]] = 0 + [y,-y] + [y,y] = 0+0+0=0$

The only ideals are 0, $\mathbb{k}y$, L.

In fact if $ax+by$ spans a nontrivial ideal $\Rightarrow [ax+by, y] = \lambda(ax+by)$

$\Rightarrow ay = \lambda ax + \lambda by \Rightarrow a = \lambda b$ and $\lambda a = 0 \Rightarrow a = 0$.

4). $\mathfrak{sl}(2,\mathbb{k})$: traceless 2×2 matrices

$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ $F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \Rightarrow \mathfrak{sl}(2,\mathbb{k}) = \mathbb{k}E \oplus \mathbb{k}H \oplus \mathbb{k}F$,

and $[H,E]=2E$, $[H,F]=-2F$, $[E,F]=H$.

Thus $\mathfrak{sl}(2,\mathbb{k})$ is a graded algebra, graded by the eigenvalue of $\mathrm{ad}(H)$ on $\mathfrak{sl}(2,\mathbb{k})$

Claim: $\mathfrak{sl}(2,\mathbb{k})$ has no non-trivial ideals other than 0 or itself.

$$\begin{tikzcd} E \arrow[d, bend left, "\mathrm{ad}F"] & 2 \\ H \arrow[u, bend left, "\mathrm{ad}E"] \arrow[d, bend left, "\mathrm{ad}F"] & 0 \\ F \arrow[u, bend left, "\mathrm{ad}E"] & -2 \end{tikzcd}$$

In fact. take $I \neq 0$ an ideal, $0 \neq x \in I$, $x = aE+bH+cF$

If $c \neq 0$, $[E[E,x]] = [E, -2bE + cH] = -2cE \in I$

$\Rightarrow E \in I \Rightarrow [E\ F] = H \in I \Rightarrow [H,F] = -2F \in I \Rightarrow I = \mathfrak{sl}(2,\mathbb{k})$

If $c=0,\ b\neq 0 \Rightarrow [E,\ X] = -2bE \in I \Rightarrow E \in I \Rightarrow I = \mathfrak{sl}(2,\Bbbk)$, as above

If $b, c = 0 \Rightarrow E \in I \Rightarrow I = \mathfrak{sl}(2,\Bbbk)$ as above.

Def. A Lie algebra L is called simple if $\dim L > 1$ and $0, L$ are the only ideals of L.

Thus, $\mathfrak{sl}(2,\Bbbk)$ is simple, in general $\mathfrak{sl}(n,\Bbbk)$ is simple $(n>1)$, by using a generalized argument as above.

Ideals in L:

Note that $[L,L]$ is an ideal of L (by Jacobi's identity). Thus if L is simple $[L,L] = L$ (simple $\Rightarrow$ non-abelian)

If $\varphi: L \longrightarrow L_1$ is a linear map and $[\varphi(x), \varphi(y)]_{L_1} = \varphi([x,y]_L)$, then φ is a homomorphism of Lie algebras and $\ker\varphi$ is an ideal of L. Ideals of L are in 1-1 correspondence with epimorphisms of Lie algebras.

$Z(L) \triangleq \{x \in L \mid [x, L] = 0\}$ is an (abelian) ideal of L. Thus if L is simple $Z(L) = 0$. e.g. $Z(\mathfrak{sl}(2,\Bbbk)) = 0$, $Z(\mathfrak{gl}(2,\Bbbk)) = \Bbbk I$, and $\mathfrak{gl}(2,\Bbbk) = \mathfrak{sl}(2,\Bbbk) \oplus \Bbbk I$.

If I, J are ideals, then so is $I+J$ and $[I, J]$ (by Jacobi's identity)

Fact: we cannot classify finite dimensional Lie algebras in general.

E.g. V, W vector spaces. $\varphi: \Lambda^2 V \longrightarrow W$ a surjective linear map.

Define $L = V \oplus W$, with $[\ ,\ ]$ defined as follows:

$v_1, v_2 \in V$, $w_1, w_2 \in W$. $[v_1, v_2] = \varphi(v_1 \wedge v_2)$; $[v_1, w] = 0$; $[w_1, w_2] = 0$.

Then $W \subseteq Z(L)$, $[L,L] \subseteq W$, $[L,[L,L]] = 0$ (Jacobi is then automatic)

Problem: Classify such Lie algebras up to isomorphism, or equivalently, classify such φ's upto the action of $GL(V) \times GL(W)$

Choose a basis for V and W respectively. $\varphi \in Mat(m \cdot \frac{n(n-1)}{2})$ (if $\frac{n(n-1)}{2} > m$, φ surj is then an open condition)

$\mathcal{M}(\varphi) \subseteq Mat(m \cdot \frac{n(n-1)}{2}) / GL(n) \times GL(m)$ and $\dim \mathcal{M}(\varphi) = m \cdot \frac{n(n-1)}{2} - n^2 - m^2$

(If $m=n$, then $\dim \mathcal{M}(\varphi) \gg 0$ if $n = m \gg 0$!)

Solvable Lie algebra

$L \rightsquigarrow L^{(1)} = [L, L] \rightsquigarrow L^{(2)} = [L^{(1)}, L^{(1)}] \rightsquigarrow \cdots \rightsquigarrow L^{(i+1)} = [L^{(i)}, L^{(i)}]$

L is called solvable if $L^{(i)} = 0$ for some i.

E.g.

1). The Lie algebra in e.g. 2) above is solvable.

$L = \Bbbk x \oplus \Bbbk y$, $[x, y] = y \Rightarrow L^{(1)} = [L, L] = \Bbbk y$ and $[L^{(1)}, L^{(1)}] = 0$

2). The Lie algebra of all upper triangular matrices.

$$t(n, \Bbbk) \triangleq \left\{ M \in \mathrm{Mat}(n, \Bbbk) \,\middle|\, M = \begin{pmatrix} * & * & * & \cdots & * \\ & * & * & \cdots & * \\ & & * & \cdots & * \\ & & & \ddots & \vdots \\ & & & & * \end{pmatrix} \right\}$$

Properties:

1). If $I \subseteq L$, an ideal, is solvable and L/I is solvable $\Rightarrow L$ is solvable.

In fact, $(L/I)^{(i)} = 0 \Rightarrow L^{(i)} \subseteq I$. $I^{(j)} = 0 \Rightarrow L^{(i+j)} \subseteq I^{(j)} = 0$.

2). If I, J are solvable $\Rightarrow I + J$ is solvable.

In fact, $0 \to I \to I + J \to J/I\cap J \to 0$ and I, $J/I\cap J$ are solvable.

2) $\Rightarrow$ Any finite dimensional Lie algebra contains a unique maximal solvable ideal, called its radical, and $\mathrm{Rad}(L/\mathrm{Rad}L) = 0$

Digression: A: a finite dimensional, associative algebra over $\Bbbk$.

The Jacobson radical of A: $J \triangleq \bigcap$ all maximal left ideals

$= $ maximal nilpotent 2-sided ideal

$\Rightarrow 0 \to J \to A \to A/J \to 0$ and A/J is semi-simple.

A is called semi-simple if $J = 0$.

For example $A = \mathbb{R}[G]$, $|G| < \infty$, A is semi-simple.

Semi-simple A's are direct sums of matrix algebras over $\Bbbk$-division algebras.

In Lie algebra cases, semi-simple Lie algebras can also be classified.

• Representations of Lie Algebras

L: finite dimensional Lie algebra over $\Bbbk$.

In case L arises as the Lie algebra of a Lie group. Then a homomorphism $G \to GL(V)$

$\rightsquigarrow L = \mathrm{Lie}(G) \longrightarrow \mathfrak{gl}(V) \cong \mathrm{Mat}(n, \Bbbk)$

Def: A rep. of L is a homomorphism $L \longrightarrow \mathfrak{gl}(V)$ of Lie algebras. Or equivalently $L \otimes V \longrightarrow V$, $x \otimes v \longmapsto xv$, $[x, y] \otimes v \longmapsto x(yv) - y(xv)$. Note that xy is not necessarily an element of L.

We shall study the category of L-modules (L-rep's)

Basic properties: (compare with finite group representations)

1). Trivial rep: $L \longrightarrow 0 \subseteq \mathfrak{gl}(V)$, $V \cong \Bbbk$ $\quad x \cdot v = 0$, $\forall x \in L$, $v \in V$.

2). $V \xrightarrow{\varphi} W$ a homomorphism of L-modules, i.e. φ commutes with L-actions. $\varphi(x \cdot v) = x \cdot \varphi(v)$ $\forall v \in V$. Then, $\ker\varphi$, $\mathrm{Im}\varphi$ are L-submodules of V and W respectively.

3). $0 \longrightarrow V \longrightarrow W$, (submodule), W/V is an L-module.

4). $V \oplus W$ is an L-module

5). $V \otimes W$ is an L-module

In case of (Lie) groups, $G \circlearrowright V \otimes W$: $g(v \otimes w) = gv \otimes gw$. Now Lie algebra is an infinitesimal approximation of Lie groups (linear approximation), i.e. $g = 1 + tx + O(t^2)$, for some $x \in L$.

$$\Rightarrow (1 + tx + O(t^2))(v \otimes w) = g(v \otimes w) = gv \otimes gw = (1 + tx + O(t^2))v \otimes (1 + tx + O(t^2))w$$
$$= v \otimes w + t(xv \otimes w + v \otimes xw) + O(t^2)$$

$$\Rightarrow x \cdot (v \otimes w) \triangleq xv \otimes w + v \otimes xw.$$

This is well-defined, as it's easily checked that $[x, y](v \otimes w) = ([x, y] \cdot v) \otimes w + v \otimes ([x, y] w)$

$$\begin{aligned}
[x, y](v \otimes w) &= x(y(v \otimes w)) - y(x(v \otimes w)) \\
&= x(yv \otimes w + v \otimes yw) - y(xv \otimes w + v \otimes xw) \\
&= xyv \otimes w + \cancel{yv \otimes xw} + \cancel{xv \otimes yw} + v \otimes xyw - yxv \otimes w - \cancel{xv \otimes yw} - \cancel{yv \otimes xw} - v \otimes yxw \\
&= ((xy - yx)v) \otimes w + v \otimes ((xy - yx)w) \\
&= ([x, y]v) \otimes w + v \otimes ([x, y]w)
\end{aligned}$$

Moreover, we have canonical isomorphisms: $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$.

$x \cdot (u \otimes v \otimes w) = xu \otimes v \otimes w + u \otimes xv \otimes w + u \otimes v \otimes xw$

$V \otimes \Bbbk \cong V$ ($\Bbbk$: trivial rep): $x \cdot (v \otimes 1) = (x \cdot v) \otimes 1 + v \otimes x \cdot 1 = x \cdot v \otimes 1$

$\varphi: V \otimes W \xrightarrow{\sim} W \otimes V$ $\quad \varphi(v \otimes w) = w \otimes v$ is an intertwiner:

$$\varphi(x(v \otimes w)) = \varphi(xv \otimes w + v \otimes xw) = w \otimes xv + xw \otimes v = x(w \otimes v) = x \cdot (\varphi(v \otimes w))$$

6). Conjugate rep: $V^* = \mathrm{Hom}_{\Bbbk}(V, \Bbbk)$ $f \in V^*$. Again for group elements $g \in G$.
$g = 1 + tx + O(t^2)$ $(g \cdot f)(v) = f(g^{-1}v) \Rightarrow ((1+tx+O(t^2))f)(v) = f((1-tx+O(t^2)) \cdot v)$
$\Rightarrow (x \cdot f)(v) \triangleq f(-xv) = -f(xv)$.

7). Adjoint rep: $L \curvearrowright V = L$: $L \otimes L \longrightarrow L$. $x \otimes y \longmapsto [x, y]$
or equivalently, $L \longrightarrow \mathrm{End}\, L$, $x \longmapsto \mathrm{ad}x$, $(\mathrm{ad}x(y) = [x, y])$
It's a representation since $[\mathrm{ad}x, \mathrm{ad}y] = \mathrm{ad}_{[x,y]}$.
This is a consequence of Jacobi's identity: $\forall z \in L$

$$\begin{aligned}
[\mathrm{ad}x, \mathrm{ad}y](z) - \mathrm{ad}_{[x,y]}(z) &= [x,[y,z]] - [y,[x,z]] - [[x,y],z] \\
&= [x,[y,z]] + [y,[z,x]] + [z,[x,y]] \\
&= 0.
\end{aligned}$$

Subrep's of L : $I \subseteq L$ is a subrep $\Leftrightarrow [L, I] \subseteq I \Leftrightarrow I$ is an ideal of L

E.g. $L \cong \Bbbk x \oplus \Bbbk y$, $[x, y] = y$. $I = \Bbbk y$

$$\Rightarrow \quad 0 \longrightarrow I \longrightarrow L \longrightarrow L/I \longrightarrow 0$$

I is a non-trivial 1-dim'l rep; L/I is the trivial rep of L.

This is a short exact sequence of L-modules, but not split, otherwise L would have a 1-dim'l center. As an L-module, L is reducible, but not completely reducible.

How to classify 1-dim'l rep's of L?
$V \cong \Bbbk v$, $x, y \in L \Rightarrow [x, y] \cdot v = x(yv) - y(xv) = 0 \Rightarrow [L, L] \cdot v = 0$.
$\Rightarrow L/[L, L] \curvearrowright V$. Thus 1-dim'l rep's of $L \Leftrightarrow \mathrm{Hom}_{\Bbbk}(L/[L,L], \Bbbk)$
1-dim'l rep's are irreducible.

• Universal Enveloping Algebra $\mathcal{U}(L)$

Treat L as a vector space. $T(L) = \bigoplus_{n=0}^{\infty} L^{\otimes n}$, with multiplication $V^{\otimes n} \times V^{\otimes m} \longrightarrow V^{\otimes(n+m)}$
$(a, b) \longmapsto a \otimes b$.

$T(L)$ is an associative, non-commutative algebra. If $\{x_1, \cdots, x_n\}$ is a basis of L, then $\{x_{i_1} \otimes \cdots \otimes x_{i_k} \mid k \geq 0, 1 \leq i_j \leq n, \forall j\}$ is a basis of $T(L)$.

$\mathcal{U}(L) \triangleq T(L)/I$, where I is the two sided ideal generated by elements of the form $x \otimes y - y \otimes x - [x, y]$. Then if L is abelian, $\mathcal{U}(L) \cong S(L)$, symmetric algebra over L.

$T(L) \cong \Bbbk\langle x_1, \cdots, x_n\rangle$. A $T(L)$-module is a $\Bbbk$-vector space with n endomorphisms on it. A $U(L)$-module must satisfy in addition: $(xy - yx - [x,y])\cdot v = 0$.

An L-module is the same as a left $U(L)$-module: $U(L) \times V \longrightarrow V$

For an arbitrary ring A (non-commutative), A^{op} is the abelian group A with a new multiplication: $a * b \triangleq b \cdot a$. $T(L) \xrightarrow{\sim} T(L)^{op}$: $x \otimes y \mapsto (-y) \otimes (-x)$ (or $x_{i_1} \otimes \cdots \otimes x_{i_k} \longmapsto (-1)^k x_{i_k} \otimes \cdots \otimes x_{i_1}$.) Moreover $x \otimes y - y \otimes x - [x,y] \mapsto y \otimes x - x \otimes y + [x,y] \in I$
$\Rightarrow$ The isomorphism descends down to $U(L) \xrightarrow{\sim} U(L)^{op}$. Moreover, for any ring A
$A \cong A^{op} \Rightarrow \{\text{left } A\text{-modules}\} \cong \{\text{left } A^{op}\text{-modules}\} \cong \{\text{right } A\text{-modules}\}$.

Size of $U(L)$:

Take a basis of L: $\{x_1, \cdots, x_n\}$, define an ordering of the basis: $x_i < x_j$ if $i < j$. Then $\{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \geq 0\}$ is a spanning set of $U(L)$.

Indeed, it suffices to check for elements of the form $x_{i_1}^{a_{i_1}} \cdots x_{i_k}^{a_{i_k}}$. If it's like $y x_j x_i z$, then $y x_j x_i z = y x_i x_j z + y [x_j, x_i] z$, and we can change all $x_j x_i$ into $x_i x_j$.

Thm (PBW) $\{x_1^{a_1} \cdots x_n^{a_n} \mid a_i \geq 0\}$ is a basis of $U(L)$.

Categorical Interpretation:

$$\text{Cat}\{\text{Associative algebras}\} \underset{U(L) \leftarrow L}{\overset{A \mapsto A^{Lie}}{\rightleftarrows}} \text{Cat}\{\text{Lie algebras}\}$$

Then: $\text{Hom}_{ass}(U(L), A) \cong \text{Hom}_{LA}(L, A^{Lie})$

i.e. The universal enveloping algebra functor U is left adjoint to the "Lie" functor which is a forgetful functor. (any associative algebra has a natural Lie algebra structure, taking A^{Lie} "forgets" its associative algebra structure.)

Usually, free object functors are left adjoint to some forgetful functor.

§4. Nilpotent and Solvable Lie Algebras

Goal: to classify all simple Lie algebras and their rep's.

Def: Derived series of L: $L^{(1)}=[L,L]$, $L^{(2)}=[L^{(1)},L^{(1)}]$, ... $L^{(k+1)}=[L^{(k)},L^{(k)}]$, ...

Lower central series of L: $L^1=[L,L]$, $L^2=[L,L^1]$, ..., $L^{k+1}=[L,L^k]$, ...

L is called solvable if $L^{(k)}=0$ for some k; nilpotent if $L^k=0$ for some k.

By def. every nilpotent Lie algebra is solvable, but not conversely:

E.g. $L=\Bbbk x\oplus\Bbbk y$. $L^{(1)}=L^1=\Bbbk y$; $L^{(2)}=0$ $L^2=L^3=\cdots=\Bbbk y$.

The Fact on page 29 shows that it's impossible to classify all nilpotent Lie algebras.

E.g. $t(n)$ = L.A. of all upper triangular $n\times n$ matrices is solvable but not nilpotent.

$t_0(n)$ = L.A. of all strict upper triangular matrices is nilpotent.

The above $\Bbbk x\oplus\Bbbk y\subseteq t(2)\subseteq \mathfrak{gl}(2)$: $x=\begin{pmatrix}1&0\\0&0\end{pmatrix}$ $y=\begin{pmatrix}0&1\\0&0\end{pmatrix}$

Prop. 1) If L is nilpotent, so is any subalgebra or quotient algebra.

2). If L is nilpotent $\Rightarrow Z(L)\neq 0$

3). If $L/Z(L)$ is nilpotent, so is L.

Pf: 1). is easy.

2). Take the last non-zero term of the lower central series: $L^n\neq 0$, $L^{n+1}=0$

$\Rightarrow [L,L^n]=0 \Rightarrow 0\neq L^n\subseteq Z(L)$.

3). $(L/Z(L))^n=0 \Rightarrow L^n\subseteq Z(L)\Rightarrow [L,L^n]=0 \Rightarrow L^{n+1}=0$. □

Rmk: We see from 2) and 3) that a nilpotent Lie algebra is constructed from abelian Lie algebras by doing central extensions.

We have the notion of nilpotent groups too. If G is a p-group, then it's nilpotent: prove by letting $G \curvearrowright G$ by conjugation, the isolated orbits are in $Z(G)$ and $|Z(G)|>1$ and $\equiv 0 \mod p \Rightarrow Z(G)$ is non-trivial $\Rightarrow G/Z(G)$ has non-trivial center since it's nilpotent ...

For the relation between p-groups and nilpotent matrices over $\mathbb{F}_p$, see Malcev.

Now ad: $L\longrightarrow \mathfrak{gl}(L)$ the adjoint representation. If $Z(L)=0$, ad is a faithful rep : $L \hookrightarrow \mathfrak{gl}(L)$. In fact, $\ker \mathrm{ad} = Z(L)$

Thm. (Ado) Any finite dimensional Lie algebra is linear.
(For a proof, see Neretin, arxiv 2007, 2-page proof)

E.g. $\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}\{E,H,F\}$: $[E,F]=H$, $[H,E]=2E$, $[H,F]=-2F$
In the adjoint rep, since $E=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $F=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ are nilpotent matrices, $\mathrm{ad}E$, $\mathrm{ad}F$ must be nilpotent. Indeed, in the basis of $\{E,H,F\}$

$\mathrm{ad}E$: $E \mapsto 0$, $H \mapsto -2E$, $F \mapsto H$ $\Rightarrow$ $\mathrm{ad}E = \begin{pmatrix}0&-2&0\\0&0&1\\0&0&0\end{pmatrix}$

$\mathrm{ad}F$: $E \mapsto -H$, $H \mapsto 2F$, $F \mapsto 0$ $\Rightarrow$ $\mathrm{ad}F = \begin{pmatrix}0&0&0\\-1&0&0\\0&2&0\end{pmatrix}$

$\mathrm{ad}H$: $E \mapsto 2E$, $H \mapsto 0$, $F \mapsto -2F$ $\Rightarrow$ $\mathrm{ad}H = \begin{pmatrix}2&0&0\\0&0&0\\0&0&-2\end{pmatrix}$

Note that H diagonalizable $\Rightarrow$ $\mathrm{ad}H$ is also diagonalizable (semi-simple). In general $A = \mathrm{diag}(\lambda_1,\cdots,\lambda_n) \Rightarrow \mathrm{ad}A(E_{ij}) = (\lambda_i-\lambda_j)E_{ij}$ i.e. A semi-simple $\Rightarrow$ $\mathrm{ad}A$ semi-simple. The converse is also true, i.e. $\mathrm{ad}A$ semi-simple $\Rightarrow$ A is semi-simple. Indeed, in Jordan canonical form, $A = \mathrm{diag}(\lambda_1,\cdots,\lambda_n)+N$ with N nilpotent $\Rightarrow$ $\mathrm{ad}A = \mathrm{ad}(\mathrm{diag}) + \mathrm{ad}N$, $\mathrm{ad}A$ s.s. $\Rightarrow N=0$.

Prop: X is ad-nilpotent iff $X = \lambda I + y$ with y nilpotent.

Pf: Consider $X \in \mathfrak{gl}(V)\otimes\overline{\mathbb{k}}$, $X = \begin{pmatrix}J_1&&\\&\ddots&\\&&J_k\end{pmatrix}$ in Jordan canonical form.

$$X = \begin{pmatrix}\lambda_1 & 1 & & & & \\ & \ddots & \ddots & & & \\ & & \lambda_1 & & & \\ & & & \lambda_2 & 1 & \\ & & & & \ddots & \ddots \\ & & & & & \lambda_2 \end{pmatrix} \begin{array}{l} \\ \\ \leftarrow k \\ \leftarrow k+1 \\ \\ \\ \end{array} \Rightarrow \mathrm{ad}X(E_{k,k+1}) = (\lambda_1-\lambda_2)E_{k,k+1}$$

Thus $\mathrm{ad}X$ nilpotent $\Rightarrow (\lambda_1-\lambda_2)^k = 0 \Rightarrow \lambda_1=\lambda_2$. □

L nilpotent $\Rightarrow L^n = 0 \Leftrightarrow \forall x_1,\cdots,x_{n-1}, y$, $[x_1,[\cdots[x_{n-1},y]]] = 0$
Take $x_1 = \cdots = x_{n-1} = x \Rightarrow (\mathrm{ad}\,x)^{n-1}(y) = 0$, $\forall y \in L \Rightarrow \mathrm{ad}\,x$ is a nilpotent operator on L. $x \in L$ is called ad-nilpotent if $\mathrm{ad}\,x$ is a nilpotent operator on L.

E.g. Consider the Lie algebra of $L = \left\{ \begin{array}{c} \overbrace{\quad}^{k}\overbrace{\quad}^{n-k} \\ \left(\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right) \end{array} \begin{array}{l} \}k \\ \}n-k \end{array} \right\}$, what's $\mathrm{Rad}(L)$?

By definition, $0 \to Rad(L) \to L \to L/Rad \to 0$ is exact and $L/Rad\,L$ is semi-simple. We can define a map: $L \to \mathfrak{sl}(k) \times \mathfrak{sl}(n-k)$ by:

$$\begin{pmatrix} A & B \\ 0 & C \end{pmatrix} \mapsto \left(A - \frac{trA}{k} Id_k,\ C - \frac{trC}{n-k} Id_{n-k}\right)$$

with kernel $= \begin{pmatrix} \lambda Id_k & B \\ 0 & \mu Id_{n-k} \end{pmatrix}$ i.e. $0 \to ker \to L \to \mathfrak{sl}(k) \times \mathfrak{sl}(n-k) \to 0$ is exact with semi-simple quotient, the sequence splits as vector spaces (not as rep's of L)

$\Rightarrow Rad\,L = \begin{pmatrix} \lambda Id & B \\ 0 & \mu Id \end{pmatrix} \cong \mathbb{C} \oplus \mathbb{C} \oplus \left\{\begin{pmatrix} 0 & * \\ 0 & 0 \end{pmatrix}\right\}$

This is also an example of $Rad(L)$ being solvable but not nilpotent.

Rmk: $0 \to Rad\,L \to L \xrightarrow{s} L^{s.s} \to 0$ always splits as vector spaces. We call $0 \to I \to L \underset{p}{\overset{s}{\rightleftarrows}} K \to 0$ with $p \circ s = id_K$ a split extension of K by I, in which case $L \cong I \oplus K$, but $[K, I] \subseteq I$, i.e. $K \to Der(I)$: $k \mapsto d_k$, and $d_k[a, b]_I = [d_k a, b]_I + [a, d_k b]_I$. For example:

E.g. $0 \to T \to Iso(\mathbb{R}^n) \rightleftarrows O(n) \to 1$ (split group extension. T: translations in $\mathbb{R}^n$).

$\overset{L.A.}{\Longrightarrow}$ $0 \to \mathfrak{t} \to \mathfrak{iso}(\mathbb{R}^n) \rightleftarrows \mathfrak{so}(n) \to 1$

$\mathfrak{t} \cong \mathbb{R}^n$, $\mathfrak{iso}(\mathbb{R}^n) \cong \left(\begin{array}{c|c} \mathfrak{so}(n) & \mathbb{R}^n \\ \hline 0 & 0 \end{array}\right)$

If $L = \mathfrak{gl}(V)$. The the usual nilpotent matrices ($X^n = 0$) is ad-nilpotent: Indeed, if $y \in \mathfrak{gl}(V)$, $(ad\,x)^{2n-1}(y) = \sum_{\ell=0}^{2n-1} (-1)^\ell x^{2n-1-\ell} y x^\ell$, and at least one of $2n-1-\ell$ or $\ell \geq n$. $\Rightarrow$ the R.H.S. is $0 \Rightarrow (ad\,x)^{2n-1} = 0$.

But the converse is not true: $X = I$, $ad\,x = 0$ but $X^n = I$. And essentially this is the only conter-example due to the prop: $ad\,x$ nilpotent $\Leftrightarrow x = \lambda Id + y$, with y nilpotent.

Now if $K \hookrightarrow L$ is an inclusion of Lie algebras. $N_L(K) \triangleq \{x \in L \mid [x, K] \subseteq K\}$ $C_L(K) \triangleq \{x \in L \mid [x, K] = 0\}$. By Jacobi's identity: ($[[x,y],k] = [[x,k],y] + [x,[y,k]]$), both $N_L(K)$ and $C_L(K)$ are subalgebras of L.

Prop. $L \subseteq \mathfrak{gl}(V)$, $V \neq 0$. If L consists of nilpotent endomorphisms of V, then $L \cdot v = 0$ for some $v \in V$, $v \neq 0$.

Pf: If $L = \Bbbk \cdot x$, x nilpotent, then $\exists\, n$ s.t. $x^n \neq 0$, $x^{n+1} = 0$, take any $u \neq 0$ s.t. $v = x^n \cdot u \neq 0$, then $x \cdot v = 0$.

Now we prove by induction on $\dim L$. If $K \subsetneq L$ is a proper subalgebra then K also consists of nilpotent endomorphisms, in particular, $\mathrm{ad}_{L/K} K$ is nilpotent. Thus by induction hypothesis, we may find $z + K \in L/K$ s.t. $[K, z] \subseteq K$, i.e. $z \in L$ is in $N_L(K)$, and $K \subsetneq N_L(K)$. Thus we may enlarge this nilpotent algebra until $\dim(L/K) = 1$, in which case K must be an ideal since $[K, z] \subseteq K$. and $L = K \oplus \Bbbk v$ as vector spaces.

$K \subsetneq L \subseteq \mathfrak{gl}(V)$, $\dim K = \dim L - 1$. By induction hypothesis, $\exists\, v \in \mathfrak{gl}(V)$ s.t. $K \cdot v = 0$. Let $W = \{w \in V \mid K \cdot w = 0\}$. $W \neq 0$. Moreover, we have

$$k \cdot z w = z \cdot k w + [k, z] \cdot w = 0 + 0 = 0 \quad \text{since } [k\, z] \in K.$$

$\Rightarrow z \curvearrowright W$. Also z is nilpotent $\Rightarrow \exists\, 0 \neq v \in W$ s.t. $z \cdot v = 0$, since $K \cdot v = 0$, we have $L \cdot v = 0$ □

Note that we have no assumption on the ground field $\Bbbk$. (algebraically closed or characteristic $= 0$ etc.)

Now if $L \subseteq \mathfrak{gl}(V)$ consists of nilpotent matrices $\Rightarrow \exists\, v \in V$. $L \cdot v = 0$. Consider the action $L \curvearrowright V/\Bbbk v$ is also by nilpotent endomorphisms $\Rightarrow \exists\, v' \in V/\Bbbk v$. s.t $L \cdot v' = 0$ or equivalently, $\exists\, v' \in V$, s.t. $L \cdot v' \subseteq \Bbbk v$. By induction, we may obtain a basis $\{v_1, \cdots, v_n\}$ of V s.t. L acts by strictly upper triangular matrices in this basis. i.e. $L \subseteq \mathfrak{t}_0 \subseteq \mathfrak{gl}(n)$. In particular, L is nilpotent.

Thm. (Engel) If all elements of L are nilpotent, then L is nilpotent.
Pf: $L \xrightarrow{\mathrm{ad}} \mathfrak{gl}(L)$ factors through $L/Z(L) \xhookrightarrow{\mathrm{ad}} \mathfrak{gl}(L)$. By the previous proposition $L/Z(L)$ is nilpotent. Thus so is L since it's a central extension of a nilpotent algebra. □

Rmk: As a representation, ($L \hookrightarrow \mathfrak{gl}(V)$), we have $0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_n = V$ and $V_i = \Bbbk v_1 + \cdots + \Bbbk v_n$, $\dim V_i/V_{i-1} = 1$ and $L \curvearrowright V_i/V_{i-1}$ trivially. (a composition series of L-modules, and each neighboring quotients are trivial modules).

Similarly, we can prove:

Thm. Let L be a solvable subalgebra of $\mathfrak{gl}(V)$, $V\neq 0$ over an algebraically closed. char $=0$ field. Then V contains a common eigen-vector for all endomorphisms in L.

i.e. $\exists\, 0\neq v\in V$, $L\cdot v\subseteq \Bbbk\cdot v$, $x\cdot v=\lambda(x)v$, $\lambda: L\longrightarrow \Bbbk$ a linear functional.

Rmk: As rep's, $\Bbbk v \hookrightarrow V$ are inclusions of L-modules, with (possible) non-trivial actions of L on $\Bbbk v$. Note that such 1-dim'l rep's are classified by $(L/[L,L])^*$

Inductively, we may obtain a basis $v_1,\cdots,v_n$ of V and $\lambda_1,\cdots\lambda_n\in(L/[L,L])^*$ s.t. L acts as $\begin{pmatrix}\lambda_1(x) & & * \\ & \ddots & \\ 0 & & \lambda_n(x)\end{pmatrix}$, $\forall\, x\in L$. i.e. any finite dimensional rep of L has a filtration $0\subsetneq V_1\subsetneq V_2\subsetneq\cdots\subsetneq V_n=V$, with $\dim V_i/V_{i-1}=1$, V_i/V_{i-1} an irrep of L. (In particular, the adjoint rep of L is not irreducible; while if L is simple, $L\xrightarrow{ad}\mathfrak{gl}(n)$ is irreducible.

As a corollary of the thm, we have:

Thm. (Lie) A solvable subalgebra of $\mathfrak{gl}(n,\mathbb{C})$ is conjugate to a subalgebra in $\mathfrak{t}(n)\subseteq\mathfrak{gl}(n)$. $\square$

In particular $L\subseteq\mathfrak{t}(n)\Rightarrow[L,L]\subseteq[\mathfrak{t}(n),\mathfrak{t}(n)]=\mathfrak{t}_0(n)\Rightarrow[L,L]$ is nilpotent.

Proof of thm.

Again by induction on $\dim L$. $\dim L=1$ is trivial. (abelian, alg. closed)

$K'=[L,L]\Rightarrow L/K'$ is abelian, take any preimage of a codim 1 subspace, say K of L, then $L=K+\Bbbk z$, $[z,K]\subseteq[L,L]\subseteq K\Rightarrow K$ is an ideal.

By induction hypothesis, $\exists\, v\in V$ s.t. $K\cdot v\subseteq\mathbb{C}\cdot v$, and this defines a linear functional $\lambda: K\longrightarrow\mathbb{C}$, $x\cdot v=\lambda(x)v$, $\forall\, x\in K$.

Define $W=\{w\mid x\cdot w=\lambda(x)w,\ \forall x\in K\}$. Claim: z preserves W. Then take any eigen-vector of z in W suffices, which exists since $\Bbbk$ algebraically closed.

Proof of claim: (from Humphreys)

$\forall x \in K$, $x \cdot zw = z \cdot xw + [x, z]w = \lambda(x) \cdot zw + \lambda([x, z])w$.

It suffices to show that $\lambda([x, z]) = 0$.

Fix $w \in W$. Let $n > 0$ be the smallest integer $n > 0$ s.t. $w, zw, \cdots, zw^n$ are linearly dependent. Let W_i be the space spanned by $w, zw, \cdots, z^{i-1}w$, $(W_0 = 0)$ $\forall x \in K$, $x \cdot W_i \subseteq W_i$. Relative to this basis of W_n, any $x \in K$ is represented by an upper triangular matrix, i.e. we can show by induction that.

$$x z^i w \equiv \lambda(x) z^i w \pmod{W_i}$$

$i = 0$ is trivial.

If $i < k$ is true $\quad x z^k w = z x z^{k-1} w + [x, z] z^{k-1} w$

$$= z(\lambda(x) z^{k-1} w + w_{k-1}) + [x, z] z^{k-1} w \quad (w_{k-1}, z^{k-1}w \in W_{k-1})$$

$$\equiv \lambda(x) z^k w \pmod{W_k}$$

It follows that, as endomorphisms of W_n, $x \in K$ are all upper triangular with eigenvalue $\lambda(x) \Rightarrow tr(x) = n \cdot \lambda(x)$. Since x, z are both endomorphisms of W_n $tr([x, z]) = 0 \Rightarrow n\lambda([x, z]) = 0$. char $k = 0 \Rightarrow \lambda([x, z]) = 0$

$\square$

§5. Representation of $\mathfrak{sl}(2,\mathbb{C})$

$\mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}\{e,f,h\}$ $e=\begin{pmatrix}0&1\\0&0\end{pmatrix}$ $f=\begin{pmatrix}0&0\\1&0\end{pmatrix}$ $h=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$

$[e,f]=h$, $[h,e]=2e$, $[h,f]=-2f$.

Let V be a finite dimensional rep of $\mathfrak{sl}(2,\mathbb{C})$.

Take an eigen-vector v of h with eigen-value $\lambda\in\mathbb{C}$ (alway possible over any algebraically closed field): $hv=\lambda v$

$\Rightarrow hev = ehv + [h,e]v = e(\lambda v) + 2ev = (\lambda+2)ev$

$\quad hfv = fhv + [h,f]v = f(\lambda v) - 2fv = (\lambda-2)fv$.

i.e. if we define $V(\lambda) \triangleq \{v\in V \mid hv=\lambda v\} \Rightarrow eV(\lambda)\subseteq V(\lambda+2)$, $f\cdot V(\lambda)\subseteq V(\lambda-2)$

$$\begin{tikzcd} \cdots \arrow[r, bend left, "e"] & V(\lambda-2) \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V(\lambda) \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & V(\lambda+2) \arrow[l, bend left, "f"] \arrow[r, bend left, "e"] & \cdots \arrow[l, bend left, "f"] \end{tikzcd}$$

$\bigoplus_{\lambda\in\mathbb{C}} V_\lambda \subseteq V$. $\dim V<\infty \Rightarrow$ only finitely many $V_\lambda\neq 0$

Now. let V be an irrep of $\mathfrak{sl}(2,\mathbb{C})$. Take λ with the largest real part.

Then: $V(\lambda)\neq 0$, $V(\lambda+2)=0$. Take $v_0\in V(\lambda)$. $ev_0\in V(\lambda+2) \Rightarrow ev_0=0$

$hv_0=\lambda v_0$. By PBW. thm, $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C})) \cong \mathbb{C}\langle f^i h^j e^k\rangle$

$\Rightarrow V = \mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))\cdot v_0 = \mathbb{C}\langle f^k v_0\rangle$, $k\in\{0,1,2,3,\cdots\}$, by irreducibility of V.

Define $v_k \triangleq \frac{f^k v_0}{k!}$. Then we have:

Lemma: $hv_k = (\lambda-2k)v_k$, $fv_k=(k+1)v_{k+1}$, $ev_k = (\lambda-k+1)v_{k-1}$.

Pf: Only the last one is not by def. Proof by induction.

$ev_1 = efv_0 = fev_0 + [e,f]v_0 = hv_0 = \lambda v_0$.

Suppose the hypothesis is true for $<k$. Now:

$kev_k = efv_{k-1} = fev_{k-1} + [e,f]v_{k-1} = f(\lambda-k+2)v_{k-2} + hv_{k-1}$

$\quad = (\lambda-k+2)(k-1)v_{k-1} + (\lambda-2k+2)v_{k-1}$ (by induction hypothesis)

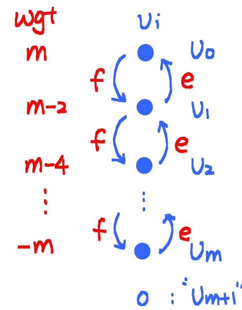
$\quad = k(\lambda-k+1)v_{k-1}$

$\Rightarrow ev_k = (\lambda-k+1)v_{k-1}$. $\square$

V is finite dimensional $\Rightarrow \exists\, m$ s.t. $v_m \neq 0$, $v_{m+1} = 0$.

$\Rightarrow 0 = e v_{m+1} = (\lambda - m) v_m \Rightarrow \lambda = m$, where $m \in \mathbb{N} \cup \{0\}$.

V irrep $\Rightarrow V = \mathbb{C}\{v_0, \cdots, v_m\}$ with v_m the lowest weight.



V_m : an irrep of dim $m+1$, with basis $\{v_0, \cdots, v_m\}$. with highest weight v_0. lowest v_m : $hv_0 = mv_0$, $hv_m = -mv_0$

Moreover, any finite dimensional irrep V of $\mathfrak{sl}(2,\mathbb{C}) \cong V_m$, $m = \dim V - 1$.

$m = 0$: the trivial rep.

$m = 1$: the defining rep of $\mathfrak{sl}(2,\mathbb{C}) \curvearrowright \mathbb{C}^2$. wgt vectors $v_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $v_1 = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$m = 2$: the adjoint rep of $\mathfrak{sl}(2,\mathbb{C}) \curvearrowright \mathfrak{sl}(2,\mathbb{C}) \cong \mathbb{C}^3$.

Note that the eigenspace decomposition of V_m is NOT canonical, and we may choose different basis of $\mathfrak{sl}(2,\mathbb{C})$, for instance, $\{geg^{-1}, gfg^{-1}, ghg^{-1}\}$ for $g \in SL(2,\mathbb{C})$.

Also note that for solvable Lie algebras, there may be uncountablely many irrep's in each dim: dim 1 irreps $\Leftrightarrow (L/[L,L])^*$. For instance, $L = \mathbb{C}\{X\}$. $\mathcal{U}(L) = \mathbb{C}[X]$, 1 dim'l irrep's $\overset{1:1}{\longleftrightarrow} \mathbb{C}$.

Thm. Any finite dim'l rep of $\mathfrak{sl}(2,\mathbb{C})$ is completely reducible.

Reminder: A: a ring. rep's of A are completely reducible

$\Leftrightarrow A = \bigoplus_{i=1}^{k} \mathrm{Mat}(n_i, D_i)$ D_i division algebras.

Thus it's very rare to have complete reducibility. Moreover, many rings even don't have finite dim'l rep's:

E.g. (First Weyl algebra): $\mathbb{C}\langle x, \partial_x \rangle / \langle \partial_x \cdot x - x\partial_x - 1 \rangle$ has no non-0 finite dim'l rep's. Indeed, $\dim V < \infty$ a rep $\Rightarrow 0 = tr(\partial_x \cdot x) - tr(x \cdot \partial_x)$ $= tr(\partial_x \cdot x - x \cdot \partial_x) = tr(1) = \dim V$.

An infinite dim'l rep on $\mathbb{C}[x]$: $\partial_x x^n = nx^{n-1}$ $x \cdot x^n = x^{n+1}$

In general, if A is a ring and L, L' are irreducible A-modules, then

$$0 \longrightarrow L \longrightarrow V \longrightarrow L' \longrightarrow 0 \quad (\text{s.e.s of } A\text{-modules})$$

need not split, i.e. there may be non-trivial extension of L' by L. (Compare with the following lemma).

Proof of Thm.

Lemma: If pairs of simple modules have only trivial extensions, then any finite length module is semi-simple.

(Recall that length(M) $\triangleq$ the number of inclusions in a composition series.
$0 = V^0 \subseteq V^1 \subseteq \cdots \subseteq V^n = V$)

Pf: By induction on the length n of the module.
$n=1$. trivial.

Suppose the hypothesis is true for length $\leq n$

length$(V) = n+1$: $\exists\, 0 = V^0 \subseteq V^1 \subseteq \cdots \subseteq V^n \subseteq V^{n+1} = V$

Take V^n, which is of length n. By induction hypothesis $V^n = \bigoplus_{i=1}^n L_i$, L_i simple.

$\Rightarrow 0 \to \bigoplus_{i=1}^n L_i \to V \to W \to 0$. $(*)$ (L_i $W = V/V^n$ simple)

mod $L_1 \Rightarrow 0 \to \bigoplus_{i=2}^n L_i \to V/L_1 \to W \to 0$.

Again by induction, $V/L_1 = W \oplus \bigoplus_{i=2}^n L_i$. Thus by considering $V \xrightarrow{pr} V/L_1 \xrightarrow{pr} \bigoplus_{i=2}^n L_i$ and $\bigoplus_{i=2}^n L_i \hookrightarrow V$, we see that $\bigoplus_{i=2}^n L_i$ is a direct sumand of V. Choose a complementary subspace V' of V, then taking quotient of $(*)$ by $\bigoplus_{i=2}^n L_i$ gives

$$0 \to L_1 \to V' \to W \to 0. \quad W \text{ simple}$$

By assumption, we have $V' \cong L_1 \oplus W \Rightarrow V \cong L_1 \oplus L_2 \oplus \cdots \oplus L_n \oplus W$. □

Now, for finite dimensional $\mathfrak{sl}(2,\mathbb{C})$-modules (finite length), we will show that:

Lemma: $0 \to V_n \to V \to V_m \to 0$ always splits as $\mathfrak{sl}(2,\mathbb{C})$-modules.

Pf: 1) $0 \to V_n \to V \xrightarrow{\psi} V_0 \to 0$ always splits, $\forall n$.

Indeed, let $V_0 = \mathbb{C}u$.

If n is odd, then V_n contains no weight 0 vector for h. Take any preimage of u', then by considering the weight vectors $\{v_0, \cdots, v_n\}$ of V_n, we obtain a basis $\{u', v_0, \cdots, v_n\}$ of V and $\psi(hu') = h\psi(u') = 0 \Rightarrow h(u') = \sum a_i v_i$

$\Rightarrow h(u' - \sum_{i=0}^n \frac{a_i}{(n-2i)} v_i) = 0$. Let $\tilde{u} = u' - \sum_{i=0}^n \frac{a_i}{n-2i} v_i$, then $h\tilde{u} = 0$, and if $e\tilde{u}$ or $f\tilde{u}$ is not zero, then $\psi(e\tilde{u}) = e\psi(\tilde{u}) = 0 \Rightarrow e\tilde{u} \in V_n$ and $he\tilde{u} = eh\tilde{u} + [h,e]\hat{u}$ $= 2e\tilde{u} \Rightarrow V_n$ contains even weights $\Rightarrow V_n$ contains the zero weight. Contradiction.

$\Rightarrow e\tilde{u} = f\tilde{u} = h\tilde{u} = 0 \Rightarrow u \mapsto \tilde{u}$ is a splitting of $\mathfrak{sl}(2,\mathbb{C})$-modules.

In case n is even, take a preimage u' of u in V as before. Again consider the weight vectors $\{v_0, \cdots, v_{2k}\}$, $2k=n$. Since $\psi(hu') = h\psi(u') = 0 \Rightarrow hu' \in V_n$ and $hu' = \sum a_i v_i \Rightarrow h(u' - \sum_{i\neq k} \frac{a_i}{n-2i} v_i) = 0$. Let $u'' = u' - \sum_{i\neq k} \frac{a_i}{n-2i} v_i$. Then similarly $eu'' \in V_n$. and $heu'' = ehu'' + [h,e]u'' = 2eu''$. Then there are two cases:

(i). $eu'' \neq 0$, then $eu'' = \lambda v_{k-1} \Rightarrow e(u'' - \frac{\lambda}{k+1} v_k) = 0$, $h(u'' - \frac{\lambda}{k+1} v_k) = 0$. Then we let $\tilde{u} = u'' - \frac{\lambda}{k+1} v_k$

(ii). $eu'' = 0$, $hu'' = 0$. Let $\tilde{u} = u''$.

Claim: $f\tilde{u} = 0$. and thus $u \mapsto \tilde{u}$ defines a splitting of the s.e.s.

Indeed, if $f\tilde{u} \neq 0$. Then similar as in (i), $f\tilde{u} = \lambda v_{k+1} \Rightarrow 0 \neq k\lambda v_k = e(\lambda v_{k+1})$ $= ef\tilde{u} = fe\tilde{u} + [e,f]\tilde{u} = 0 + h\tilde{u} = 0$. Contradiction. Hence $f\tilde{u} = 0$.

2). Now we prove that any s.e.s. $0 \to U \to V \to V_0 \to 0$ splits by induction on $\dim U$. If U is irreducible, then 1) $\Rightarrow$ the claim is true. Otherwise, $U \supseteq V_i$ for some i. $\Rightarrow 0 \to U/V_i \to V/V_i \to V_0 \to 0$ is s.e. and splits by induction hypothesis. $\Rightarrow V/V_i \cong V_0 \oplus U/V_i$. Take the preimage V_0' of V_0 in V, then we have a s.e.s: $0 \to V_i \to V_0' \to V_0 \to 0$ V_i irrep $\Rightarrow V_0' \cong V_i \oplus V_0$ by 1). $\Rightarrow \frac{V}{V_i} \cong \frac{V_0'}{V_i} \oplus \frac{U}{V_i}$
$\Rightarrow V \cong U \oplus V_0$ since $U \cap V_0' = V_i$ implies $U \cap V_0 = 0$.

3). Consider $0 \to V_n \to V \to V_m \to 0$ $(*)$

Note that $\mathrm{Hom}_{\mathbb{C}}(V, V_n)$ is an $\mathfrak{sl}(2,\mathbb{C})$-module: $x \in \mathfrak{sl}(2,\mathbb{C})$, $f \in \mathrm{Hom}_{\mathbb{C}}(V, V_n)$ $v \in V$, then $(x\cdot f)(v) = x(f(v)) - f(x\cdot v)$

Let $U \subseteq \mathrm{Hom}_{\mathbb{C}}(V, V_n)$ be the subspace of maps which are multiples of the identity on V_n $= \{f \in \mathrm{Hom}(V, V_n) \mid f|_{V_n} = \lambda \, \mathrm{Id}_{V_n}, \lambda \in \mathbb{C}\}$.

We claim that U is a submodule of $\mathrm{Hom}_{\mathbb{C}}(V, V_n)$. $f \in U$, $x \in \mathfrak{sl}(2,\mathbb{C})$ $v \in V_n \Rightarrow (x\cdot f)(v) = x(f(v)) - f(x\cdot v) = x(\lambda v) - \lambda x v = 0$. i.e. $x: U \to U_0$ where $U_0 = \{f \in U \mid f|_{V_n} = 0\}$.

$$\Rightarrow 0 \to U_0 \to U \to \mathbb{C} \to 0 \text{ is exact.}$$

By 2) $\Rightarrow$ $U \cong U_0 \oplus \mathbb{C}$.

Take $f \in \mathbb{C} \subseteq U$, $f|_{V_n} = 1$, then f serves as a splitting of the s.e.s. $(*)$

$$0 \to V_n \underset{\xrightarrow{\quad}}{\xleftarrow{f}} V \to V_m \to 0 \qquad \square$$

Remark: One crucial step in the proof is that $sl(2,\mathbb{C}) \curvearrowright \mathrm{Hom}_{\mathbb{C}}(V,W) \cong V^* \otimes W$ or more generally any $V \otimes W$ for V, W $sl(2,\mathbb{C})$-modules. In general, if V, W are A-modules, $V \otimes W$ is not a priori an A-module. If $A \cong A^{op}$, then V, W can be made into right A-modules: $x \circ v \triangleq x^{op} \cdot v$

Combining the previous lemmas, we obtain the proof of the thm, stated again:

Thm ($sl(2,\mathbb{C})$). 1) $sl(2,\mathbb{C})$ has a unique irrep in each dimension $n+1$, $n = 0, 1, 2, \cdots$ denoted V_n, with heighest weight n.

2). Any finite dimensional representation of $sl(2,\mathbb{C})$ is completely reducible, i.e. $\forall$ V an $sl(2,\mathbb{C})$-module, $V \cong \oplus V_{n_i}^{a_i}$. □

Remark: (base change). One cool thing about finite dimensional $sl(2,\mathbb{C})$ rep's is that the eigen-values of h are in $\mathbb{Z}$. Consider any finite dimensional $sl(2,\mathbb{Q})$ module V. Then $V \otimes_{\mathbb{Q}} \mathbb{C}$ is an $sl(2,\mathbb{C})$ module, and h acts diagonally with integer eigenvalues $\Rightarrow V = \bigoplus_{n \in \mathbb{Z}} V(n)$ $h|_{V(n)} = n$. and everything works as for $sl(2,\mathbb{C})$. Compare with $L = \mathbb{Q}x$ $\mathcal{U}(L) \cong \mathbb{Q}[x]$, x is not always diagonalizable on finite dimensional (irreducible) modules!

Now take any $sl(2,\mathbb{C})$-module. V can be decomposed as weight spaces of h. Since $V \cong \oplus V_{n_i}^{a_i}$ and $V_n = \bigoplus_{m=-n}^{n} V_n(m)$. Now

$$\begin{tikzcd} V: & \cdots & V(m+1) \arrow[rr, bend left, "f"] & V(m) \arrow[rr, bend right, "f"'] & V(m-1) \arrow[rr, bend left, "f"] & V(m-2) & V(m-3) & \cdots \end{tikzcd}$$

Take $V(m)$ to be of weight m, then $f^m: V(m) \xrightarrow{\sim} V(-m)$ and $e^m: V(-m) \xrightarrow{\sim} V(m)$ are isomorphisms, and $f: V(m) \longrightarrow V(m-2)$ is injective if $m > 0$ (since $ef(v_m) = m \cdot v_m$)

Thus $V(m-2) = f \cdot V(m) \oplus \ker e$, $\bigoplus_{k=0}^{m} f^k \ker e$ is a submodule of V.

Graphically: (take $V(N)$, N the heighest weight)

$$\begin{array}{ccccccccc}
V(N) & \underset{f}{\overset{e}{\rightleftarrows}} & V(N-2) & \underset{f}{\overset{e}{\rightleftarrows}} & V(N-4) & \underset{f}{\overset{e}{\rightleftarrows}} & \cdots & & \\
\cong & & \cong & & \cong & & & & \\
V(N) & \rightarrowtail & f\cdot(V(N)) & \rightarrowtail & f^2(V(N)) & \rightarrowtail & \cdots & \bigoplus_{k=0}^{N} f^k(V(N)) & \\
 & & \oplus & & \oplus & & & & \\
 & & \ker(e) & \rightarrowtail & f(\ker e) & \rightarrowtail & \cdots & \bigoplus_{k=0}^{N-2} f^k(\ker e) & \text{submodules} \\
 & & & & \oplus & & & & \\
 & & & & \ker e & \rightarrowtail & \cdots & \bigoplus_{k=0}^{N-4} f^k(\ker e) &
\end{array}$$

Characters of $sl(2,\mathbb{C})$.

- V: an $sl(2,\mathbb{C})$-module, $ch(V) \triangleq \sum_{m\in\mathbb{Z}} \dim V(m)\, q^m \in \mathbb{Z}[q]$.

Basic properties of $ch(V)$:

1). $ch(V_n) = q^n + q^{n-2} + \cdots + q^{-n} = \dfrac{q^{n+1} - q^{-(n+1)}}{q - q^{-1}} \triangleq [n+1]$ (the "quantum $n+1$", because its limit as $q \to 1$ is $(n+1)$, i.e. a deformation of $n+1$)

2). Since $V(m) \cong V(-m)$, $ch(V)(q) = ch(V)(q^{-1})$

3). $ch(V\oplus W) = ch(V) + ch(W)$

4). $ch(V\otimes W) = ch(V)\cdot ch(W)$. This is because $V\otimes W(m) = \sum_{k\in\mathbb{Z}} V(k)\otimes W(m-k)$:

$h(v_k\otimes w_{m-k}) = (hv_k)\otimes w_{m-k} + v_k\otimes h w_{m-k} = k\cdot v_k\otimes w_{m-k} + (m-k)\, v_k\otimes w_{m-k} = m\, v_k\otimes w_{m-k}$.

5). Up to isomorphism, V is uniquely determined by its character: $ch(V) = \sum_{k\in\mathbb{Z}} a_k q^k$.

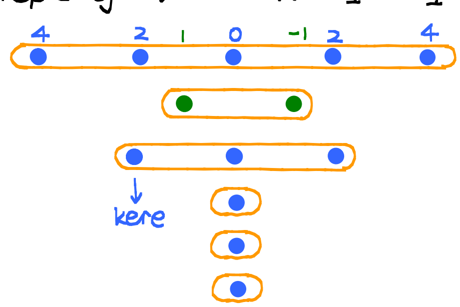
What's the multiplicity of V_m in $V \cong \oplus V_m^{b_m}$

$$\cdots \underset{f}{\overset{e}{\rightleftarrows}} \underset{\dim = a_{m+2}}{V(m+2)} \underset{f}{\overset{e}{\rightleftarrows}} \underset{\dim = a_m}{V(m)} \underset{f}{\overset{e}{\rightleftarrows}} V(m-2) \underset{f}{\overset{e}{\rightleftarrows}} \cdots$$

$$\underset{f}{\overset{e}{\rightleftarrows}} V(m+1) \underset{f}{\overset{e}{\rightleftarrows}} V(m-1) \underset{f}{\overset{e}{\rightleftarrows}}$$

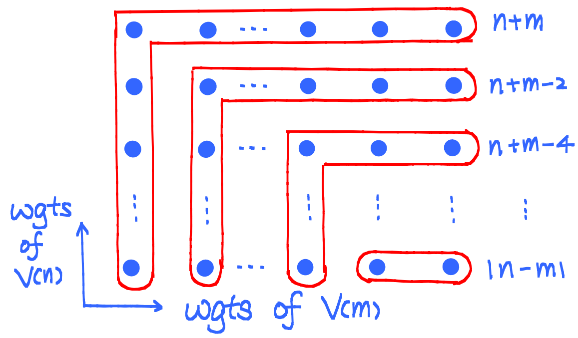
Then since $V(m) = f(V(m+2)) \oplus \ker e \Rightarrow b_m = a_m - a_{m+2} = \dim \mathrm{Hom}_{sl(2,\mathbb{C})}(V_m, V)$.

E.g. Calculate the irrep's of V: $ch(V) = q^4 + 2q^2 + q + 5 + q^{-1} + 2q^{-2} + q^{-4}$



Then $V \cong V_4 \oplus V_2 \oplus V_1 \oplus V_0^3$

E.g. Decomposition of $V_n \otimes V_m$



From the diagram, we see that $V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|}$

$V_k \subseteq V_n \otimes V_m \iff k+m+n \equiv 0 \mod 2$ and n, m, k satisfies the triangle inequality

$\iff \mathbb{C} \hookrightarrow V_k^* \otimes V_n \otimes V_m \cong V_k \otimes V_n \otimes V_m$;

$V_k \cong V_k^*$: just flip the wgts : $(V(m))^* = V^*(-m)$

$\iff \mathrm{Inv}(V_k \otimes V_n \otimes V_m) \triangleq \mathrm{Hom}_{\mathfrak{sl}(2,\mathbb{C})}(V_0, V_k \otimes V_n \otimes V_m) \neq 0$

(and the multiplicity is at most 1).

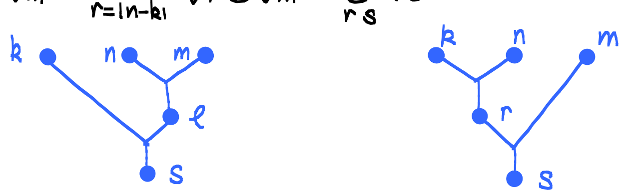
(In general we can also define $\mathrm{Inv}_G(V)$ (V^G) $\triangleq \mathrm{Hom}_G(\mathbb{C}, V)$. But there is no such concept of invariants for arbitrary A-modules, since there need not be trivial rep)

If $V_k \subseteq V_m \otimes V_n$, then it's unique. $v_j \in V_k(j) \mapsto \sum C_{jj'j''} v_{j'} \otimes v_{j''}$, $(C_{jj'j''})$ are determined up to a scalar, which can be fixed by $v_0 \mapsto u \in \ker e \subseteq V_m \otimes V_n(k)$

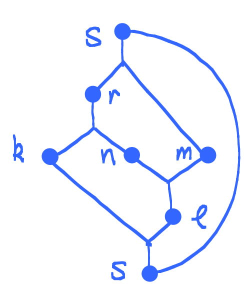
These numbers $(jj'j'')$, appearing in physics litterature, are called $3j$-symbols.

Furthermore, since $V_k \otimes (V_n \otimes V_m) \cong (V_k \otimes V_n) \otimes V_m$ canonically, we have:

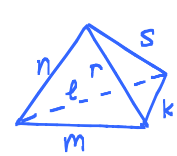
$$\left.\begin{aligned} V_k \otimes (V_n \otimes V_m) &= \bigoplus_{\ell=|m-n|}^{m+n} V_k \otimes V_\ell = \bigoplus_{\ell,s} V_s \\ (V_k \otimes V_n) \otimes V_m &= \bigoplus_{r=|n-k|}^{n+k} V_r \otimes V_m = \bigoplus_{r,s} V_s \end{aligned}\right\} \Rightarrow (n, m, k, r, \ell, s) : 6j\text{-symbols}.$$



$\Downarrow$ glueing k, n, m, s



dual $\Rightarrow$

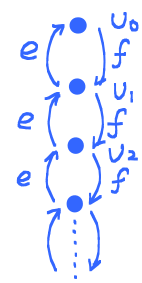


Ref: J. Roberts, Classical 6j-symbols and the tetrahedron)

The above decomposition also works over $\mathbb{R}$, $\mathbb{Q}$, or any $\mathbb{k}$, char$\mathbb{k} = 0$.

Infinite dimensional rep's.

Take v_0, and let $hv_0 = \lambda v_0$, $ev_0 = 0$. Define $v_k \triangleq \frac{f^k v_0}{k!}$. Then by a previous lemma, $hv_k = (\lambda - 2k)v_k$, $fv_k = (k+1)v_{k+1}$, $ev_k = (\lambda - k + 1)v_{k-1}$.



$M_\lambda \triangleq \bigoplus_{k=0}^{\infty} \mathbb{C} v_k$: the Verma module with H.W. λ.

Lemma: M_λ is irreducible for $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\geq 0}$

Pf: By def. $M_\lambda = \bigoplus_{\mu \in \lambda - 2\mathbb{Z}_{\geq 0}} M_\lambda(\mu)$. If $N \hookrightarrow M_\lambda$ is a submodule, take $v \in N$, $v = \sum_{i=1}^{n} v_i$, $v_i \in M_\lambda(\mu_i)$ Apply h, h^2, ..., h^{n-1}, we obtain

$$v_1 + \cdots + v_n \in N$$

$$\mu_1 v_1 + \cdots + \mu_n v_n \in N$$

$$\vdots$$

$$\mu_1^{n-1} v_1 + \cdots + \mu_n^{n-1} v_n \in N$$

But the matrix $\begin{pmatrix} 1 & 1, & \cdots, & 1 \\ \mu_1 & \mu_2, & \cdots, & \mu_n \\ \vdots & \vdots & & \vdots \\ \mu_1^{n-1} & \mu_2^{n-1} & \cdots & \mu_n^{n-1} \end{pmatrix}$ is invertible since $\mu_i \neq \mu_j$

$\Rightarrow v_i \in N$. Apply e enough times, we obtain $v_0 \in N$, unless $e \cdot v_{m+1} = 0$ for some m, in which case $\lambda = m \in \mathbb{N}$. Furthermore, if $\lambda = m \in \mathbb{N}$, then consider

$$0 \longrightarrow M_{-m-2} \longrightarrow M_m \qquad u_0 \longmapsto v_{m+1} \quad u_1 \longmapsto (\ \cdot\ ) v_{m+2}, \cdots$$

The quotient is the finite dimensional irrep V_m. M_{-m-2} is irred since $-m-2 \notin \mathbb{Z}_{\geq 0}$

i.e. $0 \longrightarrow M_{-m-2} \xrightarrow{i} M_m \longrightarrow V_m \longrightarrow 0$ s.e.s. of $\mathfrak{sl}(2,\mathbb{C})$-modules, where the inclusion is given by $i : u_{k-1} \longmapsto \frac{(m+k)!}{(k-1)!} v_{m+k}$, which is an $\mathfrak{sl}(2,\mathbb{C})$-module homomorphism:

$$i(eu_k) = i(-(m+k+1)u_{k-1}) = -(m+k+1)\frac{(m+k)!}{(k-1)!} v_{m+k} = -\frac{(m+k+1)!}{(k-1)!} v_{m+k} = -\frac{(m+k+1)!}{(k-1)!}\left(-\frac{1}{k}\right) e v_{m+k+1}$$

$$= e\frac{(m+k+1)!}{k!} v_{m+k+1} = e\, i(u_k);$$

$$i(fu_k) = i((k+1)u_{k+1}) = (k+1)\frac{(m+k+2)!}{(k+1)!} v_{m+k+2} = (k+1)\frac{(m+k+2)!}{(k+1)!}\frac{1}{(m+k+2)} f v_{m+k+1} = f\left(\frac{(m+k+1)!}{k!} v_{m+k+1}\right)$$

$$= f\, i(u_k);$$

$$i(hu_k) = i((-m-2-2k)u_k) = (-m-2-2k)\frac{(m+k+1)!}{k!} v_{m+k+1} = \frac{(m+k+1)!}{k!} h v_{m+k+1} = h\, i(u_k).$$

$\square$

In general, to construct M_λ for any simple Lie algebra L, consider $L = L_1 \oplus L_2$, sum of subalgebras, not necessarily ideals.

E.g. $L = \mathfrak{gl}(n) =$ Upper triangular matrices $\oplus$ Strictly lower triangular matrices.

Take a basis of $L_1: \{v_1, \cdots, v_n\}$, $\quad L_2: \{w_1, \cdots, w_m\}$

$$\text{PBW} \Rightarrow \mathcal{U}(L) \cong \mathbb{C}\{ v_1^{a_1} \cdots v_n^{a_n} w_1^{b_1} \cdots w_m^{b_m} \mid a_i, b_j \geq 0 \}$$

$$\mathcal{U}(L_1) \cong \mathbb{C}\{ v_1^{a_1} \cdots v_n^{a_n} \} \quad \mathcal{U}(L_2) = \mathbb{C}\{ w_1^{b_1} \cdots w_n^{b_n} \}.$$

$\Rightarrow \mathcal{U}(L) \cong \mathcal{U}(L_1) \otimes_{\mathbb{C}} \mathcal{U}(L_2)$, the isomorphism being a bimodule isomorphis: as a left $\mathcal{U}(L_1)$-module and a right $\mathcal{U}(L_2)$-module.

Now take a rep of $\mathcal{U}(L_2)$, then $\mathrm{Ind}_{L_2}^{L}(V) \triangleq \mathcal{U}(L) \otimes_{\mathcal{U}(L_2)} V$. It's size can be seen via $\mathcal{U}(L) \otimes_{\mathcal{U}(L_2)} V \cong (\mathcal{U}(L_1) \otimes_{\mathbb{C}} \mathcal{U}(L_2)) \otimes_{\mathcal{U}(L_2)} V \cong \mathcal{U}(L_1) \otimes_{\mathbb{C}} V$ (as $\mathcal{U}(L_1)$-modules)

E.g. $\mathfrak{sl}(2,\mathbb{C}) = n_- \oplus b_+$

n_-: strictly lower triangular matrices $= \mathbb{C}\langle f \rangle$

b_+: traceless upper triangular matrices $= \mathbb{C}\langle e, h \rangle$

Take the $\mathcal{U}(b_+)$-module $V = \mathbb{C}v_0$, $\quad e \cdot v_0 = 0$, $h v_0 = \lambda v_0$

$\Rightarrow \mathrm{Ind}_{b_+}^{\mathfrak{sl}(2,\mathbb{C})}(V) = \mathcal{U}(n_-) \otimes \mathcal{U}(b_+) \otimes_{\mathcal{U}(b_+)} V \cong \mathcal{U}(n_-) \otimes_{\mathbb{C}} v_0 \cong \mathbb{C}[f] \cdot v_0$ as $\mathcal{U}(n_-)$-modules.

$\Rightarrow M(\lambda) = \mathrm{Ind}_{b_+}^{\mathfrak{sl}(2,\mathbb{C})}(V)$.

Consider the s.e.s. and its dual:

$$0 \longrightarrow M_{-n-2} \longrightarrow M_n \longrightarrow V_n \longrightarrow 0 \quad \overset{*}{\rightsquigarrow} \quad 0 \longrightarrow V_n^* \longrightarrow M_n^* \longrightarrow M_{-n-2}^* \longrightarrow 0$$

i.e. we change the highest weight modules to lowest wgt modules, and $V_n \cong V_n^*$.

There is also another way around: let $\mathfrak{sl}(2,\mathbb{C})$ act by $h' \cdot v = -hv$, $e' \cdot v = -fv$, $f' \cdot v = -ev$ (i.e. by first composing with a Cartan involution of $\mathfrak{sl}(2,\mathbb{C})$)

$\Rightarrow M_n \xrightarrow{*} M_n^* \longrightarrow M_n'$ ("twisted action")

$\Rightarrow 0 \longrightarrow V_n \longrightarrow M_n' \longrightarrow M_{-n-2}' \longrightarrow 0$ ($M_{-n-2}' \cong M_{-n-2}$, since M_{-n-2} is simple)

Casimir element.

In $\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))$, define $c = ef + fe + \frac{1}{2}h^2$.

Lemma: $c \in Z(\mathcal{U}(\mathfrak{sl}(2,\mathbb{C})))$.

Pf: It's enough to check for $[h,c]=0$, $[e,c]=0$, $[f,c]=0$

$$[h,c] = [h,ef] + [h,fe] = [h,e]f + e[h,f] + f[h,e] + [h,f]e$$
$$= 2ef - 2ef + 2fe - 2fe = 0$$

$$[e,c] = [e,ef] + [e,fe] + \tfrac{1}{2}[e,h^2] = e[e,f] + [e,f]e + \tfrac{1}{2}h[e,h] + \tfrac{1}{2}[e,h]h$$
$$= eh + he - he - eh = 0$$

$$[f,c] = [f,ef] + [f,fe] + \tfrac{1}{2}[f,h^2] = [f,e]f + f[f,e] + \tfrac{1}{2}[f,h]h + \tfrac{1}{2}h[f,h]$$
$$= -hf - fh + fh + hf = 0. \qquad \square$$

Remark: In $\mathcal{U}(L)$, $\mathrm{ad}\,a$ acts as differentiation: $[a,bc] = [a,b]c + b[a,c]$. So is for any Lie algebra acting on an associative algebra.

Now, Shur's lemma $\Rightarrow$ c acts as a scalar on any irrep V of $\mathfrak{sl}(2,\mathbb{C})$. In particular, $\forall v \in V_n$, $c\cdot v = \lambda_n v$, for some $\lambda_n \in \mathbb{C}$. To specify λ_n, take $v = v_0$, then $c\cdot v = (ef + fe + \frac{1}{2}h^2)v_0 = (2fe + [e,f] + \frac{1}{2}h^2)v_0 = (2fe + h + \frac{h^2}{2})v_0 = (n + \frac{n^2}{2})v_0$
$\Rightarrow \lambda_n = \frac{n^2+2n}{2}$.

We can use c to obtain another proof that $0 \to V_n \to V \to V_m \to 0$ always splits:

$$\begin{tikzcd}
0 \arrow[r] & V_n \arrow[r] \arrow[d, "c"] & V \arrow[r, "\varphi"] \arrow[d, "c"] & V_m \arrow[r] \arrow[d, "c"] & 0 \\
0 \arrow[r] & V_n \arrow[r] & V \arrow[r, "\varphi"] & V_m \arrow[r] & 0
\end{tikzcd}$$

Since $c = \frac{m^2+2m}{2}$ on V_m $\quad c - \frac{m^2+2m}{2}\mathrm{Id}_V : V \to V_n$. Indeed, $\forall v \in V$ $\quad \varphi((c - \frac{m^2+2m}{2})v) = (c - \frac{m^2+2m}{2})\varphi(v) = 0 \Rightarrow (c - \frac{m^2+2m}{2}\mathrm{Id}_V)v \in V_n$. Moreover, restricted to V_n, $c - \frac{m^2+2m}{2}\mathrm{Id}_V = \frac{n^2+2n}{2} - \frac{m^2+2m}{2} = N \neq 0$, thus $\frac{1}{N}(c - \frac{m^2+2m}{2}\mathrm{Id}_V)$ defines a splitting of the s.e.s. $\square$

Later we will prove:

Thm. $Z(\mathcal{U}(\mathfrak{sl}(2,\mathbb{C}))) = \mathbb{C}[c]$.

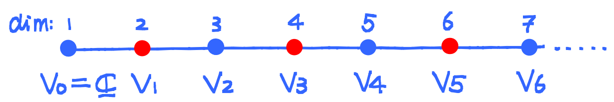
McKay correspondence for $SU(2)$.

Note that:
$$\begin{array}{ccccc} SU(2) & \hookrightarrow & SL(2,\mathbb{C}) & \hookleftarrow & SL(2,\mathbb{R}) \\ \downarrow L & & \downarrow L & & \downarrow L \\ \mathfrak{su}(2) & \longrightarrow & \mathfrak{sl}(2,\mathbb{C}) & \longleftarrow & \mathfrak{sl}(2,\mathbb{R}) \end{array},$$

and $\mathfrak{su}(2)\otimes_{\mathbb{R}}\mathbb{C} \cong \mathfrak{sl}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{R})\otimes_{\mathbb{R}}\mathbb{C}$ (the latter is obvious, while the first follows from $\mathfrak{sl}(2,\mathbb{C}) = \{\text{traceless anti-hermitian}\} \oplus \{\text{traceless hermitian}\}$
$$= \mathfrak{su}(2) \oplus i\cdot\mathfrak{su}(2) \cong \mathfrak{su}(2)\otimes_{\mathbb{R}}\mathbb{C}.\ )$$

Let $V_1 \triangleq$ the fundamental rep of $SU(2)$. ($SL(2,\mathbb{C})$, $\mathfrak{sl}(2,\mathbb{C})$, $\mathfrak{su}(2)$), then $V_n\otimes V_1 \cong V_{n-1}\oplus V_{n+1}$. (In fact, $V_n \cong S^nV_1$: write $V_1 = \mathbb{C}v_0 \oplus \mathbb{C}v_1$, then $S^nV_1 = \mathbb{C}\langle v_0^k \otimes v_1^{n-k}\rangle$, $\dim S^nV = n+1$, heighest wgt: $hv_0^n = n v_0^n$, thus $S^nV \cong V_n$)

Thus the McKay graph is:



where the blue dots represent those representations which also desend down to $SO(3)$ representations, i.e. $-I\in SU(2)$ acts as 1 on them.

Irrep's of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$.

Take arbitrary A-module V, B-module W. if V and W are irrep's, then so is the $A\otimes_{\mathbb{C}}B$ module $V\otimes_{\mathbb{C}}W$ (A, B are rings over $\mathbb{C}$, or any algebraically closed field). Indeed, $\mathrm{End}_{A\otimes B}(V\otimes W) \cong \mathrm{End}_A(V)\otimes_{\mathbb{C}}\mathrm{End}_{\mathbb{C}}(W) \cong \mathbb{C}\otimes_{\mathbb{C}}\mathbb{C} \cong \mathbb{C}$.

For example, $\mathfrak{gl}(2,\mathbb{C}) \cong \mathfrak{sl}(2,\mathbb{C})\oplus\mathbb{C}I$, the irrep's of $\mathbb{C}I$ are 1-dim'l: $\mathbb{C}_\lambda$, parametrized by $\lambda\in\mathbb{C}$: $I\cdot v_0 = \lambda\cdot v_0$. Thus irrep's of $\mathfrak{gl}(2,\mathbb{C})$ are $V_n\otimes\mathbb{C}_\lambda$.

In particular, take $A = U(\mathfrak{sl}(2,\mathbb{C})) = B$, $A\otimes_{\mathbb{C}}B \cong U(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C}))$ thus $\forall\ V_n, V_m$ irrep's of $\mathfrak{sl}(2,\mathbb{C})$, $V_n\otimes V_m$ is an irrep of $\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$.

Since $\mathfrak{sl}(2,\mathbb{C}) \xrightarrow{\Delta} \mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$, $x\mapsto(x,x)$, and under this map,
$U(\mathfrak{sl}(2,\mathbb{C})) \longrightarrow U(\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})) \cong U(\mathfrak{sl}(2,\mathbb{C}))\otimes_{\mathbb{C}}U(\mathfrak{sl}(2,\mathbb{C}))$, $x\mapsto x\otimes 1 + 1\otimes x$
$x\cdot(v\otimes w) = (x\otimes 1)(v\otimes w) + (1\otimes x)(v\otimes w) = xv\otimes w + v\otimes xw$, which agrees with our previous definition.

We may similarly consider $\mathrm{Rep}(\mathfrak{sl}(2,\mathbb{C}))$:

$\mathrm{Rep}(\mathfrak{sl}(2,\mathbb{C})) = \mathbb{Z}[[V_0],\cdots,[V_n],\cdots]$: a commutative, unital ring with a set of basis elements: $[V_i]$, $i = 0, 1, 2, \cdots$. The multiplication rule is given by:

$$[V_n] \cdot [V_m] = \sum_{\substack{k=|n-m| \\ k \equiv n+m \bmod 2}}^{n+m} [V_k]$$

§6. Semisimple Lie Algebras

Killing form

Consider $L \xrightarrow{ad} gl(L)$. The Killing form B is defined as:

$$B: L \times L \to \mathbb{k}, \quad B(x,y) \triangleq tr(adx \circ ady).$$

Here $\mathbb{k}$ may be any field, $char\,\mathbb{k} = 0$.

Lemma: $B(x,y) = B(y,x)$ (symmetic)

$B([x,y],z) = B(x,[y,z])$ (associative invariant).

Pf: The first one is easy.

$B([x,y],z) = Tr(ad[x,y] \circ adz) = Tr([adx, ady] \circ adz)$

$= Tr(adx\,ady\,adz - ady\,adx\,adz)$

$B(x,[y,z]) = Tr(adx \circ ad[y,z]) = Tr(adx \circ [ady, adz])$

$= Tr(adx\,ady\,adz - adx\,adz\,ady)$

$\Rightarrow B([x,y],z) - B(x,[y,z]) = Tr(adx\,adz\,ady) - Tr(ady\,adx\,adz)$. $\square$

Rmk: For G finite, $G \curvearrowright V$, we can always have an invariant bilinear form $(\ ,\ )$: $(gv, gw) = (v,w)$. Now if G is a Lie group, we may have the infinitesimal invariance of the bilinear form: $(xv, w) + (v, xw) = 0$. In particular, if $V = L$ and $(\cdot,\cdot) = B(\cdot,\cdot)$ on L, the above lemma just says that the infinitesimal action $L \xrightarrow{ad} gl(L)$ preserves $(\ ,\ )$: $([x,v],w) + (v,[x,w]) = 0$.

E.g.

1). $sl(2,\mathbb{R})$: w.r.t. e,h,f, we have $B = \begin{array}{cc} \begin{array}{ccc} e & h & f \end{array} & \\ \begin{pmatrix} & & 4 \\ & 8 & \\ 4 & & \end{pmatrix} & \begin{array}{c} e \\ h \\ f \end{array} \end{array} \sim \begin{pmatrix} 4 & & \\ & 8 & \\ & & -4 \end{pmatrix}$

Thus the Killing form is indefinite.

2). $su(2) \cong \mathbb{R}^3 = \mathbb{R}\langle a,b,c \rangle$, $a \times b = c$ $b \times c = a$, $c \times a = b$.

A basis for $su(2)$ is $\alpha = \begin{pmatrix} & 1 \\ -1 & \end{pmatrix}$ $\beta = \begin{pmatrix} & i \\ i & \end{pmatrix}$ $\gamma = \begin{pmatrix} i & \\ & -i \end{pmatrix}$, satisfying $[\alpha,\beta] = 2\gamma$, $[\beta,\gamma] = 2\alpha$, $[\gamma,\alpha] = 2\beta$. Thus $\frac{\alpha}{2} \mapsto a$, $\frac{\beta}{2} \mapsto \beta$, $\frac{\gamma}{2} \mapsto c$ is an isomorphism of LA's: $su(2) \cong \mathbb{R}^3$. It's easy to check that $tr((ada)^2) = -2$ $= tr((adb)^2) = tr((adc)^2)$, and all other terms are 0. Thus $B = \begin{pmatrix} -2 & & \\ & -2 & \\ & & -2 \end{pmatrix}$ w.r.t. a,b,c. and it's negative definite.

In particular, $\mathfrak{su}(2) \ncong \mathfrak{sl}(2,\mathbb{R})$ since the Killing form is intrinsically defined, yet the signatures of $B_{\mathfrak{su}(2)}$, $B_{\mathfrak{sl}(2,\mathbb{R})}$ are different.

Recall that L is semi-simple iff $\dim L > 1$ and $0 = \mathrm{Rad}\,L =$ the maximal solvable ideal in L iff L has no abelian ideals other than 0.

Thm. (Cartan). L is semi-simple iff the Killing form is non-degenerate.
i.e. $\mathrm{Rad}\,B \triangleq \{x \mid B(x,y)=0,\ \forall y \in L\} = 0$.

Note that $\mathrm{Rad}\,B$ is an ideal of L: If $x \in \mathrm{Rad}\,B$, $y, z \in L$, $B([x,y],z)$
$= B(x,[y,z]) = 0 \Rightarrow [x,y] \in \mathrm{Rad}\,B$.

To prove the thm, we need the following thm. of Cartan, whose proof is in Humphreys:
Thm. (Cartan Criterion) $L \subseteq \mathfrak{gl}(V,\mathbb{k})$, $\mathrm{char}\,\mathbb{k} = 0$. Then L solvable $\Leftrightarrow \mathrm{Tr}(xy) = 0$, $\forall x \in L$, $y \in [L,L]$.

(One side of the thm is easy, by $\otimes \bar{\mathbb{k}}$, we may assume $L \subseteq \mathfrak{gl}(\bar{V},\bar{\mathbb{k}})$ and L is contained in $\mathfrak{t}(n)$, then $\forall y \in [L,L]$, $y \in \mathfrak{t}_0(n) \Rightarrow \mathrm{tr}(xy) = 0$).

Now we can prove Cartan's thm using this Criterion.
"$\Rightarrow$": Observe that $\mathrm{Rad}\,B \subseteq \mathrm{Rad}\,L$. Let $S = \mathrm{Rad}\,B$, then $\mathrm{ad}: S \to \mathfrak{gl}(S)$
$\ker \mathrm{ad} = Z(S)$, then we may assume $S \xhookrightarrow{\mathrm{ad}} \mathfrak{gl}(S)$ since a central extension of S will still be solvable if S is. Now $\forall x \in S$, $y \in [S,S]$, $\mathrm{tr}(xy) = 0$ by definition of $S = \mathrm{Rad}\,B$. $\Rightarrow S$ is solvable by Cartan criterion.

Now L is semi-simple $\Rightarrow 0 = \mathrm{Rad}\,L \supseteq \mathrm{Rad}\,B \Rightarrow \mathrm{Rad}\,B = 0 \Rightarrow B$ is non-degenerate.

Remark that it may happen that $\mathrm{Rad}\,B \subsetneq \mathrm{Rad}\,L$.

"$\Leftarrow$": B non-degenerate. It suffices to show that L has no abelian ideals. i.e. those $I \subseteq L$ s.t. $[I,L] \subseteq I$, $[I,I] = 0$.

If yes, take $x \in I$, $y \in L$, then consider $\mathrm{ad}x \circ \mathrm{ad}y$

$$L \xrightarrow{\mathrm{ad}y} L \xrightarrow{\mathrm{ad}x} I \xrightarrow{\mathrm{ad}y} I \xrightarrow{\mathrm{ad}x} 0$$

$\Rightarrow (\mathrm{ad}x \circ \mathrm{ad}y)^2 = 0$, (nilpotent) $\Rightarrow \mathrm{Tr}(\mathrm{ad}x \circ \mathrm{ad}y) = B(x, y) = 0$, contradiction with B being non-degenerate. $\square$

Jordan - Chevalley decomposition

Take $X: V \longrightarrow V$, an endomorphism of $V/\mathbb{k}$, $\mathbb{k} = \bar{\mathbb{k}}$. Then in some basis of V, X is given by Jordan matrices $\mathrm{diag}(J_1, \cdots, J_r)$, $J_i = \begin{pmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_i \end{pmatrix}$, $i = 1, \cdots, r$.

With X, V is turned into a $\mathbb{k}[x]$-module, $x \cdot v = Xv$. $\Rightarrow V \cong \oplus_i \mathbb{k}[x]/(x-\lambda_i)^{n_i}$

$J_i = \lambda_i Id + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}$ = semi-simple part + nilpotent part. $\Rightarrow X = X_s + X_n$, X_s: the semi-simple part of X; X_n: the nilpotent part of X. Surely $[X_s, X] = 0 \Rightarrow [X_n, X] = 0$ and $[X_s, X_n] = 0$.

Claim: X_s is a polynomial in X.

Indeed, write $V = \oplus_\lambda \ker(X-\lambda)^N$: decomposition of V into generalized weight spaces of X, where $N = \dim V$. Take $f(x) \in \mathbb{k}[x]$ s.t. $f(x) \equiv \lambda_i \mod ((x-\lambda_i)^N)$. (It's possible since $(x-\lambda)^N$, $(x-\mu)^N$ are coprime if $\lambda \neq \mu$). Let $X_s = f(X)$, then $X_s \equiv \lambda_i \mod (x-\lambda_i)^N$ and X_s acts on $V(\lambda_i)$ as $\lambda_i Id$. This is a more intrinsic way to define X_s, and we may set $X_n = X - X_s$. Now $X_n \equiv X - \lambda_i \mod (X-\lambda_i)^N \Rightarrow X_n^N \equiv 0 \mod (X-\lambda_i)^N$, and thus X_n is nilpotent.

Such decompositions are unique. If $X = X_s + X_n = X_s' + X_n' \Rightarrow X_s - X_s' = X_n' - X_n$. But the left hand side is semisimple and the right hand side is nilpotent $\Rightarrow 0 = X_s - X_s' = X_n' - X_n$.

In case $\mathbb{k}$ is not algebraically closed, take $X \curvearrowright V \otimes_{\mathbb{k}} \bar{\mathbb{k}} \triangleq \bar{V}$, and $Gal(\bar{\mathbb{k}}/\mathbb{k})$ acts on coefficients of matrices in $\mathfrak{gl}(\bar{V}, \bar{\mathbb{k}})$ and G fixes the entries of X (w.r.t a basis taken from V). $\Rightarrow \bar{X}_s + \bar{X}_n = X = gX = g\bar{X}_s + g\bar{X}_n$. But $g\bar{X}_s$ and $g\bar{X}_n$ are still semi-simple and nilpotent respectively $\Rightarrow g\bar{X}_s = \bar{X}_s$ and $g\bar{X}_n = \bar{X}_n$ $\forall g \in Gal(\bar{\mathbb{k}}/\mathbb{k})$ $\Rightarrow \bar{X}_s, \bar{X}_n \in Mat(n, \mathbb{k})$.

In char p, if $\mathbb{k}$ is not seperable, then it may happen that $X_s, X_n \notin Mat(n, \mathbb{k})$. Indeed, take $\mathbb{k}$, $\mathrm{char}\,\mathbb{k} = p$, $a \in \mathbb{k}$, $\sqrt[p]{a} \notin \mathbb{k}$. Take the $\mathbb{k}[x]$-module $V \cong \mathbb{k}[x]/(x^p - a)$. Then $\bar{\mathbb{k}} \otimes V \cong \bar{\mathbb{k}}[x]/(x - \sqrt[p]{a})^p$, $x = \sqrt[p]{a}\, Id + \begin{pmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{pmatrix}$ in $\bar{V}$. However $X_s \notin Mat(p, \mathbb{k})$

since w.r.t. any basis of $\bar{V}$ $X_s = \sqrt[p]{a}\,Id$ and $\sqrt[p]{a} \notin \Bbbk$.

Now take $X \in gl(V)$, and $X \curvearrowright gl(V)$ by adjoint representation: $adX \in gl(gl(V)) = Mat(n^2, \Bbbk)$ $adX = (adX)_s + (adX)_n = ad(X_s) + ad(X_n)$. Moreover, X_s semi-simple $\Rightarrow adX_s$ is semi-simple; X_n nilpotent $\Rightarrow adX_n$ is nilpotent. Thus by uniqueness $(adX)_s = adX_s$, $(adX)_n = adX_n$.

Characterization of semi-simple Lie algebras.

Lemma: $I \subseteq L$ is an ideal, then $B_I = B_L|_{I\times I}$.

Pf: $\forall\, x, y \in I$, $adx \circ ady : L \longrightarrow I$ and $adx \circ ady = \begin{array}{c} \begin{array}{cc} I & \end{array} \\ \left[\begin{array}{c|c} A & B \\ \hline 0 & 0 \end{array}\right] I \end{array} \Rightarrow tr_L(adx \circ ady) = trA = tr_I(adx \circ ady)$. i.e. $B_L|_{I\times I} = B_I$. □

Thm. L: Lie algebra, $L \neq 0$. Then the following are equivalent:

1). L is semi-simple

2). $Rad\, L = 0$

3). $Rad\, B = 0$, where B is the Killing form of L.

4). L has no solvable ideals other than 0.

5). L has no abelian ideals other than 0.

6). $L = \oplus$ simple Lie algebras.

Pf: It only suffices to check 6) now, the other equivalences were established before.

1) $\Rightarrow$ 6). If L is semi-simple, choose $I \subsetneq L$ a proper ideal, and look at $I^{\perp}$ w.r.t. the Killing form, i.e. $I^{\perp} = \{x \in L \mid B(x,y) = 0,\ \forall y \in I\}$. Then by the associative invariance property of B, we know that $I^{\perp}$ is an ideal.

Claim: $I \cap I^{\perp} = 0$.

Indeed $\forall\, x, y \in I \cap I^{\perp}$, lemma $\Rightarrow B_{I\cap I^{\perp}}(x,y) = B_L(x,y) = 0 \Rightarrow I \cap I^{\perp}$ is solvable

Cartan's criterion $\Rightarrow I \cap I^{\perp}$ is solvable $\Rightarrow I \cap I^{\perp} = 0$. by assumption that L s.s.

It follows that $L = I \oplus I^{\perp}$. We may keep on decomposing I until it's simple and obtain that $L = \oplus_i I_i$ as direct sums of simple ideals (Lie algebras).

6) $\Rightarrow$ 1) is easy. □

Remark: The decomposition in 6) is unique in the strongest sense: up to permutation of the I_i's! Indeed if I is any ideal of L, $[I, I_i] \subseteq I_i$, thus could only be 0 or I_i itself. In the latter case $I \supseteq [I, I_i] = I_i$. Inductively, one obtains that $I = \oplus_j I_j$ where $j \in J = \{i \mid I_i \subseteq I\}$. Thus if I is any simple ideal of L, $I = I_i$ for some I. This is more rigid than the decomposition of representations, where one decomposes a module $V = V_1^{a_1} \oplus \cdots \oplus V_n^{a_n}$, and if $a_i > 1$, $V_i^{a_i} \cong V_i \oplus \cdots \oplus V_i$ is not a canonical decomposition!

In case of decomposition of semi-simple rings: $A = \oplus \mathrm{Mat}(n_i, D_i)$, D_i: division algebras / $\Bbbk$. The $\mathrm{Mat}(n_i, D_i)$ are minimal 2-sided ideals of L, and the only minimal two sided ideals. This is similar as for s.s. Lie algebras.

Thm $\Rightarrow$ to classify s.s. Lie algebras, it suffices to classify the simple Lie algebras.

Cor. L s.s. $\Rightarrow [L, L] = L$. $\square$

Conversely, $[L, L] = L \not\Rightarrow L$ s.s. We can only say that L is not solvable.

E.g. $0 \to \Bbbk^n \to L \to \mathfrak{sl}(n, \Bbbk) \to 0 \quad (n > 1)$

$$L = \left\{ \left(\begin{array}{c|c} * & \Bbbk^n \\ \hline 0 & 0 \end{array} \right) \right\}$$

L is not s.s: $\mathrm{rad} L = \Bbbk^n$, $[L, L] = L$, and L is not solvable.

For L, consider $\mathrm{Der} L = \{ d \mid d[x,y] = [dx, y] + [x, dy],\ \forall x, y \in L \}$, the L.A. of all derivations on L. $\Rightarrow \quad 0 \to \mathrm{Inn} L \to \mathrm{Der} L \to \mathrm{Out}(L) \to 0$, where

$$0 \to Z(L) \to L \xrightarrow{ad} \mathrm{Inn} L \to 0 \ , \quad L \ni x \mapsto \mathrm{ad}x \in \mathrm{Inn}(L)$$

$\mathrm{Inn} L$ is an ideal in $\mathrm{Der} L$, since $[d, \mathrm{ad}x](y) = d[x, y] - [x, dy] = [dx, y] = \mathrm{ad}(dx)(y)$.
$\Rightarrow [\mathrm{Der} L, \mathrm{Inn} L] \subseteq \mathrm{Inn} L$.

The concept of inner derivations is the infinitesimal version of the fact that $\mathrm{Inn}(G)$ sits in $\mathrm{Aut}(G)$ as a normal subgroup.

E.g. L abelian $\Rightarrow \mathrm{Inn} L = 0$, $\mathrm{Der}(L) \cong \mathrm{Out}(L)$.

Take any linear map $f: L \to L \Rightarrow [f(x), f(y)] = 0 = f([x, y])$
$\Rightarrow \mathrm{Der} L = \mathrm{Out}(L) = \mathrm{End} L$.

On the other extreme, we have:

Cor. L s.s. $\Rightarrow \mathrm{Inn}L \cong \mathrm{Der}L$. ($\Rightarrow \mathrm{Out}L = 0$)

Pf: Take $d \in \mathrm{Der}L$, $tr(d \circ adx)$ defines a linear functional on L. Since the Killing form B of L is non-degenerate, $tr(d \circ adx) = B(y, x)$ for some $y \in L$.

Claim: $d(z) = [y, z]$, $\forall z \in L$, thus d is inner.

Indeed, $[d, adz] = ad(dz) \Rightarrow B(dz, x) = tr(ad\, dz \circ adx) = tr([d, adz] \circ adx)$ $= tr(d, [adz, adx]) = tr(d, ad[z,x]) = B(y, [z, x]) = B([y, z], x)$, and B is non-degenerate. □

Remark: In finite group case, there is no such analogue. Indeed, even if G is a simple group, $\mathrm{Out}(G)$ may not be trivial. For instance $\mathrm{Out}(A_n) \cong \mathbb{Z}/2$, $n>6$ (coming from conjugation by S_n). However, in case of Lie groups, we have the fact that all automorphisms of G close to identity is inner. (since $\mathrm{Der}\,\mathfrak{g} = \mathrm{Lie}(\mathrm{Aut}(G)_0)$, where $\mathfrak{g}$ = L.A. of G.)

Casimir element.

L: simple L.A. V: an irrep of L. Then $\varphi: L \to \mathfrak{gl}(V)$ is either trivial or faithful.

Now suppose V an irrep, non-trivial. We may define a symmetric bilinear form B_V on L: $B_V: L \times L \to \mathbb{k}$: $B_V(x, y) = tr(\varphi(x) \circ \varphi(y))$. Then

1). $B_V(x, y) = B_V(y, x)$, symmetric

2). $B_V([x,y], z) = B_V(x, [y, z])$ associative invariant.

These two properties actually works for any Lie algebra with any rep. In particular

3). B_L is the Killing form, where $L \to \mathfrak{gl}(L)$ is the adjoint rep.

4). B_V is non-degenerate in case L simple and V irrep. Indeed $\mathrm{Rad}B_V$ is an ideal of L (by 2)), thus could only be L or 0. But by Cartan's criterion, if V is non-trivial, $\varphi: L \xrightarrow{\sim} \varphi(L) \subseteq \mathfrak{gl}(V)$, and $tr(\varphi(x)\varphi(y)) \not\equiv 0$. Thus $\mathrm{Rad}B_V = 0$.

5). The B_V's are propotional to each other. This follows from the more general fact that L simple $\Rightarrow$ all associative bilinear maps on L are propotional. (Indeed, as rep's of L, $L \xrightarrow[\sim]{B_1^*} L^* \xrightarrow{B_2^*} L$, L irreducible $\Rightarrow B_2^* \circ B_1^* = $ const, by Shur's lemma).

Choose a basis of L : $\{x_i\}$, and take its dual basis w.r.t. V, say $\{y_j\}$ i.e. $B(x_i, y_j) = \delta_{ij}$.

Def: (Casimir element) $C_V \triangleq \sum_i x_i y_i \in \mathcal{U}(L)$.

Lemma: C_V is central in $\mathcal{U}(L)$.

Pf: $\forall\, x \in L$, $[x, x_i] = a_{ij} x_j$. B_V associative $\Rightarrow a_{ij} = B_V([x, x_i], y_j) = -B_V(x_i, [x, y_j])$

$\Rightarrow [x, y_j] = -a_{ij} y_i$.

$$\begin{aligned}\Rightarrow [x, C_V] &= \textstyle\sum_i ([x\ x_i] y_i + x_i [x, y_i]) \\ &= \textstyle\sum_i (a_{ij} x_j y_i + x_i (-a_{ji}) y_j) \\ &= \textstyle\sum_i (a_{ij} x_j y_i - a_{ij} x_j y_i) \\ &= 0. \end{aligned}$$ □

Now, by Shur's lemma, $C_V : V \longrightarrow V$ as an endomorphism of V must be constant since V is irrep. $\Rightarrow C_V = \lambda\, Id_V \Rightarrow tr_V(C_V) = \lambda \cdot \dim V$, also $tr_V(C_V) = B_V(x_i, y_i) = \dim L$ $\Rightarrow \lambda = \frac{\dim L}{\dim V}$. Presumablely, C_V might be a constant in $\mathcal{U}(L)$, but this computation and the fact that $C_V = 0$ on the trivial rep shows that this won't happen.

Lemma: C_V is invariant under changes of basis.

Pf: Take $\{x_i'\}$, $\{y_j'\}$ new basis and dual basis. Then $x_i' = a_{ij} x_j$, for some invertible matrix (a_{ij}). If $y_j' = \sum b_{ji} y_i$, then $\delta_{ij} = B(x_i', y_j') = B(a_{ik} x_k, b_{j\ell} y_\ell) = a_{i\ell} b_{j\ell}$

$\Rightarrow \sum x_i' y_i' = \sum a_{ij} x_j b_{i\ell} y_\ell = \sum \delta_{j\ell} x_j y_\ell = \sum x_j y_j$. □

Thus C_V is intrinsically associated with (L, V). Moreover, since for non-trivial irrep's of L, B_V are propotional to each other by 5) above, such C_V's differ only by a non-zero constant.

Complete reducibility

Thm. Any finite dimensional representation of a simple LA is completely reducible.

Pf: The proof is almost identical to the $sl(2, \mathbb{C})$ case.

Step 1. We show that the trivial repn can always be split off.

By a reduction argument as in step 2 of $\mathfrak{sl}(2,\mathbb{C})$, it suffices to check that

$$0 \longrightarrow W \longrightarrow V \longrightarrow \mathbb{C} \longrightarrow 0 \quad (*)$$

where W is an irrep of L, always splits.

If $W \cong \mathbb{C}$, then L acts on V nilpotently: $V \xrightarrow{L} W \xrightarrow{L} 0$, thus trivially $\Rightarrow (*)$ splits.

If $W \not\cong \mathbb{C}$, then $C_W \in Z(\mathcal{U}(L))$ acts as a non-zero scalar on W $(= \frac{\dim L}{\dim W})$. Thus

$$\begin{tikzcd} 0 \arrow[r] & W \arrow[r] \arrow[d, "C_W"] & V \arrow[r] \arrow[d, "C_W"] \arrow[dl] & \mathbb{C} \arrow[r] \arrow[d, "C_W"] & 0 \\ 0 \arrow[r] & W \arrow[r] & V \arrow[r] & \mathbb{C} \arrow[r] & 0 \end{tikzcd}$$

$C_W : V \longrightarrow W$ since C_W acts as 0 on $\mathbb{C}$ $\Rightarrow \frac{\dim W}{\dim L} \cdot C_W$ is a splitting of $(*)$ and $V \cong W \oplus \ker C_W$.

Step 2. Any $0 \longrightarrow W \longrightarrow V \longrightarrow V/W \longrightarrow 0$ splits.

Again consider the L-module $R = \mathrm{Hom}_{\mathbb{C}}(V, W)$ and its submodules $S = \{f \in R \mid f|_W = \lambda\, \mathrm{Id}_W\}$, $S_0 = \{f \in S \mid f|_W = 0\}$, $S \xrightarrow{L} S_0$ as in $\mathfrak{sl}(2,\mathbb{C})$ case. The sequence:

$$0 \longrightarrow S_0 \longrightarrow S \longrightarrow \mathbb{C} \longrightarrow 0$$

is exact, thus splits by step 1. Thus we may find $f \in S$, $f|_W = \mathrm{Id}_W$, $f(xv) = x f(v)$, $\forall\, x \in L$, $v \in V$. Thus f serves as a splitting map. $\square$

Thm. Any finite dimensional representation of a semi-simple L.A. is completely reducible.

Pf: $L = L_1 \oplus \cdots \oplus L_k$, where each L_i is simple $\Rightarrow \mathcal{U}(L) \cong \mathcal{U}(L_1) \otimes \cdots \otimes \mathcal{U}(L_k)$.

To give a rep of L on $V \Leftrightarrow L \longrightarrow \mathfrak{gl}(V)$ as Lie algebras $\Leftrightarrow \mathcal{U}(L) \longrightarrow \mathrm{End}(V)$ as associative algebras. In this language, any finite dim'l rep of L is completely reducible $\Leftrightarrow$ any finite dim'l quotient algebra of $\mathcal{U}(L) \cong \bigoplus_i \mathrm{Mat}(n_i, \mathbb{C})$.

Now, given a finite dim'l quotient A of $\mathcal{U}(L)$, then $A \cong A_1 \otimes \cdots \otimes A_n$ where A_i is a finite dim'l quotient of $\mathcal{U}(L_i)$. By the previous thm, $A_i \cong \bigoplus_j \mathrm{Mat}(n_{ij}, \mathbb{C})$

$\Rightarrow A \cong \bigotimes_{i=1}^{k} (\bigoplus_j \mathrm{Mat}(n_{ij}, \mathbb{C})) \cong \bigoplus_{j_1, \ldots, j_k} \mathrm{Mat}(n_{1j_1} \cdots n_{kj_k}, \mathbb{C})$, thus completely reducible.

Here we used $\mathrm{Mat}(n, \mathbb{C}) \otimes \mathrm{Mat}(m, \mathbb{C}) \cong \mathrm{Mat}(nm, \mathbb{C})$. $\square$

Semisimple and nilpotent elements.

Let A be a finite dimensional algebra over $\mathbb{C}$, d a derivation on A. Then A can be decomposed as generalized weight spaces of d: $A = \oplus_{\lambda\in\mathbb{C}} A(\lambda)$, where $A(\lambda) = \ker(d-\lambda Id)^N$, $N \gg 0$, and $d = d_s + d_n$, where $d_s|_{A(\lambda)} = \lambda Id$.

Claim: d_s is also a derivation. i.e. $d_s(xy) = (d_s x)y + x d_s y$.

Indeed, we have $A(\lambda)\cdot A(\mu) = A(\lambda+\mu)$: $\forall x\in A(\lambda)$, $y\in A(\mu)$

$$(d-(\lambda+\mu)Id)^N(xy) = \sum_{k=0}^{N}\binom{N}{k}(d-\lambda)^k x\cdot(d-\mu)^{N-k}y = 0 \quad \text{for } N\gg 0.$$

Thus $d_s(xy) = (\lambda+\mu)xy = (\lambda x)\cdot y + x\cdot(\mu y) = (d_s x)\cdot y + x\cdot d_s y \Rightarrow d_s \in Der(A)$.

Now if L is semi-simple, $DerL \cong L$. $\forall x\in L$, $adx\in DerL \Rightarrow adx = (adx)_s + (adx)_n$
$(adx)_s \in DerL \cong L$, $(adx)_n = adx - (adx)_s \in DerL = L$. Since moreover, the adjoint rep is faithful, $(adx)_s = adx_s$, $(adx)_n = adx_n$ for some $x_s, x_n \in L$, and $x = x_s + x_n$.

Def: $h\in L$ is called semisimple if $adh = (adh)_s$; h is called nilpotent if $adh = (adh)_n$.

Lemma: A semisimple element acts semisimplely on any finite dim'l rep of L.

Pf: $h\in L$, $L \longrightarrow gl(V)$ a rep. $V = \oplus V(\lambda)$, decomposition of V as generalized eigenspaces of h: $v\in V(\lambda)$ iff $(h-\lambda)^N v = 0$. Take $0\neq V'(\lambda)\subseteq V(\lambda)$ be the subspace of eigenvectors of h. Then $\oplus V(\lambda) = V \supseteq V' = \oplus V'(\lambda)$.

Claim: V' is a subrep of V.

Since we may also decompose L as adh-weight spaces: $L = \oplus L(\lambda)$, where $x\in L(\lambda)$ iff $[h,x] = \lambda x$, it suffices to check that $L(\mu)V'(\lambda) \subseteq V'(\mu+\lambda)$: $\forall x\in L(\mu)$, $v\in V'(\lambda)$

$$hxv = xhv + [h\,x]v = x\lambda v + \mu xv = (\lambda+\mu)xv.$$

The claim follows. Finally, since V is completely reducible, $V \cong V'\oplus V''$ for some $V''\subseteq V$. The lemma follows by induction on $\dim V$. $\square$

Furthermore if L is a simple LA, $\varphi: L\to gl(V)$ a non-trivial rep. If h acts semisimply on V, then $h = h_s + h_n \Rightarrow \varphi(h)_n = \varphi(h_n) = 0$, but φ faithful $\Rightarrow h_n = 0$. It follows that h is semisimple.

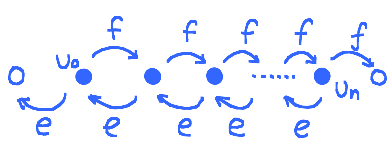
Combing the above discussion with the lemma, we obtain the following characterization of elements of L:

- L: a simple LA, $h \in L$.

h is semisimple in L
$\Leftrightarrow$ $\operatorname{ad} h$ is semisimple
$\Leftrightarrow$ h acts semisimply in all finite dim'l rep of L.
$\Leftrightarrow$ h acts semisimply in some non-trivial finite dim'l irrep of L.

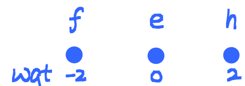
h is nilpotent in L
$\Leftrightarrow$ $\operatorname{ad} h$ is nilpotent
$\Leftrightarrow$ h acts nilpotently in all finite dim'l rep of L.
$\Leftrightarrow$ h acts nilpotently in some non-trivial finite dim'l irrep of L.

E.g. $L = \mathfrak{sl}(2,\mathbb{C}) = \mathbb{C}\{e, f, h\}$. h is semisimple and acts semisimply in any V_n
e, f are nilpotent and act nilpotently on any V_n:
$e^{n+1} = 0 = f^{n+1}$, $h = \operatorname{diag}(n, n-2, \cdots, -n)$



Application on Lie algebras
L: LA, $h \in L$ semisimple, $\Rightarrow L = \oplus_{\lambda \in \mathbb{C}} L_\lambda$, $[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}$, i.e. L is graded by eigenvalues of h. If I is an ideal in L, then I respects the weight decomposition of L: $I_\lambda \triangleq I \cap L_\lambda$, then $I = \oplus_\lambda I_\lambda$.
Indeed, if $x \in I$, $x = \sum x_{\lambda_i}$. $[h, x] = \sum \lambda_i x_{\lambda_i} \in I, \cdots, (\operatorname{ad} h)^{n-1} x = \sum \lambda_i^{n-1} x_{\lambda_i} \in I$
$\Rightarrow x_{\lambda_i} \in I$, $\forall i$.

E.g. $\mathfrak{sl}(3,\mathbb{C})$, $(\mathfrak{sl}(n,\mathbb{C}))$
$\mathfrak{sl}(n,\mathbb{C}) \supseteq H \triangleq \operatorname{Span}\{h_i \mid h_i = e_{ii} - e_{i+1,i+1}\}$: diagnal, ad-semisimple, commutative.
$\Rightarrow L = \oplus_{\lambda \in H^*} L_\lambda$, $L_\lambda = \{x \mid [h, x] = \lambda(h) x, \forall h \in H\}$. Such λ can be described by $(n-1)$-numbers, since we have fixed a distinguished basis of H.
$$\lambda = (\lambda_1, \cdots, \lambda_{n-1}), \quad \lambda_i = \lambda(h_i), \quad i = 1, \cdots, n-1.$$
Since $[h_k, e_{ij}] = [e_{kk}, e_{ij}] - [e_{k+1,k+1}, e_{ij}] = (\delta_{ki} - \delta_{kj} - \delta_{k+1,i} + \delta_{k+1,j}) e_{ij}$
$\Rightarrow \mathfrak{sl}(n,\mathbb{C}) = L_0 \oplus (\oplus_{\lambda \in H^*} L_\lambda) = H \oplus (\oplus_{i \neq j} \mathbb{C} e_{ij})$ and these $\lambda \in H^*$ seperate e_{ij}, $(i \neq j)$
i.e. $[h\ e_{ij}] = \lambda(h) e_{ij}$, $[h\ e_{kl}] = \lambda'(h) e_{kl}$. $\lambda(h) = \lambda(h') \Rightarrow e_{ij} = e_{kl}$.

$sl(2,\mathbb{C}) = \mathbb{C}h \oplus \mathbb{C}e \oplus \mathbb{C}f$ 

$sl(3,\mathbb{C})$: $H = \mathbb{C}h_1 \oplus \mathbb{C}h_2$: $h_1 = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}$, $h_2 = \begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$. $H^* \cong \mathbb{C}^2$

Take $e_{12} \in sl(3,\mathbb{C})$ and let $[h, e_{12}] = \alpha_1(h) e_{12}$, $\alpha_1 \in H^*$

then $[h_1\ e_{12}] = 2e_{12}$, $[h_2, e_{12}] = -e_{12}$ $\Rightarrow \alpha_1(h_1) = 2$, $\alpha_1(h_2) = -1$

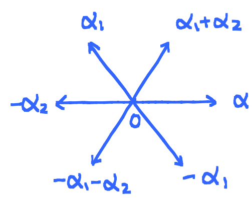
Take $e_{23} \in sl(3,\mathbb{C})$ and similarly define $\alpha_2(h)$: $[h, e_{23}] = \alpha_2(h) e_{23}$

then $\alpha_2(h_1) = -1$ $\alpha_2(h_2) = 2$.

$\Rightarrow \alpha_1, \alpha_2$ span H^*. Since $e_{13} = [e_{12}, e_{23}]$, $[h, e_{13}] = (\alpha_1 + \alpha_2)(h) e_{13}$

$e_{12}^t = e_{21}$ $h^t = h$ $\Rightarrow [h, e_{21}] = -\alpha_1(h) e_{21}$, similarly for $e_{31} \leftrightarrow -\alpha_2$, $e_{31} \leftrightarrow -\alpha_1 - \alpha_2$.

We obtain the weight diagram of $sl(3,\mathbb{C})$:



Thus $sl(3,\mathbb{C})$ consists of 3 copies of $sl(2,\mathbb{C})$, each one taking up a direction in the weight diagram: α_1: $\{e_{12}, h_1, e_{21}\}$, α_2: $\{e_{23}, h_2, e_{32}\}$, $\alpha_1+\alpha_2$: $\{e_{13}, h_1+h_2, e_{31}\}$

If I is any ideal $\neq 0$ in $sl(3,\mathbb{C})$, it will contain some weight vector. If it contains e_{ij}, then it contains all $sl(3\ \mathbb{C})$ since $[e_{ij}, e_{jk}] = e_{ik}$, $[e_{ij}\ e_{ji}] = e_{ii} - e_{jj}$.

Thus we obtain, by similar methods extended to $sl(n,\mathbb{C})$:

Thm. $sl(n,\mathbb{C})$ is simple, $\forall n > 1$. $\square$

Now if $H \subseteq L$ an (abelian) subalgebra consisting of semisimple elements. Then we can decompose L as $L = \oplus_{\alpha \in H^*} L_\alpha$. $L_\alpha = \{x \in L \mid [h, x] = \alpha(h) x,\ \forall h \in H\}$. $\Phi = \{\alpha \neq 0 \mid L_\alpha \neq 0\}$ are called roots.

Rmk: if H consists of semisimple elements, it's necessarily abelian. Indeed, if $\exists$ $x \in H$, $[x, H] \neq 0$, then $\exists y \in H$ s.t. $[x, y] = \lambda y$, $y \in H$, $\lambda \neq 0$. (H respects the weight decomposition of L w.r.t. $\mathrm{ad}x$ just as for the ideal case). Now $x \xmapsto{\mathrm{ad}y} \lambda y \xmapsto{\mathrm{ad}y} 0$ $\Rightarrow x \in$ generalized 0-eigenspace of $\mathrm{ad}y$ and $x \neq 0$. This contradicts the fact that y is semisimple.

Def: $H \subseteq L$ a subalgebra consisting of semisimple elements is called toral. Maximal toral subalgebras are called Cartan subalgebas of L.

Now take a Cartan subalgebra $H \subseteq L$ and apply the wgt space decomposition:
$= \bigoplus_{\alpha \in H^*} L_\alpha = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$ where $L_0 = \{y \in L \mid [h, y] = 0, \forall h \in H\} = C_L(H)$ and
$\Phi = \{\alpha \in H^* \setminus \{0\} \mid L_\alpha \neq 0\}$.

Lemma: $B(L_\alpha, L_\beta) = 0$ if $\alpha + \beta \neq 0$. Consequently, since B is non-degenerate, we have: $L_\alpha \neq 0$ iff $L_{-\alpha} \neq 0$.

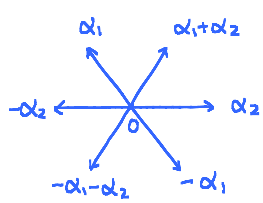
Pf: $\forall x \in L_\alpha,\ y \in L_\beta,\ h \in H$, $B([x, h], y) = B(x, [h, y])$
$\Rightarrow -\alpha(h) B(x, y) = \beta(h) B(x, y) \Rightarrow (\alpha+\beta)(h) B(x, y) = 0$.
Thus if $\alpha + \beta \neq 0$, $\exists h \in H$ s.t. $(\alpha+\beta)(h) \neq 0 \Rightarrow B(x, y) = 0$ $\square$

Lemma $\Rightarrow B: L_\alpha \times L_{-\alpha} \longrightarrow \mathbb{C}$ is non-degenerate. Thus $L_\alpha \cong L_{-\alpha}^*$ canonically and $\dim L_\alpha = \dim L_{-\alpha}$. Furthermore $B|_{C_L(H) = L_0}$ is non-degenerate.

Prop. $C_L(H) = H$, where $H \subseteq L$ is maximal toral.
Idea of proof: Keep track of $x = x_s + x_n$, and if $x = x_s \in C_L(H)$, $x = x_s \Rightarrow x \in H$. Otherwise $H + \mathbb{k}x$ is toral and $H + \mathbb{k}x \supsetneq H$, contradiction with H being maximal.
For the proof see Humphreys. $\square$

Cor. $B|_{H \times H}$ is non-degenerate. $\square$

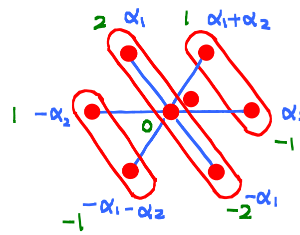
E.g. Restriction of B to $H = \mathbb{C}\{h_1, h_2\}$, $h_1 = \begin{pmatrix}1&&\\&-1&\\&&0\end{pmatrix}$, $h_2 = \begin{pmatrix}0&&\\&1&\\&&-1\end{pmatrix}$.



To calculate $B(h_i, h_j)$ $(i, j = 1, 2)$, we use the following adjoint actions of $\mathfrak{sl}(2)$'s on $\mathfrak{sl}(3)$ (an 8-dim'l rep of $\mathfrak{sl}(2)$):

$\mathfrak{sl}(2)_1 = \left\{ \begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 0 \end{pmatrix} \right\}$ $\qquad$ $\mathfrak{sl}(2)_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ 0 & * & * \\ 0 & * & * \end{pmatrix} \right\}$

As an $\mathfrak{sl}(2)_1$-module, $\mathfrak{sl}(3)$ decomposes as $\mathfrak{sl}(2)$ representations: $V_2 \oplus V_1^{\oplus 2} \oplus V_0$

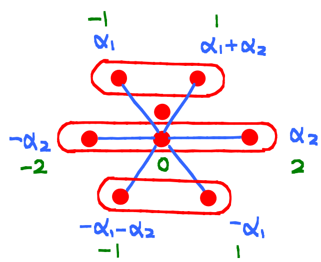


$\alpha_1(h_1) = 2$

$\alpha_2(h_1) = -1$

The $\mathrm{ad}h_1$- weight decomposition of $\mathfrak{sl}(3)$

and $\mathrm{ad}h_1$ acts with weights as shown above. Similarly for the copy $\mathfrak{sl}(2)_2$



$\alpha_2(h_2) = 2$

$\alpha_1(h_2) = -1$

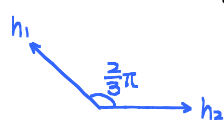
The $\mathrm{ad}h_2$- weight decomposition of $\mathfrak{sl}(3)$

Thus $B(h_1, h_1) = \mathrm{Tr}(\mathrm{ad}h_1 \circ \mathrm{ad}h_1) = 2^2 + (-2)^2 + 1^2 + (-1)^2 + 1^2 + (-1)^2 + 0^2 + 0^2 = 12$

and similarly $B(h_1 h_2) = 2\cdot(-1) + 1\cdot 1 + (-1)\cdot 2 + (-2)\cdot 1 + (-1)\cdot(-1) + 1\cdot(-2) = -6$

$B(h_2, h_2) = (-1)^2 + 1^2 + 2^2 + 1^2 + (-1)^2 + (-2)^2 = 12.$

It follow that in H, h_1, h_2 makes an angle $\frac{2}{3}\pi$ and are of the same length.



and we can see that $B(h, h') = \sum_{\alpha\in\Phi} \alpha(h)\alpha(h')$, $\Phi = \{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$.

Def: Since B is non-degenerate on H, $H \cong H^*$ canonically via B. $\forall \nu \in H^*$, we may define t_ν via: $B(h, t_\nu) = \nu(h)$, $\forall h \in H$.

E.g.

For $\mathfrak{sl}(2) = \mathbb{C}\{e, h, f\}$ $B = \begin{pmatrix} & & 4 \\ & 8 & \\ 4 & & \end{pmatrix}$ $\Rightarrow B(h,h) = 8$. $H = \mathbb{C}h$ $H^* = \mathbb{C}\alpha$ where $\alpha(h) = 2$. $t_\alpha \in H$. $B(t_\alpha h) = \alpha(h) = 2 \Rightarrow t_\alpha = \frac{h}{4}$.

For $\mathfrak{sl}(3)$, $H = \mathbb{C}\{h_1, h_2\}$ $H^* = \mathbb{C}\{\alpha_1, \alpha_2\}$. $\alpha_1(h_1) = 2$, $\alpha_1(h_2) = -1$; $\alpha_2(h_1) = -1$ $\alpha_2(h_2) = 2$, and $B(h_1, h_1) = 12$ $B(h_1, h_2) = -6$ $B(h_2, h_2) = 12$.

$B(t_{\alpha_1}, h) = \alpha_1(h) \Rightarrow t_{\alpha_1} = \frac{h_1}{6}$; $B(t_{\alpha_2}, h) = \alpha_2(h) \Rightarrow t_{\alpha_2} = \frac{h_2}{6}$.

Now fix maximal H and $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$. We list some basic properties:

a). Φ spans H^*

Pf: If $\mathbb{C}\Phi \neq H^*$, then $\exists h \in H$, $[h, x_\alpha]=0$, $\forall x_\alpha \in L_\alpha \Rightarrow [h, L]=0$, then $h \in Z(L)$, contradiction. $\square$

b). $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$

Pf: Since B is non-degenerate $B(L_\alpha, L_\beta)=0$ for $\beta \neq -\alpha$. $\square$

c). $x \in L_\alpha$, $y \in L_{-\alpha} \Rightarrow [x,y] \in L_0 = H$. Moreover $[x,y] = B(x,y)t_\alpha$.

Pf: $B(h\ [x,y]) = B([h,x], y) = \alpha(h) B(x,y) = B(t_\alpha, h) B(x,y)$

$= B(h, B(x,y)t_\alpha)$, $\forall h \in H$

B non-degenerate on $H \Rightarrow [x,y] = B(x,y)t_\alpha$. $\square$

d). $\dim [L_\alpha, L_{-\alpha}] = 1$ and $[L_\alpha, L_{-\alpha}] = \mathbb{C}t_\alpha$. $\square$

e). $\alpha(t_\alpha) = B(t_\alpha, t_\alpha) \neq 0$

Pf: Suppose $\alpha(t_\alpha)=0$. Choose $x \in L_\alpha$, $y \in L_{-\alpha}$, s.t. $[x,y]=t_\alpha$ (this can be done since B is non-degenerate on $L_\alpha \times L_{-\alpha}$.) $[t_\alpha, x] = \alpha(t_\alpha)x = 0$, $[t_\alpha, y]=0$.
$\Rightarrow S = \mathbb{C}\{x, y, t_\alpha\}$ is a 3 dimensional subalgebra of L. $[S,S] = \mathbb{C}t_\alpha$ and $[[S,S],[S,S]]=0 \Rightarrow S$ is nilpotent. Thus $ad: S \longrightarrow gl(L)$ can be conjugated to strictly upper triangular matrices. $\Rightarrow t_\alpha = (t_\alpha)_n$, contradiction. $\square$

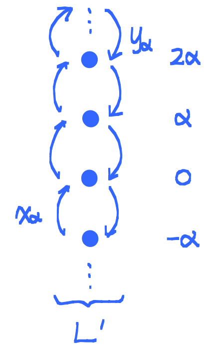
f). $L_\alpha \oplus L_{-\alpha} \oplus \mathbb{C}t_\alpha$ spans a copy of $sl(2,\mathbb{C})$ in L.

Pf: Take $0 \neq x_\alpha \in L_\alpha$ ($\leftrightarrow e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$). Let $h_\alpha = \frac{2t_\alpha}{B(t_\alpha, t_\alpha)}$ ($\leftrightarrow h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$)
Then $[h_\alpha, x_\alpha] = \frac{2}{\alpha(t_\alpha)}[t_\alpha\ x_\alpha] = \frac{2\alpha(t_\alpha)}{\alpha(t_\alpha)} x_\alpha = x_\alpha$. Finally, define $y_\alpha \in L_{-\alpha}$ by $[x_\alpha, y_\alpha] = h_\alpha$. (which is possible since $B|_{L_\alpha \times L_{-\alpha}}$ is non-degenerate. $\square$

g). $\dim L_\alpha = 1$.

Pf: Consider the $sl(2)$ constructed above and $L' = \bigoplus_{\lambda\in\mathbb{Z}} L_{\lambda\alpha}$.

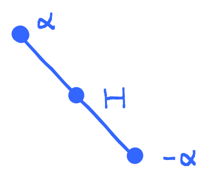
$\mathfrak{sl}(2) \curvearrowright L'$ and L' decomposes as L' wgt spaces:
Since $[L_\alpha, L_{-\alpha}] = \mathbb{C}h_\alpha$ and $\mathrm{ad}\, y_\alpha : L_\alpha \longrightarrow L_0 = H$ is always injective $\Rightarrow L' \cong \mathfrak{sl}(2) \oplus \ker x_\alpha$, where $\mathfrak{sl}(2) = \mathbb{C}\{x_\alpha, h_\alpha, y_\alpha\}$.
$\Rightarrow L_\alpha = \mathbb{C}x_\alpha$ and $\dim L_\alpha = 1$.



□

In particular, we have:
Cor. Multiples of roots are not roots. □

Rmk: The idea is to study L via these copies of $\mathfrak{sl}(2)$'s constructed as in f), one for each pair of $\{\alpha, -\alpha\}$. Also note that although the choices of x_α, h_α



→ looks like $\mathfrak{sl}(2) \oplus$ an abelian LA.

y_α are not canonical, $\mathfrak{sl}(2)_\alpha = L_\alpha \oplus L_{-\alpha} \oplus [L_\alpha, L_{-\alpha}]$ is canonically associated with α.

However, potentially, $\frac{\alpha}{2}$ might be in Φ, but we may start by working with $\mathfrak{sl}(2)_{\frac{\alpha}{2}}$
$\Rightarrow \alpha \notin \Phi$ which is a contradiction. More generally, consider the subspace of L: $\bigoplus_{n \in \mathbb{Q}} L_{n\alpha}$, which is also an $\mathfrak{sl}(2)_\alpha$-module, but by similar reasoning, $\frac{\alpha}{\ell} \notin \Phi$, for any $\ell \in \mathbb{N}$, $\ell > 1$, and $\bigoplus_{n \in \mathbb{Q}} L_{n\alpha} = L_\alpha \oplus L_{-\alpha} \oplus H$.

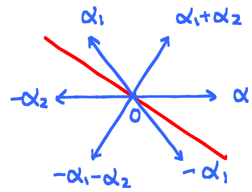
Now, if $\alpha, \beta \in \Phi$, we may consider $\bigoplus_{n \in \mathbb{Z}} L_{\beta + n\alpha}$ as an $\mathfrak{sl}(2)_\alpha$ module, which must in fact be an irrep of $\mathfrak{sl}(2)_\alpha$. Indeed, $\dim L_\gamma = 1$ $\forall \gamma \in \Phi$ and the weights $\beta + n\alpha$ on h_α always shift by 2. In particular:

Cor. If $\alpha, \beta, \alpha+\beta \in \Phi$, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ (previously we only know that $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$) since e_α always maps L_β isomorphically onto $L_{\beta+\alpha}$ in this irrep $\bigoplus_{n \in \mathbb{Z}} L_{\beta+n\alpha}$. □

Cor. $\beta(h_\alpha) = \frac{2\langle \alpha, \beta \rangle}{\langle \alpha, \alpha \rangle} \in \mathbb{Z}$. □

Fix $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$, we can always take a plane to seperate Φ into Φ^+ and Φ^- so that $\alpha \in \Phi^+$ then $-\alpha \in \Phi^-$ ($\alpha, -\alpha$ lie on different sides of the plane) and if $\alpha, \beta \in \Phi^+$, $\alpha+\beta \in \Phi \Rightarrow \alpha+\beta \in \Phi^+$.

E.g. $sl(3)$.



Then $L^+ \triangleq \oplus_{\alpha\in\Phi^+} L_\alpha$ is a (nilpotent) subalgebra of L since $[L_\alpha\ L_\beta] = L_{\alpha+\beta}$ or 0. (In $sl(3)$ case as above, $[L^+, L^+] = L_{\alpha_1+\alpha_2}$ and $[L^+, [L^+, L^+]] = 0$.) This is an analogue of the subalgebra of strictly upper triangular matrices in $sl(n)$ for arbitrary simple LA's.

Similarly, $H \oplus L^+$ is a (solvable) subalgebra of L which is not nilpotent ($[H\oplus L^+, H\oplus L^+] = L^+$, but $[H\oplus L^+, L^+] = L^+$). This is an analogue of upper triangular matrices in $sl(n)$ for arbitrary simple LA's.

Def: $L^+ \oplus H$ is called a Borel subalgebra of L. $L = \overbrace{L^+ \oplus \underbrace{H}_{}}^{\text{positive Borel}} \underbrace{\oplus L^-}_{\text{negative Borel}}$

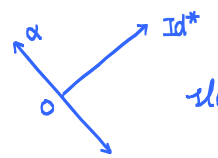
Root (Weight) decomposition of classical LA's.

- $sl(n)$ ($\subseteq gl(n)$)

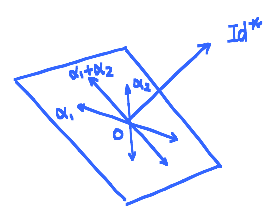
Let $\tilde{H} \subseteq gl(n)$ ($H \subseteq sl(n)$) : (traceless) diagonal matrices.

$\tilde{H} = \mathbb{C}\langle h_1, \cdots, h_n\rangle$ $h_i = e_{ii}$, and $[h_i, e_{ij}] = e_{ij}$ $[h_j\ e_{ij}] = -e_{ij}$ $(i \neq j)$, and $[h_k, e_{ij}] = 0$ if $k \neq i, j$. Thus in $\tilde{H}^*$, we obtain a dual basis $\varepsilon_1 = h_1^*, \varepsilon_2 = h_2^* \cdots \varepsilon_n = h_n^*$ and the weight of e_{ij} is $\varepsilon_i - \varepsilon_j$.

Now $\tilde{H} = H \oplus \mathbb{C}\mathrm{Id} \Rightarrow \tilde{H}^* = H^* \oplus \mathbb{C}\mathrm{Id}^*$, $H^* = \mathbb{C}\{\varepsilon_i - \varepsilon_j\}_{i\neq j}$, which is an $(n-1)$-dim'l hyperplane of $\tilde{H}^*$.



$sl(2)$



$sl(3)$

Note that $[e_{ij}, e_{jk}] = e_{ik}$ $(i\neq k)$ corresponds in H^* $\varepsilon_i - \varepsilon_j + \varepsilon_j - \varepsilon_k = \varepsilon_i - \varepsilon_k$.
$\Phi = \{\varepsilon_i - \varepsilon_j\}_{i\neq j}$ can be partitioned into $\Phi^+ = \{\varepsilon_i - \varepsilon_j\}_{i<j}$ and $\Phi^- = \{\varepsilon_i - \varepsilon_j\}_{i>j}$ and using this fact, we may prove the thm that $\mathfrak{sl}(n)$ is simple:

Take $I \subseteq \mathfrak{sl}(n)$ an ideal, then I is homogeneous in the sense that

$$I = I\cap H \oplus \bigoplus_{\alpha\in\Phi} I \cap L_\alpha$$

whose proof is similar as before. Here we may take $h\in H$ s.t. $\alpha\neq\beta \Rightarrow \alpha(h)\neq\beta(h)$. ($(\alpha-\beta)(h)=0$ are finitely many hyperplanes, $\alpha,\beta\in\Phi$)
If $I \supseteq L_{\varepsilon_i-\varepsilon_j}$ or $e_{ij}\in I$, then by subsequent actions of $\mathrm{ad}\, e_{jk}$ shows that $I\supseteq L_\alpha$, $\forall \alpha\in\Phi$. If $I\ni h$, then take $\varepsilon_i-\varepsilon_j$ s.t. $(\varepsilon_i-\varepsilon_j)(h)\neq 0$, then $[h, e_{ij}] = (\varepsilon_i-\varepsilon_j)(h) e_{ij} \in I \Rightarrow I\supseteq L_{\varepsilon_i-\varepsilon_j}$.

- $\mathfrak{so}(n)$

Usually $\mathfrak{so}(n)$ refers to LA of antisymmetric matrices, but it's not so convenient to work with since it contains no diagonal matrices. Instead, we work with another version of $\mathfrak{so}(n)$.

- Orbits of $GL_n(\Bbbk) \curvearrowright Sym(n) = \{$symmetric $n\times n$ matrices$\}$ $T\in GL_n(\Bbbk)$, $J\in Sym(n)$ $T.J \triangleq TJT^t$.
  $\Bbbk = \mathbb{R}$, $O = \left\{\begin{pmatrix} I_k & & \\ & -I_\ell & \\ & & 0\end{pmatrix} \,\middle|\, 0\le k+\ell\le n\right\}$
  $\Bbbk = \mathbb{C}$, $O = \left\{\begin{pmatrix} I_k & \\ & 0\end{pmatrix} \,\middle|\, 0\le k\le n\right\}$

Consider the even dimensional case first. $SO(2n,\mathbb{C}) = \{A \mid A^tA = Id\}$ can be conjugated to the group $SO(2n,\mathbb{C}) = \{A^tJA = J\}$, where $J = \begin{pmatrix} 0 & I_n \\ I_n & 0\end{pmatrix}$. It's LA $\{X \mid X^tJ + JX = 0\}$ = infinitesimal symmetries of the bilinear form $\langle x,y\rangle = x^tJy$.

Write $X = \begin{pmatrix} A & B \\ C & D\end{pmatrix}$. $X^tJ + JX = 0 \Leftrightarrow C^t = -C,\ B^t = -B,\ A^t + D = 0$. Now this LA have a large subalgebra of diagonal matrices $H = \left\{\begin{pmatrix} D & 0 \\ 0 & -D\end{pmatrix} \,\middle|\, D \text{ diagonal } n\times n \text{ matrices}\right\}$ which is in fact a C.S.A. $\dim H = n$. (In case $\Bbbk = \mathbb{R}$, $J\sim\begin{pmatrix} I_n & \\ & -I_n\end{pmatrix}$, and $\mathfrak{so}(2n, J)$ is isomorphic to the LA $\mathfrak{so}(n,n;\mathbb{R})$, a real form of $\mathfrak{so}(2n,\mathbb{C})$).

Now, as an example, consider the special case when $n=2$:

$\mathfrak{so}(4) \supseteq H = \mathbb{C}\langle h_1, h_2\rangle$. $h_1 = \left(\begin{array}{cc|cc} 1 & & & \\ & 0 & & \\ \hline & & -1 & \\ & & & 0 \end{array}\right) = e_{11} - e_{33}$, $h_2 = \left(\begin{array}{cc|cc} 0 & & & \\ & 1 & & \\ \hline & & 0 & \\ & & & -1 \end{array}\right) = e_{22} - e_{44}$

and we can take weight vectors:

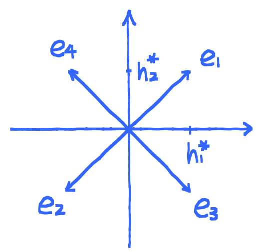
$$e_1 = \left(\begin{array}{cc|cc} & & 0 & 1 \\ & & -1 & 0 \\ \hline & & & \\ & & & \end{array}\right) \quad e_2 = \left(\begin{array}{cc|cc} & & & \\ & & & \\ \hline 0 & 1 & & \\ -1 & 0 & & \end{array}\right) \quad e_3 = \left(\begin{array}{cc|cc} 0 & 1 & & \\ 0 & 0 & & \\ \hline & & 0 & 0 \\ & & -1 & 0 \end{array}\right) \quad e_4 = \left(\begin{array}{cc|cc} 0 & 0 & & \\ 1 & 0 & & \\ \hline & & 0 & -1 \\ & & 0 & 0 \end{array}\right)$$

We have:

$[h_1, e_1] = e_1$ $\quad [h_1, e_2] = -e_2$ $\qquad [h_2, e_1] = e_1$ $\quad [h_2, e_2] = -e_2$

$[h_1, e_3] = e_3$ $\quad [h_1, e_4] = -e_4$ $\qquad [h_2, e_3] = -e_3$ $\quad [h_2, e_4] = e_4$

diagramatically, we have



weight diagram of $\mathfrak{so}(4)$

In particular, the diagram shows that $e_1, e_2, [e_1, e_2]$; $e_3, e_4, [e_3, e_4]$ span two copies of non-interfering $\mathfrak{sl}(2)$'s, and

$$\mathfrak{so}(4) = \mathbb{C}\langle e_1, e_2, [e_1, e_2]\rangle \oplus \mathbb{C}\langle e_3, e_4, [e_3, e_4]\rangle \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$$

(Over $\mathbb{R}$, $\mathfrak{so}(4;\mathbb{R}) \cong \mathfrak{su}(2) \oplus \mathfrak{su}(2) \xrightarrow{\otimes\mathbb{C}} \mathfrak{so}(4, \mathbb{C}) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$. Indeed, we have the Lie group homomorphisms: $1 \to \mathbb{Z}/2 \to SU(2) \times SU(2) \longrightarrow SO(4) \to 1$, $\mathbb{Z}/2$ generated by $(-I, -I)$ of $SU(2) \times SU(2)$).

In general, $h_i = e_{ii} - e_{n+i,n+i}$ $(1 \leq i \leq n)$, $H = \mathbb{C}\langle h_1, \cdots, h_n\rangle$, and we obtain a weight decomposition of $\mathfrak{so}(2n) = H \oplus \sum_{\alpha\in\Phi} L_\alpha$: $(\lambda_i \triangleq h_i^*, 1 \leq i \leq n)$

weight spaces L_α	weights : $\alpha \in \Phi \cup \{0\}$
H	0
$e_{ij}^1 = e_{i,n+j} - e_{j,n+i}$	$\lambda_i + \lambda_j$
$e_{ij}^2 = e_{n+i,j} - e_{n+j,i}$	$-\lambda_i - \lambda_j$
$e_{ij}^3 = e_{ij} - e_{n+j,n+i}$	$\lambda_i - \lambda_j \quad (i<j)$
$e_{ij}^4 = e_{ji} - e_{n+i,n+j}$	$\lambda_j - \lambda_i \quad (i<j)$

In particular, $\#\{\text{roots}\} = 2n(n-1)$ and $\dim \mathfrak{so}(2n) = n + 2n(n-1) = 2n^2 - n$.

We can check from the above decomposition that if $\alpha, \beta, \alpha+\beta \in \Phi$, which implies:

Cor. $\mathfrak{so}(2n)$ is simple for $n>2$. $\square$

- Aside: $\mathfrak{so}(6) \cong \mathfrak{sl}(4)$ (they have the same root system)

Now let's consider the odd dimensional case : $\mathfrak{so}(2n+1)$

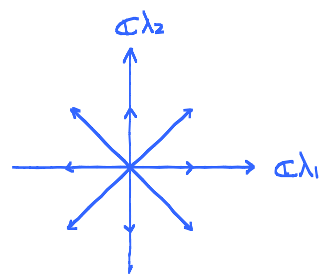
Similar as above, we may use $J = \left(\begin{array}{c|cc} 1 & & \\ \hline & 0 & I_n \\ & I_n & 0 \end{array}\right)$. $SO(2n+1) = \{AJA^t = J\}$ and $\mathfrak{so}(2n+1) = \{XJ + JX^t = 0\}$. Write $X = \left(\begin{array}{c|cc} e & a & b \\ \hline c & A & B \\ d & C & D \end{array}\right) \Rightarrow e=0,\ c=-b^t,\ d=-a^t$ and A, B, C, D as in $\mathfrak{so}(2n)$.

For example, consider $n=2$: $\mathfrak{so}(5)$, $\mathfrak{so}(4) \subseteq \mathfrak{so}(5)$ with the same CSA: $H = \mathbb{C}\langle h_1, h_2 \rangle$. The weight decomposition are the weights of $\mathfrak{so}(4)$ plus $\pm\lambda_i$

For instance :

$$e' = \left(\begin{array}{c|cc|cc} 0 & 1 & 0 & 0 & 0 \\ \hline 0 & & & & \\ 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \\ \hline -1 & & & & \\ 0 & \multicolumn{2}{c|}{0} & \multicolumn{2}{c}{0} \end{array}\right) \Rightarrow [h_1, e'] = -e' \quad [h_2, e'] = 0$$

i.e wgt $-\lambda_1$



weight diagram of $\mathfrak{so}(5)$
wgts : $\{\pm\lambda_1 \pm \lambda_2,\ \pm\lambda_1, \pm\lambda_2\}$

The weight diagram also shows that $\mathfrak{so}(5)$ is simple.

In general, $\mathfrak{so}(2n) \subseteq \mathfrak{so}(2n+1)$ have the same CSA, and $\mathfrak{so}(2n+1)$ has $2n$ more roots than those in $\mathfrak{so}(2n)$, namely $\Phi = \{\pm\lambda_i \pm \lambda_j,\ \pm\lambda_i,\ 1 \le i \le n,\ 1 \le j \le n\}$, and the weight vectors for $\pm\lambda_i$ are $e_{1,n+i+1} - e_{i+1,1}$ for λ_i and $e_{1,i+1} - e_{n+i+1,1}$ for $-\lambda_i$.

weight spaces L_α	weights : $\alpha \in \Phi \cup \{0\}$
H	0
$e^1_{ij} = e_{i+1,n+j+1} - e_{j+1,n+i+1}$	$\lambda_i + \lambda_j$
$e^2_{ij} = e_{n+i+1,j+1} - e_{n+j+1,i+1}$	$-\lambda_i - \lambda_j$

$e^3_{ij} = e_{i+1,j+1} - e_{n+j+1,n+i+1}$ $\qquad \lambda_i - \lambda_j \quad (i<j)$

$e^4_{ij} = e_{j+1,i+1} - e_{n+i+1,n+j+i}$ $\qquad \lambda_j - \lambda_i \quad (i<j)$

$e^5_i = e_{1,n+i+1} - e_{i+1,1}$ $\qquad \lambda_i$

$e^6_i = e_{1,i+1} - e_{n+i+1,1}$ $\qquad -\lambda_i$

- Aside: $\mathfrak{so}(3) \cong \mathfrak{sl}(2)$, $\mathfrak{so}(5) \cong \mathfrak{sp}(2)$.

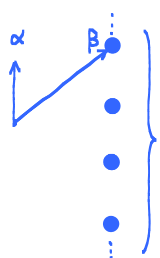
Rationality:

Question: why the real pictures are legitimate for $H^*/\mathbb{C}$?

Recall that for L simple $\rightsquigarrow$ H CSA, $L \cong H \oplus_{\alpha\in\Phi} L_\alpha$, $B|_{H\times H}$ non-degenerate.

$\rightsquigarrow$ $H^* \cong H$, $\alpha \mapsto t_\alpha$: $B(h, t_\alpha) = \alpha(h)$.

$\rightsquigarrow$ For each α, we can construct a copy of $\mathfrak{sl}(2)_\alpha$: $x_\alpha \in L_\alpha$, $h_\alpha = \frac{2t_\alpha}{\alpha(t_\alpha)}$ and $y_\alpha \in L_{-\alpha}$. W.r.t. the adjoint action of this copy of $\mathfrak{sl}(2)$, L decomposes into simple submodules:

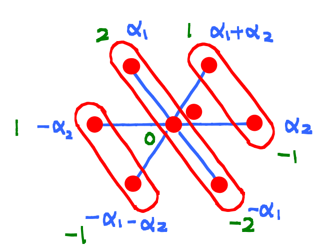


$\oplus_{n\in\mathbb{Z}} \beta + n\alpha$: a simple $\mathfrak{sl}(2)_\alpha$ submodule of ad $\mathfrak{sl}(2)_\alpha$-action

$\forall x \in L_\beta$, $[h_\alpha, x] = \beta(h_\alpha)x$ and $\beta(h_\alpha) = \frac{2\beta(t_\alpha)}{(t_\alpha, t_\alpha)} = \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ by properities of $\mathfrak{sl}(2)$ representations. Moreover, for $\mathfrak{sl}(2)$-modules, if m is a wgt, so is $-m$

$\Rightarrow$ $\beta - \beta(h_\alpha)\alpha \in \Phi$ $\quad \left((\beta - \beta(h_\alpha)\alpha)(h_\alpha) = \beta(h_\alpha) - \beta(h_\alpha)\frac{2(\alpha,\alpha)}{(\alpha,\alpha)} = -\beta(h_\alpha) \right)$.

E.g. ad $\mathfrak{sl}(2)_{\alpha_1}$-decomposition of $\mathfrak{sl}(3)$



Notation: $\langle \beta, \alpha \rangle \triangleq \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ (only for roots).

Note that $\langle \cdot, \alpha \rangle$ is linear in the first spot.

Now we shall show that $H, H^*/\mathbb{C} \rightsquigarrow \mathbb{Q}$ (defined over $\mathbb{Q}$). Take a basis in Φ of H^* (possible since Φ spans H^*), say. $\alpha_1, \cdots, \alpha_\ell \in \Phi$. Then $\forall \beta \in \Phi$
$\beta = \sum_{i=1}^{\ell} c_i \alpha_i$

Claim: $c_i \in \mathbb{Q}$

Indeed, $\langle \beta, \alpha_j \rangle = \sum_{i=1}^{\ell} c_i \langle \alpha_i, \alpha_j \rangle$, and $\langle \beta, \alpha_j \rangle, \langle \alpha_i, \alpha_j \rangle \in \mathbb{Z}$. Moreover, since B is non-degenerate, $(\langle \alpha_i, \alpha_j \rangle)_{i,j=1}^{n} \in GL_n(\mathbb{Q}) \Rightarrow c_i \in \mathbb{Q}$.

Furthermore, $\forall \alpha, \beta \in \Phi$, $(\alpha, \beta) \in \Phi$. Indeed, recall that we have $\forall \gamma, \delta \in H^*$ $(\gamma, \delta) = \sum_{\lambda \in \Phi} (\gamma, \lambda)(\delta, \lambda)$. $\Rightarrow (\beta, \beta) = \sum_{\lambda \in \Phi} (\beta, \lambda)^2 \Rightarrow \frac{1}{(\beta, \beta)} = \sum \frac{(\beta, \lambda)^2}{(\beta, \beta)^2} = \sum \frac{1}{4} \langle \beta, \lambda \rangle^2 \in \mathbb{Q}$
$\Rightarrow (\beta, \beta) \in \mathbb{Q} \Rightarrow (\alpha, \beta) = \frac{1}{2} \langle \alpha, \beta \rangle \cdot (\alpha, \alpha) \in \mathbb{Q}$.

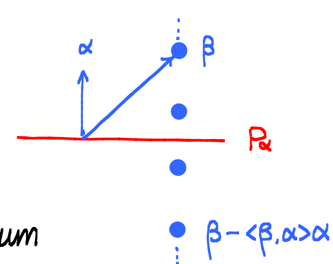
Define $E_{\mathbb{Q}} \triangleq \mathbb{Q} \cdot \Phi \subseteq H^*$, ($\mathbb{Q}$ vector space). $\dim_{\mathbb{Q}} E_{\mathbb{Q}} = \dim_{\mathbb{C}} H^*$

Properties of $E_{\mathbb{Q}}$ (which establishes the legitamacy of our drawing of H^* as a real vector space)

1). $B: E_{\mathbb{Q}} \times E_{\mathbb{Q}} \longrightarrow \mathbb{Q}$, and is positive definite ($B = \sum_\lambda (\lambda, \cdot)^2$, and $(\lambda, \alpha) \in \mathbb{Q}$).
2). $\forall \alpha, \beta \in \Phi$, $\langle \beta, \alpha \rangle \in \mathbb{Z}$; $\alpha, \beta \in \Phi \Rightarrow \beta - \langle \beta, \alpha \rangle \alpha \in \Phi$.
3). $\alpha \in \Phi \Rightarrow -\alpha \in \Phi$ and no other rational multiples of α are in Φ.
4). Φ spans $E_{\mathbb{Q}}$.

Note that $\beta \mapsto \beta - \langle \beta, \alpha \rangle \alpha$ is nothing but the reflection of $E_{\mathbb{Q}}$ about the hyperplane $P_\alpha \perp \alpha$:

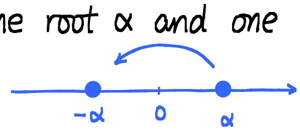
Thus 2) $\Rightarrow$ the reflections S_α about P_α preserves Φ : $S_\alpha \Phi = \Phi$



Def: (Weyl group) The Weyl group of root datum is the group generated by the reflections S_α.

E.g.

1). $\mathfrak{sl}(2)$: only one root α and one reflection : $\alpha \mapsto \alpha - \langle \alpha, \alpha \rangle \alpha = -\alpha$



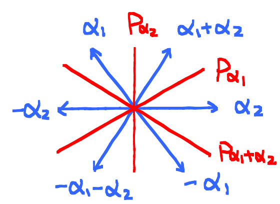
$W \cong S_2 \cong \mathbb{Z}/2$.

2). $\mathfrak{sl}(3)$: recall that B on H is given by $\begin{array}{ccc} & h_1 & h_2 \\ & \left(\begin{array}{c} 12 \\ -6 \end{array}\right. & \left.\begin{array}{c} -6 \\ 12 \end{array}\right) \end{array}\begin{array}{c} \\ h_1 \\ h_2 \end{array}$

$\Rightarrow$ $t_1 = \frac{h_1}{6}$, $t_2 = \frac{h_2}{6}$ $(t_1 \leftrightarrow \alpha_1,\ t_2 \leftrightarrow \alpha_2)$

$\Rightarrow$ B on H^* is given by $\frac{1}{36}\begin{pmatrix} 12 & -6 \\ -6 & 12 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{6} & \frac{1}{3} \end{pmatrix}$ (w.r.t. α_1, α_2).

After rescaling, B can be given by $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, in particular, $\{\pm\alpha_1, \pm\alpha_2, \pm(\alpha_1+\alpha_2)\}$ are of the same length, and also shows the angles between neighboring roots are $\frac{\pi}{3}$ (in $E_{\mathbb{Q}}$).



$W \cong S_3$

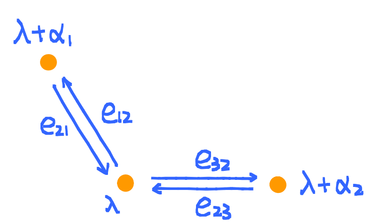
- Finite dimensional $\mathfrak{sl}(3)$-modules.

  They are:

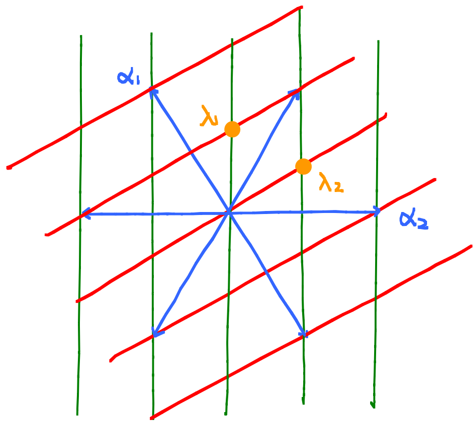
1). Completely reducible

2). Weight space decomposition w.r.t. H: $V = \oplus_{\lambda \in H^*} V(\lambda)$, where $V(\lambda) = \{v \in V \mid h.v = \lambda(h)v\}$ (recall that $x \in L$ acts semisimply iff x acts semisimply in the adjoint representation.

3). If $V(\lambda)$ is a weight space then $V(\lambda) \underset{e_{21}}{\overset{e_{12}}{\rightleftarrows}} V(\lambda+\alpha_1)$ and $V(\lambda) \underset{e_{23}}{\overset{e_{32}}{\rightleftarrows}} V(\lambda+\alpha_2)$ or graphically:



Hence, as $\mathfrak{sl}(2)_{\alpha_1}$-modules, e_{12}, e_{21} shift h_1 wgts; as $\mathfrak{sl}(2)_{\alpha_2}$-modules, e_{32}, e_{23} shift h_2 wgts. Thus the wgts of V should be on lines where α_1, α_2 take integer values:



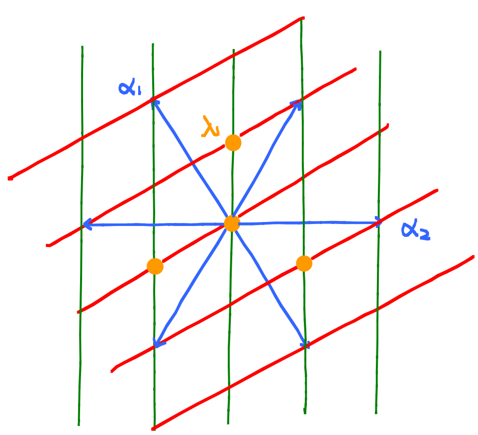
In the lattice, we may find λ_1, λ_2 of the smallest length, $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$
The lattice $\Lambda = \mathbb{Z}\lambda_1 \oplus \mathbb{Z}\lambda_2$ spans an integral lattice and $\forall \lambda \in H^*$, $\langle \lambda, \alpha_i \rangle \in \mathbb{Z}$ $\forall \alpha_i \iff \lambda \in \Lambda$.

In particular, $\Phi \subseteq \Lambda$ since $\mathfrak{sl}(3)$ is itself a rep $(\alpha_1 = 2\lambda_1 - \lambda_2, \alpha_2 = 2\lambda_2 - \lambda_1)$
Moreover, $\Lambda_r \triangleq$ root vectors $= \mathbb{Z}\Phi$, then $\Lambda/\Lambda_r \cong \mathbb{Z}/3$, since it's easy to check that $3\lambda_1 = 2\alpha_1 + \alpha_2 \in \Lambda_r$ and α_1, λ_1 also generate Λ. $\{0, \lambda_1, 2\lambda_1\}$ is a set of representatives for Λ/Λ_r.

Let's look at specific representations:

- Trivial rep: only the zero weight.
- Fundamental rep: $\mathfrak{sl}(3) \curvearrowright \mathbb{C}^3$. Highest weight vector $v_0 = \begin{pmatrix}1\\0\\0\end{pmatrix}$

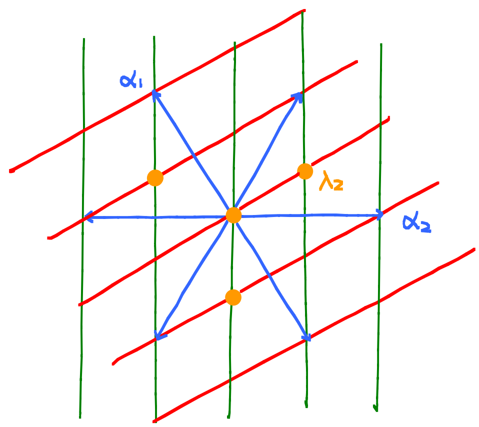
$h_1 \cdot v_0 = v_0$, $h_2 \cdot v_0 = 0$ $(h_1 = e_{11} - e_{22}, h_2 = e_{22} - e_{33})$ $\Rightarrow \lambda_1$ is the highest wgt.
Moreover $\lambda_1 \xrightarrow{e_{21}} \lambda_1 - \alpha_1 \xrightarrow{e_{32}} \lambda_1 - \alpha_1 - \alpha_2$ are the three wgts:



The 3-wgts of $\mathbb{C}^3$

• $(\mathbb{C}^3)^*$: $V(\lambda)^* = V^*(-\lambda)$ by definition of conjugate actions.

$\Rightarrow$ V^* has wgts : $\lambda_2,\ \lambda_2-\alpha_2,\ \lambda_2-\alpha_1-\alpha_2$



The 3-wgts of $(\mathbb{C}^3)^*$

Notice that $(\mathbb{C}^3)^* \cong \Lambda^2\mathbb{C}^3$, ($\Lambda^2\mathbb{C}^3$ has basis $v_0\wedge v_1,\ v_0\wedge v_2,\ v_1\wedge v_2$)

$h(v_0\wedge v_1) = (hv_0)\wedge v_1 + v_0\wedge hv_1 = (\lambda_1(h) + (\lambda_1-\alpha_1)(h))\, v_0\wedge v_1 = \lambda_2\, v_0\wedge v_1$.

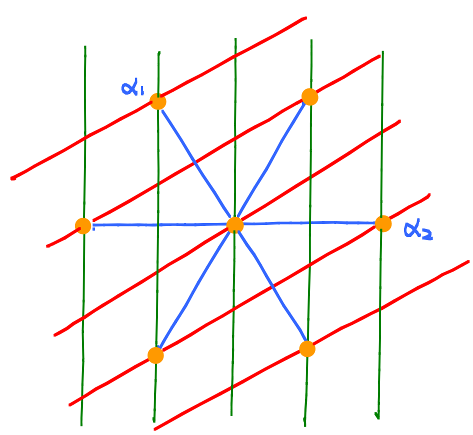
$h(v_1\wedge v_2) = (\lambda_1-\alpha_1+\lambda_1-\alpha_1-\alpha_2)\, v_1\wedge v_2 = (\lambda_2-\alpha_1-\alpha_2)(h)\, v_1\wedge v_2$

$h(v_0\wedge v_2) = (\lambda_1+\lambda_1-\alpha_1-\alpha_2)\, v_0\wedge v_2 = (\lambda_2-\alpha_2)(h)\, v_0\wedge v_2$

• Adjoint rep. and its dual

wgts : Φ, and $\mathfrak{sl}(3)^* \cong \mathfrak{sl}(3)$, since B is an intertwiner: $\mathfrak{sl}(3)\otimes\mathfrak{sl}(3)\to\mathbb{C}$.

This argument works for any simple LA : $L\otimes L \xrightarrow{B} \mathbb{C} \Rightarrow L\cong L^*$.



The wgt diagram of the adjoint rep

Notice that in this case weights are also invariant under the Weyl group actions. In general, weights of L-modules are invariant under $W(L)$ actions. Moreover, $W \subsetneq D_3 = \mathrm{Sym}(\Phi)$, and in general this is also true, that $W(L) \subsetneq \mathrm{Sym}(\Phi)$.

- Characters.

Since for any $\mathfrak{sl}(3)$-module V: $V=\oplus_{\lambda\in H^*,\lambda\in\Lambda}V(\lambda)$, we may form a group ring $\mathbb{Z}[\Lambda]$ from Λ, where we write e^λ symbolically for $\lambda\in\Lambda$; $e^\lambda\cdot e^\mu=e^{\lambda+\mu}$.

Then $\mathbb{Z}[\Lambda]$ is a unital ring with unit $e^0=1$. If $\lambda\in\mathbb{Z}[\Lambda]=\mathbb{Z}\lambda_1\oplus\mathbb{Z}\lambda_2$, $\lambda=a\lambda_1+b\lambda_2$, then $e^\lambda=(e^{\lambda_1})^a(e^{\lambda_2})^b$.

Take V an $\mathfrak{sl}(3)$-module. We define $ch(V)=\sum_{\lambda\in\Lambda}\dim V(\lambda)\cdot e^\lambda$

E.g. 1) $ch(\mathbb{C})=1$

2). Denote the fundamental rep $\mathbb{C}^3$ by $V_{1,0}$; its dual $V_{0,1}$, then

$ch(V_{1,0})=e^{\lambda_1}+e^{\lambda_1-\alpha_1}+e^{\lambda_1-\alpha_1-\alpha_2}=e^{\lambda_1}+e^{\lambda_2-\lambda_1}+e^{-\lambda_2}$

$ch(V_{0,1})=e^{-\lambda_1}+e^{\lambda_1-\lambda_2}+e^{\lambda_2}$

3). $ch(\mathfrak{sl}(3)_{ad})=e^{\alpha_1}+e^{\alpha_1+\alpha_2}+e^{\alpha_2}+2+e^{-\alpha_2}+e^{-\alpha_1-\alpha_2}+e^{-\alpha_1}=ch(\mathfrak{sl}(3)^*_{ad})$

In general, we have

1). $ch(V\oplus W)=ch(V)+ch(W)$; $ch(V\otimes W)=chV\cdot chW$.

2). $ch(V^*)=ch(V)[\lambda\mapsto-\lambda]$ since $V^*(\lambda)\cong V(-\lambda)^*$.

The rep ring of $\mathfrak{sl}(3)$ is the free abelian group with basis $\{[V]\}$ where V ranges over isomorphism classes of irreps, which is denoted $Rep(\mathfrak{sl}(3))$. We have

$$\begin{aligned}\{\mathfrak{sl}(3)\text{-modules}\}&\longrightarrow Rep(\mathfrak{sl}(3))\\ V&\longmapsto [V]\end{aligned}$$

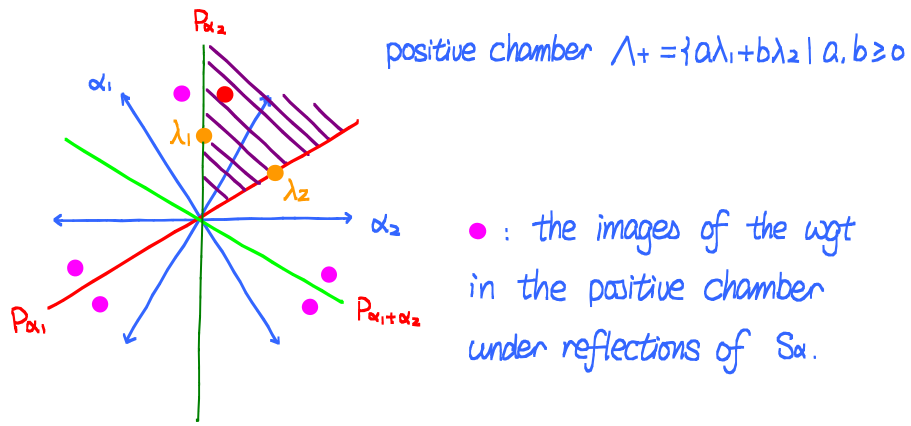
satisfying: $[V\oplus W]=[V]+[W]$, $[V\otimes W]=[V]\cdot[W]$ $[\mathbb{C}]$: unit element.

It follows that $ch: Rep(\mathfrak{sl}(3))\longrightarrow\mathbb{Z}[\Lambda]$ is a unital ring homomorphism.

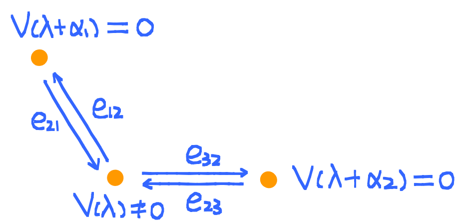
Notice that the Weyl group generated by reflections S_α, $\alpha\in\Phi$ preserves Λ, and $\dim V(\lambda)=\dim V(\lambda-\langle\lambda,\alpha\rangle\alpha)$. (consider the $\mathfrak{sl}(2)_\alpha$-action!)

E.g. S_{α_1}: $\lambda_1\mapsto\lambda_1-\alpha_1=\lambda_2-\lambda_1$, $\lambda_2\mapsto\lambda_2$. Thus $S_{\alpha_1}(e^{\lambda_1})\triangleq e^{\lambda_2-\lambda_1}$, $S_{\alpha_1}(e^{\lambda_2})=e^{\lambda_2}$ and $S_{\alpha_1}(chV_{1,0})=S_{\alpha_1}(e^{\lambda_1})+S_{\alpha_1}(e^{\lambda_2-\lambda_1})+S_{\alpha_1}(e^{\lambda_1})=e^{\lambda_2-\lambda_1}+e^{\lambda_1}+e^{\lambda_2}=ch(V_{1,0})$.

Thus $ch: Rep(\mathfrak{sl}(3))\longrightarrow\mathbb{Z}[\Lambda]^W\subseteq\mathbb{Z}[\Lambda]$ (W-invariant subalgebra). To specify an element in $\mathbb{Z}[\Lambda]^W$, it suffices to specify it inside the positive chamber.



If V is an irrep, then $\exists\ \lambda \in \Lambda$ s.t. $V(\lambda) \neq 0$, but $e_{12} \cdot V(\lambda) = 0$, $e_{23} \cdot V(\lambda) = 0$ ($\Rightarrow e_{13} = [e_{12}, e_{23}]\, V(\lambda) = 0$)



Def: λ is called the highest weight of V if $V(\lambda) \neq 0$ but $V(\lambda + a\alpha_1 + b\alpha_2) = 0$, $\forall\ a, b \geq 0$, $a+b > 0$. More generally, any $v \in W$, an $sl(3)$-module is said to have h.w. λ if $h.v = \lambda(h) v$ and $e_{12} v = e_{13} v = e_{23} v = 0$.

Since if $V(\lambda) \neq 0$, $V(S_\alpha \cdot \lambda) \neq 0$, $S_\alpha \lambda = \lambda - \langle \lambda, \alpha \rangle \alpha$. If $\langle \lambda, \alpha \rangle < 0$, we can consider $S_\alpha \cdot \lambda$ which will be of the form $\lambda + a\lambda_1 + b\lambda_2$, $a, b \geq 0$. Continue this process we end up with a h. w. in Λ_+.

Prop. There is a bijection between finite dimensional irrep's of $sl(3)$ and the elements of Λ_+, the positive integral wgts.

The proof is divided into steps.

- Construction of V_λ with h. w. $\lambda = a\lambda_1 + b\lambda_2 \in \Lambda_+$.

Notice that if V has h.w. λ, W has h.w. μ, then $V \otimes W$ has a wgt vector with h.w. $\lambda + \mu$. Indeed, $V(\lambda+\mu) = V(\lambda) \otimes W(\mu) \neq 0$.

Now, consider $V_{1,0}^{\otimes a} \otimes V_{0,1}^{\otimes b}$. Take $v_0 \in V_{1,0}(\lambda_1)$ and $v_1 \in V_{0,1}(\lambda_2)$. Then $v_\lambda = v_0^{\otimes a} \otimes v_1^{\otimes b} \in V_{1,0}^{\otimes a} \otimes V_{0,1}^{\otimes b}$ has wgt $a\lambda_1 + b\lambda_2$, and is a h.w, since e_{12} e_{23}, e_{13} kill each v_0, v_1, thus kill v_λ by Leibnitz rule.

Thus v_λ generate a subrep of $V_{1,0}^{\otimes a} \otimes V_{0,1}^{\otimes b}$, namely $\mathcal{U}(\mathfrak{sl}(3)) \cdot v_\lambda \triangleq V_\lambda$.

- V_λ is an irrep.

Indeed if $V_\lambda = V \oplus W$. Since this decomposition must preserve the wgt decomposition. $V_\lambda(\lambda) = V(\lambda) \oplus W(\lambda)$. But $V_\lambda(\lambda)$ is 1-dim'l. $W(\lambda) = 0$ and W is not generated by v_λ.

- Verma modules and uniqueness of V_λ

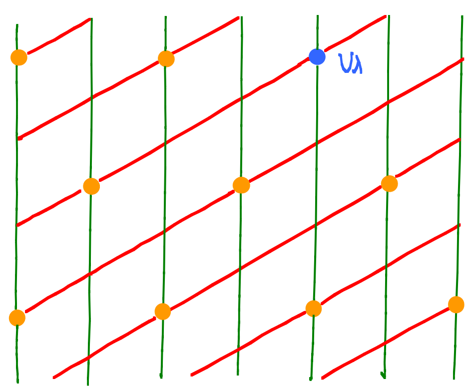
$\mathfrak{sl}(3) = n_+ \oplus H \oplus n_-$, and $b_+ = n_+ \oplus H$ is a positive Borel subalgebra. If $\lambda \in H^*$, we may construct a 1-dim'l rep of $\mathcal{U}(b_+)$: $\mathbb{C}v_\lambda$ by:

$$n_+ \cdot v_\lambda = 0, \quad h \cdot v_\lambda = \lambda(h) v_\lambda.$$

Define $M_\lambda = \mathcal{U}(\mathfrak{sl}(3)) \otimes_{\mathcal{U}(b_+)} \mathbb{C}v_\lambda$. The size of M_λ can be measured as follows

$\mathcal{U}(\mathfrak{sl}(3)) \cong \mathcal{U}(n_-) \otimes \mathcal{U}(b_+)$ (P.B.W. Thm; only as $\mathcal{U}(n_-)$-modules, not as rings)

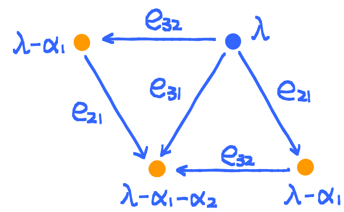
$\Rightarrow M_\lambda \cong \mathcal{U}(n_-) \otimes \mathbb{C}v_\lambda$. M_λ has a wgt decomposition: $M = \bigoplus_{a,b \geq 0} M(\lambda - a\alpha_1 - b\alpha_2)$.



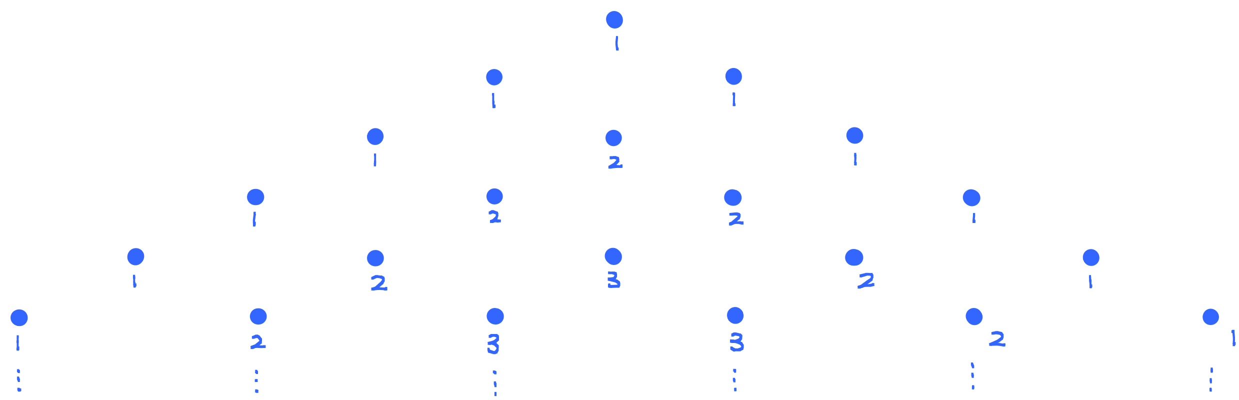
The wgt diagram of the Verma module M_λ.

Notice that $\mathcal{U}(n_-) \cong \mathbb{C}\langle e_{21}^a e_{32}^b e_{31}^c \rangle$ as vector spaces. $e_{21}^a e_{32}^b e_{31}^c$ has wgt $\lambda - a\alpha_1 - b\alpha_2 - c(\alpha_1 + \alpha_2) = \lambda - (a+c)\alpha_1 - (b+c)\alpha_2 = \lambda - d_1\alpha_1 - d_2\alpha_2$. Moreover, since $e_{32} e_{21} \cdot v_\lambda = e_{21} e_{32} \cdot v_\lambda + [e_{32}, e_{21}] v_\lambda = e_{21} e_{32} v_\lambda + e_{31} v_\lambda$, and $e_{21} e_{32} v_\lambda$ and

$e_{31}v_\lambda$ are both of wgt $\lambda-\alpha_1-\alpha_2$. It follows that $\dim V(\lambda-\alpha_1-\alpha_2)=2$



In general, $\dim M_\lambda(\lambda-d_1\alpha_1-d_2\alpha_2)=\min(d_1,d_2)+1$, (the same as $U(n_-)$, in any case), the dimension of the wgts are fitted into the diagram:



Now, consider $\mathrm{Hom}_{sl(3)}(M_\lambda, V)$, where V is any $sl(3)$-module. If $0\neq f\in \mathrm{Hom}_{sl(3)}(M_\lambda, V)$, then $f(v_\lambda)$ is a non-zero h.w. of wgt λ for V. Indeed, $h\cdot f(v_\lambda)=f(hv_\lambda)=\lambda(h)f(v_\lambda)$, $n_+ f(v_\lambda)=f(n_+\cdot v_\lambda)=0$. Thus $f(v_\lambda)$ in V has h.w. λ.

In our case, if V is finite dimensional, $V=\oplus_{\mu\in\Lambda}V(\mu)$, $\exists\, u_\lambda\in V(\lambda)$, and $e_{12}\cdot u_\lambda=0$, $e_{23}\cdot u_\lambda=0$, then $e_{13}\cdot u_\lambda=[e_{12},e_{23}]u_\lambda=0 \Rightarrow n_+\cdot v_\lambda=0 \Rightarrow$ $\exists f: M_\lambda\longrightarrow V_\lambda$, $f(v_\lambda)=u_\lambda$.

E.g. $V=sl(3)_{ad}$

$\lambda=0$. $\mathrm{Hom}_{sl(3)}(M_0, sl(3)_{ad})=0$ since there is no vector in wgt 0 killed by e_{12} and e_{23}. Similar reason applies to $\lambda=\alpha_1$, $\mathrm{Hom}(M_{\alpha_1}, sl(3)_{ad})=0$.

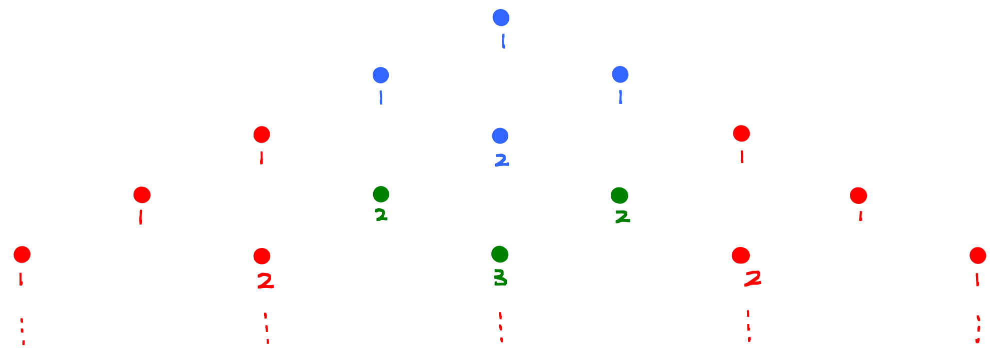
$\lambda=\alpha_1+\alpha_2$. $\mathrm{Hom}_{sl(3)}(M_{\alpha_1+\alpha_2}, sl(3)_{ad})=\mathbb{C}$.

Now if $\lambda\in\Lambda_+$ and V_λ is an irrep of h.w. λ. Then the map f_λ: $M_\lambda\longrightarrow V_\lambda$ is surjective, and $\ker f_\lambda\subseteq M_\lambda$.

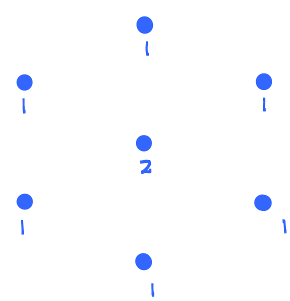
Notice that any submodule $M \subseteq M_\lambda$ must respect the weight decomposition by similar arguments as we did for $\mathfrak{sl}(2)$ case. i.e. $M = \oplus_\mu M \cap M_\lambda(\mu)$. It follows that M_λ has a unique maximal proper submodule $M_\lambda' \subsetneq M_\lambda$. M_λ' proper $\Rightarrow$ $v_\lambda \notin M_\lambda' \Rightarrow M_\lambda / M_\lambda' \cong V_\lambda$.

This argument shows that V_λ is unique, and finishes the proof of the proposition.

E.g. $M_{\alpha_1+\alpha_2} \longrightarrow \mathfrak{sl}(3)_{ad}$.



$\downarrow f_\lambda$



● : $f|_{M(\lambda)}$: isomorphism

● : $f|_{M(\lambda)}$: 0

● : $f|_{M(\lambda)}$: surjective, not injective

Previously we have found $v_\lambda \in V_{1,0}^{\otimes a} \otimes V_{0,1}^{\otimes b}$ with h. w. $a\lambda_1 + b\lambda_2$. Then to obtain $V(\lambda)$, it suffices to apply e_{21}, e_{31}, e_{32} to v_λ repeatedly.

E.g. $\mathfrak{sl}(3)_{ad}$ has h. w. $\alpha_1+\alpha_2 = \lambda_1+\lambda_2 \Rightarrow \mathfrak{sl}(3)_{ad} \subseteq V_{1,0} \otimes V_{0,1} = V \otimes V^*$

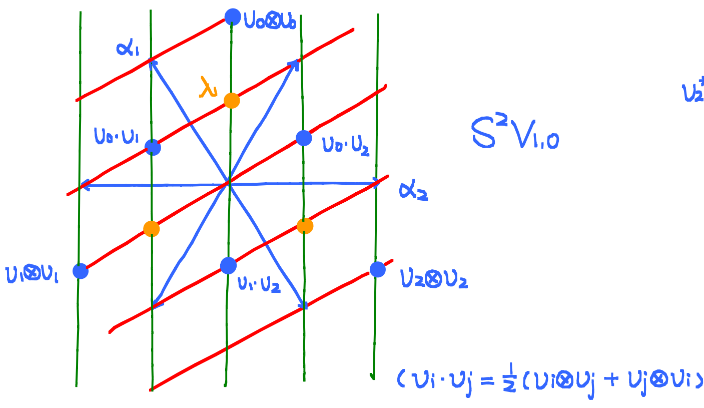
Indeed, $V \otimes V^* \cong \mathrm{Hom}_{\mathbb{C}}(V,V) \cong \mathfrak{sl}(3) \oplus \mathbb{C}$ ($\mathbb{C}$ coming from Shur's lemma !)

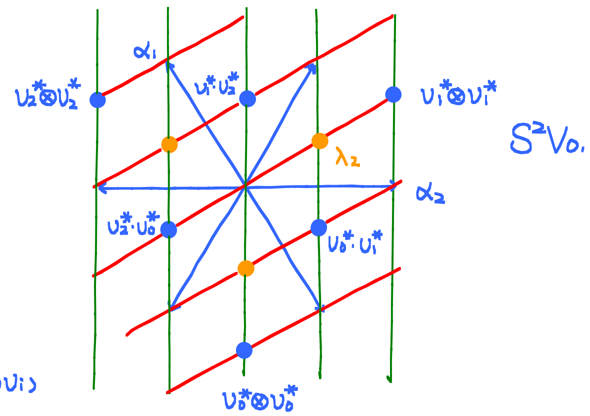
This is actually true for any $\mathfrak{sl}(n)$ and its fundamental rep. $V \cong \mathbb{C}^n$.

$V \otimes V^* = \mathrm{End}(V) \cong \mathfrak{gl}(n) \cong \mathfrak{sl}(n) \oplus \mathbb{C}$.

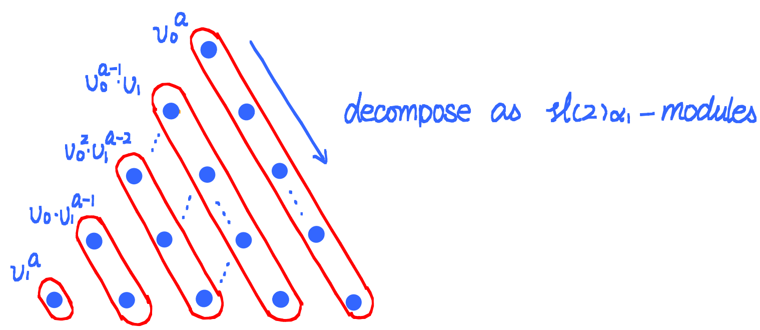
E.g. $S^2V_{1,0}$ is an irrep with h.w. $2\lambda_1$; $S^2V_{0,1}$ is an irrep with h.w. $2\lambda_2$

Indeed, if $V_{1,0} = \mathbb{C}\langle v_0, v_1, v_2\rangle$, $S^2V_{1,0}$ contains wgt vectors $v_0\otimes v_0$, $v_1\otimes v_1$, $v_2\otimes v_2$, of wgts $2\lambda_1$. By successively applying e_{ij} $(i\neq j)$ we have all 6-wgts: Dually we have the diagram for $S^2V_{0,1}$

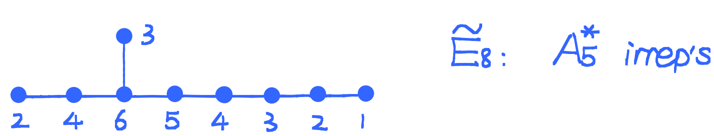




Moreover, $\forall a, b\geq 0$, S^aV, S^bV^* are irrep's of $\mathfrak{sl}(3)$ of h.w. $a\lambda_1$ and $b\lambda_2$, since each wgt space only has dimension 1. Indeed, the wgts form a triangle as below:

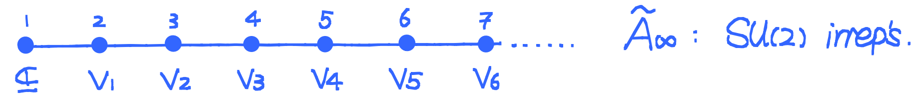


More generally, S^nV is irreducible for any defining rep of $\mathfrak{sl}(k)$, $k\geq 2$, of h.w. $n\lambda_1$. However, this is not a general phenominon. i.e. it is not true for other simple LA's, Lie groups, or finite group rep's. As a counter example, consider A_5^* and its McKay graph:

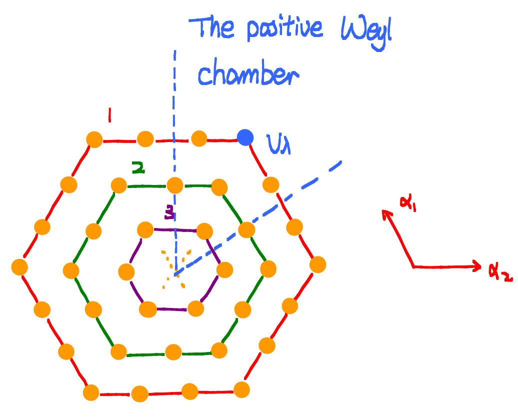


Here rep's 1 – 6 are $\mathbb{C}$, V, $S^2V, \cdots, S^5V$ resp. but $S^6V \cong \underline{3} \oplus \underline{4}$. Indeed $\underline{3} \oplus \underline{4} \oplus \underline{5} \cong V \otimes \underline{6} = V \otimes S^5V \cong S^6V \oplus S^4V \cong S^6V \oplus \underline{5}$.

Compare with $\mathfrak{sl}(2)$-modules (SU(2) - MaKay). S^nV is alway an irrep, and $V \otimes S^nV \cong S^{n-1}V \otimes S^{n+1}V$.



Hence if $\lambda = a\lambda_1 + b\lambda_2 \in \Lambda_+$ is the h.w. of V_λ, then $V_\lambda \subseteq S^a(V) \otimes S^b(V^*)$ $\subseteq V^{\otimes a} \otimes (V^*)^{\otimes b}$, since the h.w vector $v_0^{\otimes a} \otimes (v_0^*)^{\otimes a}$ lies inside $S^a(V) \otimes S^b(V^*)$. If $V_\lambda = \oplus_\mu V_\lambda(\mu)$, the wgts form a diagram as below : the multiplicity of the weights increases toward the center, and are equal on the barycentric edges. (since V_λ decomposes as various $\mathfrak{sl}(2)_{\alpha_i}$ modules ; for instance, $\{e_{21}^m \cdot v_\lambda, m \geq 0\}$ form the upper right, outer-most edge, which corresponds to an $\mathfrak{sl}(2)_{\alpha_1}$ - module thus each with multiplicity one ; so are all the outer-most edges since they are in one Weyl group orbit)



Prop : $\mathrm{Rep}(\mathfrak{sl}(3)) \xrightarrow{\cong} \mathbb{Z}[\Lambda]^W \cong \mathbb{Z}[\Lambda_+]$ as free abelian groups.

Pf : Recall that the map is induced by $[V] \mapsto \mathrm{ch}V$. It suffices to show that $\{\mathrm{ch}V_\lambda\}$ gives a basis of $\mathbb{Z}[\Lambda]^W$. An obvious basis is given by $\tilde{e}^\lambda \triangleq \sum_{g \in W} e^{g \cdot \lambda}$, $\lambda \in \Lambda^+$.

We can order Λ or Λ^+ by $\lambda > \mu$ iff $\lambda - \mu = a\alpha_1 + b\alpha_2$, $a, b \geq 0$. Then we see that :

$$\mathrm{ch}V_\lambda = \hat{e}^\lambda + \Sigma_{\mu\in\Lambda^+,\,\mu<\lambda} \dim V_\lambda(\mu)\cdot \hat{e}^\mu$$

Thus the basis formed by $\{\mathrm{ch}V_\lambda \mid \lambda\in\Lambda^+\}$ differs from that formed by $\{\hat{e}^\lambda \mid \lambda\in\Lambda^+\}$ by an infinite upper triangular matrix with 1's on the diagonal, thus is invertible.

□

Thus to specify a rep. it suffices to specify its character. For instance, to determine how $V_\lambda\otimes V_\mu$ decomposes, it suffices to check for $\mathrm{ch}(V_\lambda\otimes V_\mu) = \mathrm{ch}V_\lambda\cdot \mathrm{ch}V_\mu$.

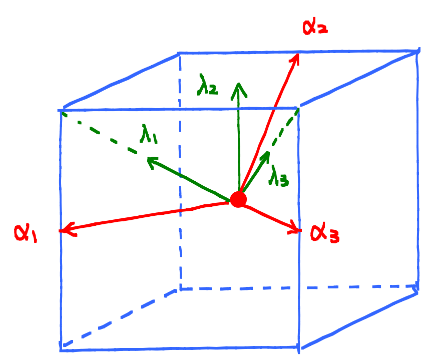
- $\mathfrak{sl}(4)$, $\mathfrak{sl}(n)$.

Recall that for $\mathfrak{sl}(n)$, $E_{\mathbb{Q}}\otimes\mathbb{R} \cong \mathbb{R}^{n-1} \subseteq \mathbb{R}^n = \mathbb{R}\{\varepsilon_1,\cdots,\varepsilon_n\}$, and is spanned by $\{\varepsilon_i - \varepsilon_{i+1}\}_{i\le n-1}$. $\Phi = \{\varepsilon_i - \varepsilon_j\} = \Phi^+ \amalg \Phi^-$, where $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i<j\}$, $\Phi^- = \{\varepsilon_i - \varepsilon_j \mid i>j\}$, and $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. Define $\{\alpha_i = \varepsilon_i - \varepsilon_{i+1} \mid i\le n-1\}$ the simple roots, then $\forall\, i>j$, $\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1}$. Note that

$$(\alpha_i, \alpha_j) = \begin{cases} 2 & i=j \\ -1 & i=j\pm1 \\ 0 & \text{otherwise}\end{cases}$$

It follows that α_i, $\alpha_{i\pm1}$ form an angle of $\frac{2}{3}\pi$, and $\alpha_i \perp \alpha_j$, $j\ne i, i\pm1$.

E.g. $\mathfrak{sl}(4)$

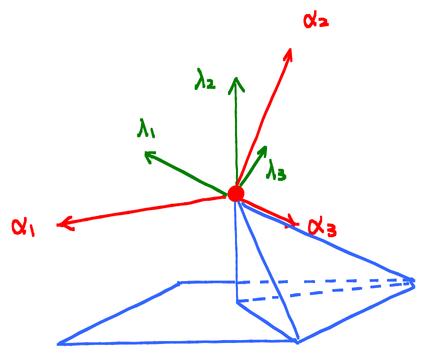


$(\alpha_i, \alpha_i) = 2$

$(\lambda_i, \alpha_j) = \delta_{ij}$

$i = 1, 2, 3$

$H^* \cong \mathbb{R}^3$. $\Phi = \{$mid-points of the cube's edges$\}$. Note that $\mathbb{R}^3$ is divided into 24 Weyl chambers, with the positive chamber $C_+ = \{\lambda \mid (\lambda, \alpha_i) \ge 0,\, i=1,2,3\}$ $= \{\lambda \mid \langle\lambda, \alpha_i\rangle = 0,\, i=1,2,3\} = \{\sum_{i=1}^{3} a_i\lambda_i \mid a_i \in \mathbb{R}_+\}$.



A Weyl chamber inside the cube

Note that the group of symmetry of the cube is $\mathrm{Rot}(\square) \times \mathbb{Z}/2 \cong S_4 \times \mathbb{Z}/2$. The S_4 factor generated by the permutation of the 4 main diagonals. However the Weyl group $W \cong S_4$ sits as the graph of sign $S_4 \to \mathbb{Z}/2$ in $\mathrm{Sym}(\square)$.

In general, if $\alpha_1, \cdots, \alpha_{n-1}$ are the simple roots of $\mathfrak{sl}(n)$, $\lambda_1, \cdots, \lambda_{n-1}$ their dual w.r.t. the Killing form, then (in $\mathbb{R}^n = \oplus \mathbb{R}\varepsilon_i$)

$$\lambda_1 = \left(\frac{n-1}{n}, -\frac{1}{n}, \cdots, -\frac{1}{n}\right)$$

$$\vdots$$

$$\lambda_k = (\underbrace{\frac{n-k}{n}, \cdots, \frac{n-k}{n}}_{k}, \underbrace{-\frac{k}{n}, \cdots, -\frac{k}{n}}_{n-k})$$

$$\vdots$$

Prop. Irrep's of $\mathfrak{sl}(n) \overset{1:1}{\longleftrightarrow} \Lambda^+ = \{\sum a_i \lambda_i \mid a_i \in \mathbb{Z}_+\}$.
The proof is almost identical to that of $\mathfrak{sl}(3)$ case

- Construction of V_{λ_i} $(i=1, \cdots, n-1)$

Recall that for $\mathfrak{sl}(3)$, we have $V_{\lambda_1} \cong V$, $V_{\lambda_2} \cong V^* \cong \Lambda^2 V$. Similarly, consider the defining rep $V \cong \mathbb{C}^n$ of $\mathfrak{sl}(3)$, and $\Lambda^k V$, $k = 1, \cdots, n-1$

$$\Lambda^k V = \mathbb{C}\{e_{i_1} \wedge \cdots \wedge e_{i_k},\ 1 \le i_1 < \cdots < i_k \le n\},$$

Note that $\Lambda^k V \times \Lambda^{n-k} V \xrightarrow{\wedge} \Lambda^n V \cong \mathbb{C}$ is a perfect pairing and $\Lambda^k V \cong \Lambda^{n-k} V^*$.

Prop. $\Lambda^k V$ $(k=1, \cdots, n-1)$ is irreducible of h.w. λ_k.
Pf: It's easy to check that $e_1 \wedge \cdots \wedge e_k$ is a h.w. vector of h.w. λ_k.
Successive application of e_{ij} $(i>j)$ gives all $e_{i_1} \wedge \cdots \wedge e_{i_k}$ $\square$

E.g. For $\mathfrak{sl}(4)$, $v_1 \wedge v_2$ is a h.w. vector for $\Lambda^2 V$.

$$\begin{tikzcd}
 & v_1\wedge v_2 \arrow[dr, "e_{32}"] & & \\
 & & v_1\wedge v_3 \arrow[dl, "e_{21}"'] \arrow[dr, "e_{43}"] & \\
 & v_2\wedge v_3 \arrow[dr, "e_{43}"'] & & v_1\wedge v_4 \arrow[dl, "e_{21}"] \\
 & & v_2\wedge v_4 \arrow[dl, "e_{32}"] & \\
 & v_3\wedge v_4 & &
\end{tikzcd}$$

By the prop., given any $\lambda \in \Lambda_+$, $\lambda = a_1\lambda_1 + \cdots + a_{n-1}\lambda_{n-1}$, there is a h.w. λ-vector

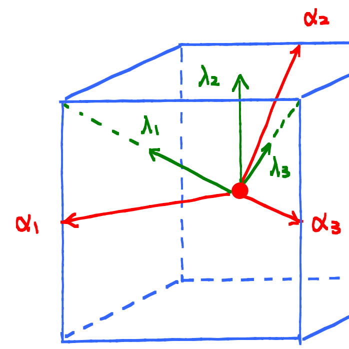
$$\begin{aligned}
v_\lambda = v_1^{\otimes a_1} \otimes (v_1\wedge v_2)^{\otimes a_2} \otimes \cdots (v_1 \wedge \cdots \wedge v_{n-1})^{\otimes a_{n-1}} &\in S^{a_1}(V) \otimes S^{a_2}(\Lambda^2 V) \otimes \cdots \otimes S^{a_{n-1}}(\Lambda^{n-1} V) \\
&\subseteq V^{\otimes a_1} \otimes (\Lambda^2 V)^{\otimes a_2} \otimes \cdots \otimes (\Lambda^{n-1} V)^{\otimes a_{n-1}} \\
&\subseteq V^{\otimes a_1} \otimes (V^{\otimes 2})^{\otimes a_2} \otimes \cdots \otimes (V^{\otimes n-1})^{\otimes a_{n-1}} \\
&= V^{\otimes (a_1 + 2a_2 + \cdots + (n-1)a_{n-1})}
\end{aligned}$$

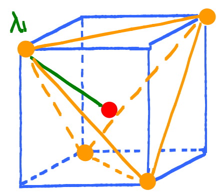
This v_λ generates an irreducible $\mathfrak{sl}(n)$-submodule : $\mathcal{U}(\mathfrak{n}_-)\cdot v_\lambda$.

Rmk: Any finite dim'l irrep of $\mathfrak{sl}(n)$ is contained in $V^{\otimes N}$ for some N. In fact, for any finite group G, and a faithful rep V of G, and W an irrep of G, we have $W \subseteq V^{\otimes n}$ for some n. Since finite dim'l irrep's of $\mathfrak{sl}(k)$ are in 1-1 correspondence with irrep's of $SU(k)$, we see that this result is an analogue of finite groups with compact Lie groups.

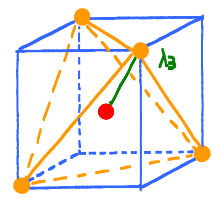
Now, similar as for $\mathfrak{sl}(3)$, let M_λ be the Verma module of $\mathfrak{sl}(n)$ with h.w. λ. It has a unique maximal proper submodule M_λ' (the one which doesn't contain v_λ). Then $\exists!$ homomorphism $M_\lambda \xrightarrow{\varphi_\lambda} V_\lambda$ carrying v_λ to a h.w. vector of V_λ. Since V_λ is irreducible, φ_λ must be surjective, and $\ker\varphi_\lambda \cong M_\lambda'$. Thus $V_\lambda \cong M_\lambda / M_\lambda'$ is unique.

E.g. Weight diagrams of V_{λ_i} for $\mathfrak{sl}(4)$

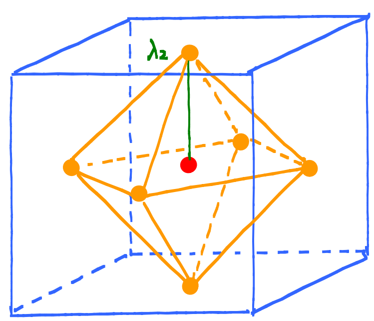




$V_{\lambda_1} \cong V$



$V_{\lambda_3} \cong \Lambda^3 V \cong V^*$



$V_{\lambda_2} \cong \Lambda^2 V$

A generic irrep of $\mathfrak{sl}(4)$ would span a 24 vertices convex hall, since $\mathbb{R}^3$ is divided into 24 Weyl chambers.

Def. An irrep V_λ of L is called miniscule if all wgts belongs to $W \cdot \lambda$

E.g.

1). Trivial rep of any L

2). $\Lambda^k V$ of $\mathfrak{sl}(n)$, $0 \le k \le n-1$. (Since $\Lambda^k V$ has h.w vector $v_1 \wedge \cdots \wedge v_k$, and all the other wgt vectors are $v_{i_1} \wedge \cdots \wedge v_{i_k}$, and are h.w vectors w.r.t. another choices of positive root system, thus on the same Weyl orbit, C.f. the next section).

Note that V miniscule $\Rightarrow$ all wgt spaces have dim 1 (since the h.w. space has dim 1). The converse is not true. For instance. $S^m V$ of $\mathfrak{sl}(n)$ has all wgt spaces dim 1, but is not miniscule unless $m=1$)

§7. Root Systems

Def. A set $\Phi \subseteq E$ ($\cong \mathbb{R}^n$ as Euclidean spaces) is called a root system if

1). Φ spans E

2). $\alpha \in \Phi \Rightarrow \mathbb{R}\alpha \cap E = \{\pm\alpha\}$

3). S_α (reflections about the hyperplane $\perp \alpha$) preserves Φ, $\forall \alpha$ i.e.

$\forall \beta \in \Phi$, $S_\alpha(\beta) = \beta - \frac{2(\beta,\alpha)}{(\alpha,\alpha)}\alpha \in \Phi$

4). $\langle\beta,\alpha\rangle \triangleq \frac{2(\beta,\alpha)}{(\alpha,\alpha)} \in \mathbb{Z}$ (Cartan integers)

The reflections generated by S_α, $\alpha \in \Phi$ is called the Weyl group.

Def. (Φ, E), (Φ', E') are called equivalent if $\exists$ a vector space isomorphism φ

$\varphi: E \longrightarrow E'$ s.t. $\varphi(\Phi) = \Phi'$ and $\langle\varphi(\alpha), \varphi(\beta)\rangle = \langle\alpha, \beta\rangle$, $\forall \alpha, \beta \in \Phi$.

Def. If (Φ, E) satisfies $\Phi = \Phi_1 \amalg \Phi_2$, $(\Phi_1, \Phi_2) = 0$, then $E = E_1 \oplus E_2$, where $E_1 = \mathbb{R}\Phi_1$, $E_2 = \mathbb{R}\Phi_2$, $W \cong W_1 \times W_2$. Such root systems are called decomposable or reducible. Otherwise it's called indecomposable or irreducible.

Rmk: L semisimple LA $\Rightarrow L = \oplus L_i$. Then the root system of L is a a sum of irreducibles, each corresponding to some L_i.

Rank n root systems

- A_n: $E \subseteq \mathbb{R}^{n+1}$ (with the standard Euclidean metric), the subspace of codim 1 and $E \perp (1, \cdots, 1)$. $\Phi = \{\varepsilon_i - \varepsilon_j \mid i \neq j\}$, where $\{\varepsilon_i \mid i = 1, \cdots, n+1\}$ is the standard basis of $\mathbb{R}^{n+1}$. $W \cong S_{n+1}$ $\alpha = \varepsilon_i - \varepsilon_j \Rightarrow S_\alpha(x_1, \cdots x_i, \cdots, x_j \cdots x_n) = (x_1, \cdots, x_j, \cdots, x_i, \cdots x_n)$

- B_n: $E = \mathbb{R}^n$, $\Phi = \{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \mid i = 1, \cdots, n, i<j\}$

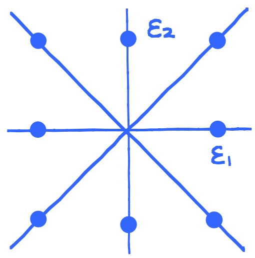
  $S_{\varepsilon_i}(x_1, \cdots, x_i, \cdots, x_n) = (x_1, \cdots, -x_i, \cdots, x_n)$ $\quad S_{\varepsilon_i+\varepsilon_j}(x_1, \cdots, x_i, \cdots x_j, \cdots x_n) = (x_1, \cdots, -x_j, \cdots, -x_i, \cdots x_n)$

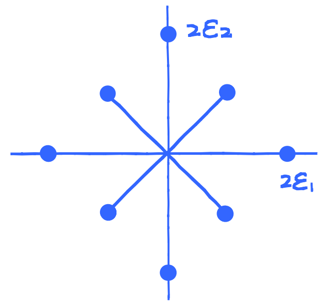
  $S_{\varepsilon_i-\varepsilon_j}(x_1, \cdots, x_i, \cdots x_j, \cdots x_n) = (x_1, \cdots, x_j, \cdots, x_i, \cdots x_n)$

  Thus we have $1 \longrightarrow (\mathbb{Z}/2)^n \longrightarrow W \longrightarrow S_n \longrightarrow 1 \Rightarrow |W| = 2^n \cdot n!$

- C_n: (Dual to B_n)

$E \cong \mathbb{R}^n$, $\Phi = \{\pm 2\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j\}$. The Weyl group is isomorphic to that of B_n.





In general, from a root system (Φ, E) we may define $(\Phi^\vee, E^\vee)$, where $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$ and $\alpha^\vee \triangleq \frac{2\alpha}{(\alpha,\alpha)}$ (shorter $\alpha \mapsto$ longer $\alpha^\vee$), $E^\vee = E$

Then $(\Phi^\vee, E^\vee)$ is still a root system. In particular, $A_n^\vee \cong A_n$, $B_n^\vee \cong C_n$

- D_n: $E = \mathbb{R}^n$, $\Phi = \{\pm\varepsilon_i \pm \varepsilon_j \mid i<j\}$

The Weyl group: $1 \longrightarrow (\mathbb{Z}/2)^{n-1} \longrightarrow W \longrightarrow S_n \longrightarrow 1$ ($(\mathbb{Z}/2)^n$: only even number of sign changes occur).

Note that $D_2 \cong A_1 \oplus A_1$ ($\mathfrak{so}(4) \cong \mathfrak{sl}(2) \oplus \mathfrak{sl}(2)$), $D_3 \cong A_3$ ($\mathfrak{so}(6) \cong \mathfrak{sl}(4)$)

$B_2 \cong C_2$ ($\mathfrak{so}(5) \cong \mathfrak{sp}(2)$), thus to avoid redandence, we require that

$$A_n \ (n \geq 1), \quad B_n \ (n \geq 2), \quad C_n \ (n \geq 3), \quad D_n \ (n \geq 4)$$

There will be 5 more exceptional cases E_6, E_7, E_8, F_4, G_2.

Def. A root system is called simply-laced if all roots have the same length.

Note that a simply-laced root system $\cong$ its own dual. (e.g. A_n, D_n, E_6, E_7, E_8).

Rank 2 systems.

Take $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$. Assume that α, β are not orthogonal and $|\alpha| \geq |\beta|$.

Then $\langle\alpha,\beta\rangle = \frac{2(\alpha,\beta)}{|\alpha|^2} = \frac{2|\beta|}{|\alpha|}\cos\theta$, $\langle\beta,\alpha\rangle = \frac{2|\alpha|}{|\beta|}\cos\theta$. ($0<\theta<\frac{\pi}{2}$, or consider $-\alpha$).

$\Rightarrow 4\cos^2\theta = \langle\alpha,\beta\rangle\langle\beta,\alpha\rangle \in \mathbb{N}$, but $0 < \cos^2\theta < 1$

$\Rightarrow \langle\alpha,\beta\rangle\langle\beta,\alpha\rangle = 4\cos^2\theta = 1, 2, 3.$

Moreover, since $\frac{\langle \alpha, \beta \rangle}{\langle \beta, \alpha \rangle} = \frac{|\alpha|^2}{|\beta|^2} \geq 1$, if we set $|\alpha| = c \cdot |\beta|$ $(c \geq 1)$

$\Rightarrow c^2 \cdot \langle \beta, \alpha \rangle^2 = 1, 2, 3.$ $\quad \begin{cases} \langle \beta, \alpha \rangle = 1,\ c^2 = 1 \Rightarrow \cos\theta = \frac{1}{2},\ \theta = \frac{\pi}{3} \\ \langle \beta, \alpha \rangle = 1,\ c^2 = 2 \Rightarrow \cos\theta = \frac{\sqrt{2}}{2},\ \theta = \frac{\pi}{4} \\ \langle \beta, \alpha \rangle = 1,\ c^2 = 3 \Rightarrow \cos\theta = \frac{\sqrt{3}}{2},\ \theta = \frac{\pi}{6} \end{cases}$

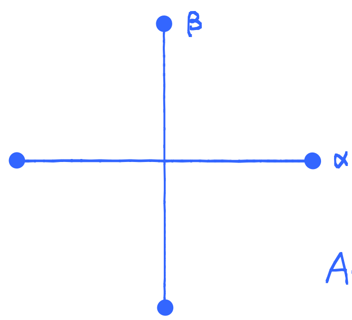
Lemma: If $\alpha, \beta \in \Phi$, $\beta \neq \pm\alpha$, $(\alpha, \beta) > 0$, then $\alpha - \beta \in \Phi$.

Pf: $S_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha \in \Phi$; $S_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle \beta \in \Phi$ and $\langle \alpha, \beta \rangle > 0$, $\langle \beta, \alpha \rangle > 0$

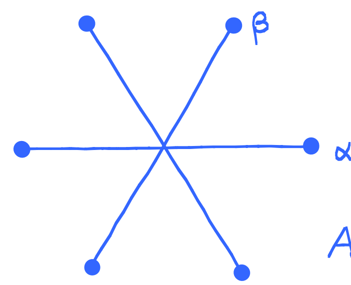
By the above computation, at least one of $\langle \alpha, \beta \rangle$, $\langle \beta, \alpha \rangle$ equals 1.

$\Rightarrow$ either $\beta - \alpha$ or $\alpha - \beta \in \Phi$. Also if $\beta - \alpha \in \Phi$, $\alpha - \beta = S_{\beta-\alpha}(\beta - \alpha) \in \Phi$. $\square$

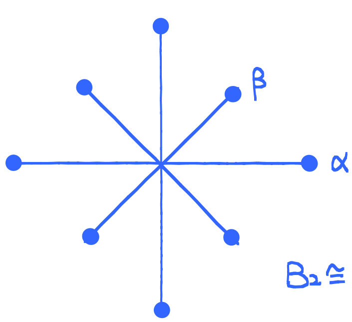
From these discussion, we conclude the only possible rank 2 root systems are:
(In case $\alpha \perp \beta$ and there are no other roots than $\pm\alpha$, $\pm\beta$, we may rescale β so that $|\beta| = |\alpha|$.)



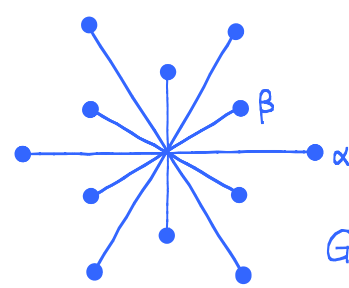
$A_1 \oplus A_1$



A_2



$B_2 \cong C_2$



G_2

Note that the Weyl groups of the systems for A_2, B_2, G_2 are dihedral groups $Di_3 \cong S_3$, Di_4, Di_6 respectively, but the automorphism group of the graphs are Di_6, Di_4, Di_6 respectively.

Simple roots

Def. $\Delta \subseteq \Phi$ is called a base if

(1). Δ is a basis of Φ

(2). $\Phi = \Phi^+ \sqcup \Phi^-$, where $\Phi^- = -\Phi^+$ and any $\beta \in \Phi^+$ can be written as

$$\beta = \textstyle\sum_{\alpha\in\Delta} k_\alpha \cdot \alpha, \quad k_\alpha \in \mathbb{Z}_+.$$

Elements $\alpha \in \Delta$ are called simple (w.r.t. Δ)

E.g. A_n: $\Delta = \{\varepsilon_i - \varepsilon_{i+1}\}$, $\Phi^+ = \{\varepsilon_i - \varepsilon_j \mid i<j\}$, $\Phi^- = \{\varepsilon_i - \varepsilon_j \mid i>j\}$.

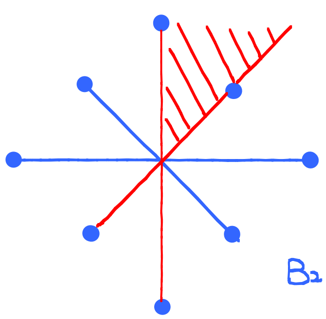
Thm. 1). Any root system has a base.

2). The Weyl group acts simply transitively on all bases

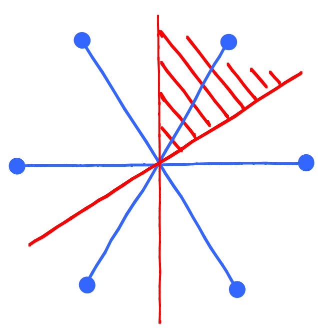
3). $\{\text{Bases}\} \overset{1:1}{\longleftrightarrow} \{\text{Weyl chambers}\}$

(Recall that $\mathbb{R}^n \setminus \bigcup_{\alpha\in\Phi} P_\alpha = \bigsqcup (\text{open})$ Weyl chambers).

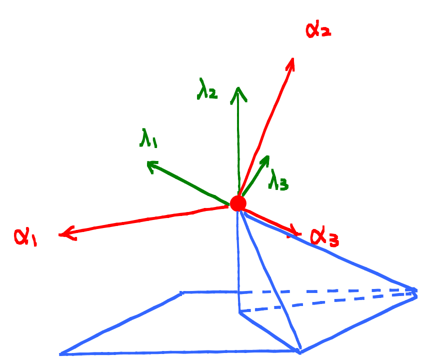
E.g.



$B_2 \cong C_2$



A_2



A_3 Some Weyl chambers

We will prove the theorem in steps.

Lemma: If Δ is a base, then $(\alpha,\beta)\leq 0$, $\forall\,\alpha,\beta\in\Delta$.

Pf: Indeed, otherwise $\alpha-\beta$ or $\beta-\alpha\in\Phi$, which contradicts condition (ii) of a base. $\square$

Let $E^{reg}=E\setminus\bigcup_{\alpha\in\Phi}P_\alpha=\coprod$ open Weyl chambers, $\forall\,\gamma\in E^{reg}$, define $\Phi^+(\gamma)\triangleq\{\alpha\in\Phi\mid(\gamma,\alpha)>0\}$, $\Phi^-(\gamma)\triangleq\{\alpha\in\Phi\mid(\gamma,\alpha)<0\}$. $\alpha\in\Phi^+(\gamma)$ is said to be decomposable if $\alpha=\alpha_1+\alpha_2$ for some $\alpha_1,\alpha_2\in\Phi^+(\gamma)$. Let $\Delta(\gamma)$ be the set of indecomposable elements. Then:

1). $\forall\,\alpha\in\Phi^+(\gamma)$, $\alpha\in\mathbb{Z}_+\Delta(\gamma)$.

Indeed, it's true for indecomposable elements. If $\exists\,\alpha\in\Phi^+(\gamma)$, $\alpha\notin\mathbb{Z}_+\Delta(\gamma)$, choose such α that (α,γ) is the smallest. Then $\alpha=\alpha_1+\alpha_2$ and $0<(\alpha,\gamma)=(\alpha_1,\gamma)+(\alpha_2,\gamma)\Rightarrow\alpha_1,\alpha_2\in\mathbb{Z}_+\Delta(\gamma)\Rightarrow\alpha\in\mathbb{Z}_+\Delta(\gamma)$, contradiction.

2). $\alpha,\beta\in\Delta(\gamma)$, $\alpha\neq\beta\Rightarrow(\alpha,\beta)\leq 0$

Otherwise, $(\alpha,\beta)>0\Rightarrow\alpha-\beta$, $\beta-\alpha$ are roots, say, $\alpha-\beta\in\Phi^+(\gamma)\Rightarrow\alpha=\alpha-\beta+\beta$ is decomposable, contradiction.

3). $\Delta(\gamma)$ is a set of linearly independent elements.

Otherwise, $\sum_{\alpha\in\Delta}k_\alpha\alpha=0$, $\Rightarrow\sum_{k_\alpha>0}k_\alpha\alpha=\sum_{k_\beta<0}(-k_\beta)\cdot\beta\triangleq\varepsilon\Rightarrow$ $(\varepsilon,\varepsilon)=\sum_{k_\alpha>0,k_\beta<0}k_\alpha(-k_\beta)(\alpha,\beta)\leq 0\Rightarrow\varepsilon=0$.

4). $\Delta(\gamma)$ is a basis of E.

Since $\mathbb{Z}_+\Delta(\gamma)\supseteq\Phi^+\Rightarrow\mathbb{Z}\cdot\Delta(\gamma)\supseteq\Phi$, and Φ spans E.

Rmk: Each base Δ gives a partial order on E: $\mu<\lambda$ iff $\lambda-\mu\in\mathbb{Z}_+\Delta$.

5). Any base Δ of $\Phi=\Phi^+\amalg\Phi^-$ has the form $\Delta(\gamma)$ for some $\gamma\in E^{reg}$.

Indeed, if $\Delta=\{\alpha_1,\cdots,\alpha_n\}$, just take any γ: $(\gamma,\alpha_i)>0$, $\forall\,\alpha_i\in\Delta$. Note that this γ decomposes Φ into $\Phi^+(\gamma)$ and $\Phi^-(\gamma)$, and $\Phi^+\subseteq\Phi^+(\gamma)$, $\Phi^-\subseteq\Phi^-(\gamma)$, but $|\Phi^+|=\frac{|\Phi|}{2}=|\Phi^+(\gamma)|\Rightarrow\Phi^+=\Phi^+(\gamma)$, $\Phi^-=\Phi^-(\gamma)$.

Cor. There is a natural bijection between bases and Weyl chambers. $\square$

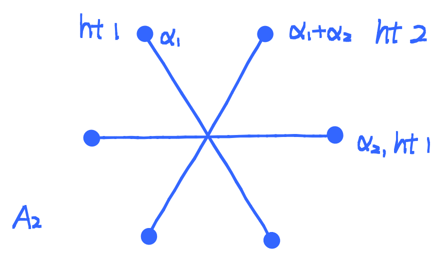
Prop. Fix Δ, if α is a positive, but not simple $\Rightarrow\exists\,\beta\in\Delta$ s.t. $\alpha-\beta\in\Phi^+$

Pf: Note that $(\alpha,\beta)>0$ for some $\beta\in\Delta$, since $0<(\alpha,\alpha)=\sum_{\gamma\in\Delta}k_\gamma(\alpha,\gamma)$ and $k_\beta>0$.
$\Rightarrow S_\beta(\alpha)=\alpha-\langle\alpha,\beta\rangle\beta=\sum_{\gamma\in\Delta,\gamma\neq\beta}k_\gamma\cdot\gamma+(k_\beta-\langle\alpha,\beta\rangle)\beta$
α not simple $\Rightarrow$ some other $k_\gamma\neq 0$. Since every element is either in $\mathbb{Z}_+\cdot\Delta$
$\Rightarrow k_\beta-\langle\alpha,\beta\rangle\geq 0$. By a previous lemma, $\alpha-\beta\in\Phi$ and $\alpha-\beta\in\mathbb{Z}_+\Delta$, thus $\alpha-\beta\in\Phi^+$. □

Cor. Every $\alpha\in\Phi$ can be written as $\alpha=\alpha_1+\cdots+\alpha_r$ (may have repetition), $\alpha_i\in\Delta$, and each $\alpha_1+\cdots+\alpha_i\in\Phi$, $1\leq i\leq r$. □

Def: Write $\alpha=\sum_{\gamma\in\Delta}k_\gamma\cdot\gamma$, the height of α is defined as $ht(\alpha)=\sum_\gamma k_\gamma$.

E.g.



In general, for A_n, $\Delta=\{\varepsilon_1-\varepsilon_2,\cdots,\varepsilon_n-\varepsilon_{n+1}\}$
$\varepsilon_1-\varepsilon_{n+1}=\varepsilon_1-\varepsilon_2+\cdots+\varepsilon_n-\varepsilon_{n+1}$ has height n.

Note that if $\alpha\in\Phi^+$ is not simple, then by the same argument as in prop. $S_\beta(\alpha)\in\Phi^+$, $\beta\in\Delta$ and $ht(S_\beta(\alpha))=ht(\alpha)-\langle\alpha,\beta\rangle$.

Cor. Any positive root can be reflected by simple reflections (i.e. reflections w.r.t. P_α, α simple) to a simple root. □

Cor. W is generated by simple reflections. (w.r.t. any Δ)
Pf: $\forall\alpha$, $\alpha=S_{\alpha_r}\cdots S_{\alpha_1}\alpha_0$ ($\triangleq\sigma(\alpha_0)$), $\alpha_0,\alpha_1,\cdots,\alpha_r\in\Delta$.
$\Rightarrow S_\alpha=\sigma S_{\alpha_0}\sigma^{-1}=S_{\alpha_r}\cdots S_{\alpha_1}S_{\alpha_0}S_{\alpha_1}\cdots S_{\alpha_r}$. □

Note that, under S_α, $\Phi^+=\Phi^+\backslash\{\alpha\}\sqcup\{\alpha\}$ and S_α only permutes $\Phi^+\backslash\{\alpha\}$ and sends α to $-\alpha$. (c.f. proof of prop. above)

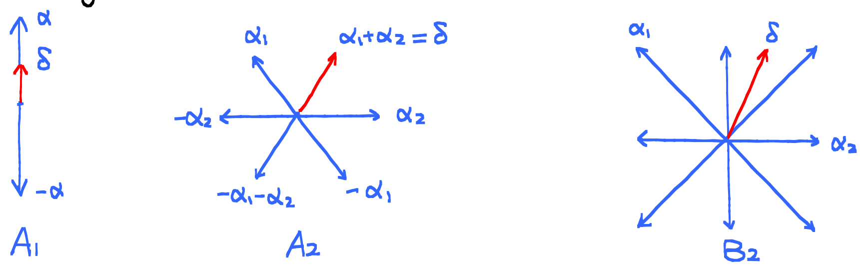
Def. $\delta=\frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$

Cor. $\forall \beta \in \Delta$, $s_\beta(\delta) = \delta - \beta$.

Pf: Indeed, $s_\beta(\delta) = \frac{1}{2}\left(\sum_{\alpha \in \Phi^+ \setminus \{\beta\}} \alpha + (-\beta)\right) = \delta - \beta$. □

Rmk: Note that if $\{\lambda_i\}$ is a dual basis of Δ, i.e. $\langle \lambda_i, \alpha_j \rangle = \delta_{ij}$.
$s_\beta\left(\sum_{i=1}^n \lambda_i\right) = \sum \lambda_i - \beta$, $\forall \beta \in \Delta$. Thus $s_\beta\left(\delta - \sum_{i=1}^n \lambda_i\right) = \delta - \sum_{i=1}^n \lambda_i$, $\forall \beta \in \Delta$
$\Rightarrow \delta = \sum_{i=1}^n \lambda_i$.

E.g. δ for some LA's.



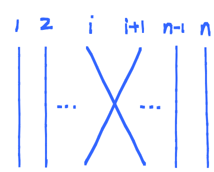
Prop. Let $\alpha_1, \cdots, \alpha_t \in \Delta$, write $s_i = s_{\alpha_i}$ for simplicity. If $s_1 \cdots s_{t-1}(\alpha_t) \in \Phi^-$
$\Rightarrow$ for some $1 \le j < t$, $s_1 \cdots s_{t-1} \cdot s_t = s_1 \cdots s_{j-1} s_{j+1} \cdots s_t$.

Pf: Let $\beta_i = s_{i+1} \cdots s_{t-1}(\alpha_t)$, $\beta_{t-1} = \alpha_t$. Find the smallest j s.t. $\beta_j > 0$
and $s_j(\beta_j) = \beta_{j-1} < 0$. $\Rightarrow \beta_j = \alpha_j = -\beta_{j-1} \Rightarrow \alpha_j = s_{j+1} \cdots s_{t-1}(\alpha_t)$. Write
$\sigma = s_{j+1} \cdots s_{t-1} \Rightarrow s_j = \sigma s_t \sigma^{-1} = s_{j+1} \cdots s_{t-1} s_t s_{t-1} \cdots s_{j+1}$
$\Rightarrow 1 = s_j s_{j+1} \cdots s_t s_{t-1} \cdots s_{j+1} \Rightarrow s_1 \cdots s_{j-1} \cdot s_{j+1} \cdots s_{t-1} = s_1 \cdots s_{j-1} \cdot s_{j+1} \cdots s_t$. □

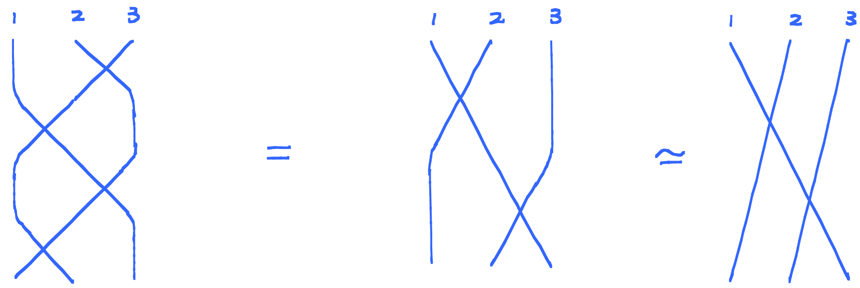
Cor. If $\sigma = s_1 \cdots s_t$ is a shortest way of writing $\sigma \in W$, then $\sigma(\alpha_t) < 0$ □

- A pictorial presentation of classical Weyl groups.

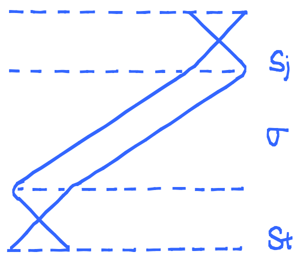
We may use the picture below to represent $s_i = (i, i+1) \in S_n$



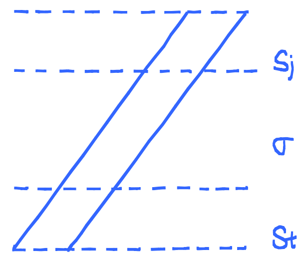
The fact that $s_2 s_1 s_2 s_1 = s_1 s_2$ is then :



Thus in the proof of the prop. above, note that s_j, s_t (in case of A_n), $s_j = s_{\varepsilon_j - \varepsilon_{j+1}} = (j, j+1)$ and $s_t = (t, t+1)$. Moreover, $\sigma(\alpha_t) = s_{j+1} \cdots s_{t-1}(\alpha_t) = \alpha \Rightarrow s_j \cdot \sigma = \sigma \cdot s_t \Rightarrow s_j \sigma s_t = \sigma$ ($j \mapsto j+1 \mapsto \sigma(j+1) \mapsto s_t(\sigma(j+1)) = \sigma(j)$; $j+1 \mapsto j \mapsto \sigma(j) \mapsto s_t(\sigma(j)) = \sigma(j+1) \Rightarrow \sigma(j) = t$, $\sigma(j+1) = t+1$, not the other way around since s_j is smallest such.)

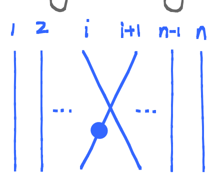


i.e. the picture changed into



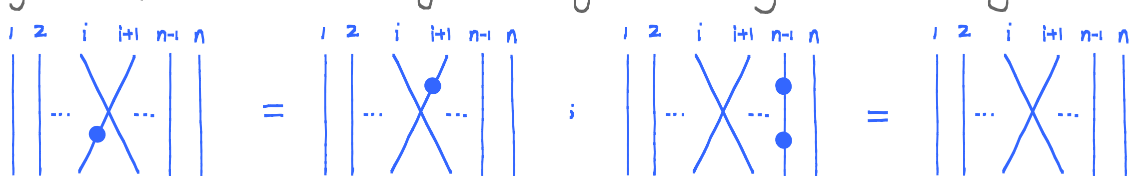
which has less crossings! Such graphs are very useful when dealing with S_n, or for A_n's whose Weyl groups are precisely S_n's.

To represent signed S_n's (Weyl groups of B_n, C_n, D_n), we can put beads on the edges to represent a sign change:

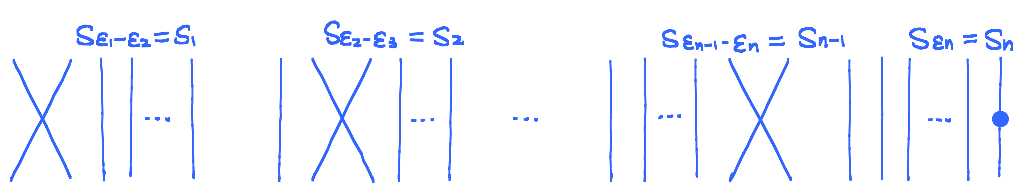


$x_i \mapsto x_{i+1}$, $x_{i+1} \mapsto -x_i$

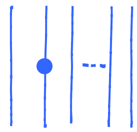
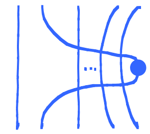
Of course, the beads can glide on edges and any two on one edge cancel:

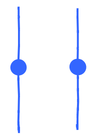
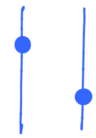
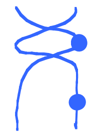


For example, $W(B_n)$: $1 \to (\mathbb{Z}/2)^n \to W \to S_n \to 1$ is generated by reflections:



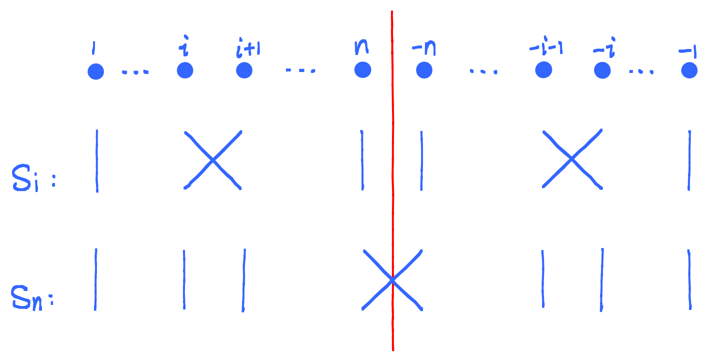
As examples, we decompose the element

1)  $= ?$ $=$  $= s_2 \cdots s_{n-1} s_n s_{n-1} \cdots s_2$

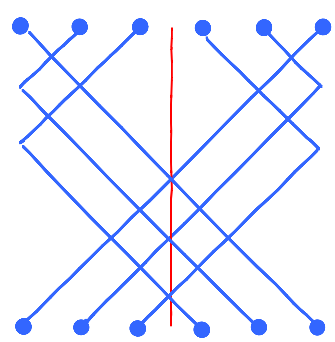
2)  $= ?$ $=$  $=$  $= s_{n-1} \cdot s_n \cdot s_{n-1} s_n$

Note that these graphical representations only work for classical LA's, whose Weyl groups are more or less S_n's with signs (beads, or even number of beads).

Another way of such graphical representation is to imbed $W(B_n)$ into S_{2n} as permuting $\{x_i \mid i = 1, 2, \cdots, n, -n, \cdots -2, -1\}$, then $s_i(x_i) = x_{i+1}$, $s_i(x_{-i}) = x_{-(i+1)}$ $i = 1, \cdots, n-1$; $s_n(x_n) = x_{-n}$, $s_n(x_{-n}) = x_n$.



i.e. symmetric graphs w.r.t the vertical line, subject to simplification rules as before for S_n (now symmetrically). As an example, we can compute the length of σ_{max} $(1,-1)(2,-2)(3,-3)\cdots(n,-n)$ (C.f. def below for σ_{max})



$$\text{length}(\sigma_{max}) = \frac{n(n-1)}{2} + \frac{n(n-1)}{2} + n = n^2$$

Road Map: We are moving toward establishing the correspondences (Coxtergroups: some finite subgroups of $O(n)$ generated by reflections. C.f. Humphreys Reflection groups and Coxeter groups)

$$\begin{array}{ccccc} \text{Simple LA's} & \xleftrightarrow{1:1} & \text{Irreducible root systems} & \overset{\text{not } 1:1}{\dashrightarrow} & \text{Coxter groups} \\ L & \mapsto & \Phi(L, H) & & \\ & & \Phi & \mapsto & W(\Phi) \end{array}$$

Not 1:1 since $W(B_n) = W(C_n)$; and the Coxter group $D_{i,n}$ $(n \neq 3, 4, 6)$ have no preimage

Moreover, we will study representations of LA's on root system point of view (combinatorics).

Def: $\sigma \in W$. The length of σ is defined as the length of the shortest presentation as a product of simple reflections. (It may not be unique since pictorially, we have:

i.e. $s_1 s_2 s_1 = s_2 s_1 s_2$)

$$n(\sigma) \triangleq \#\{\alpha \in \Phi^+ \mid \sigma(\alpha) < 0\} = \#\{\sigma(\Phi^+) \cap \Phi^-\}.$$

Prop. W acts simply transitively on Weyl chambers.

Pf: 1). Transitivity.

Fix a base Δ, with associated Weyl chamber $C(\Delta)$. Take any γ regular, and consider the numbers $\{(\sigma(\gamma), \delta) \mid \sigma \in W\}$. Choose σ such that this number is maximal.

$\Rightarrow (\sigma(\gamma), \delta) \geq (s_i \cdot \sigma(\gamma), \delta) = (\sigma(\gamma), s_i(\delta)) = (\sigma(\gamma), \delta - \alpha_i) = (\sigma(\gamma), \delta) - (\sigma(\gamma), \alpha_i)$.

$\Rightarrow (\sigma(\gamma), \alpha_i) \geq 0$, and "$>0$" holds since γ is chosen to be regular.

$\Rightarrow \sigma(\gamma) \in C(\Delta) \Rightarrow \gamma \in \sigma^{-1}(C(\Delta)) = C(\sigma^{-1}(\Delta))$

2). Simply-transitivity.

$\forall \sigma \in \mathrm{Stab}_W(\Delta)$, if $\sigma \neq 1$, then σ necessarily permutes simple roots in Δ. Take a shortest representation of σ, $\sigma = S_1 \cdots S_m \Rightarrow \sigma(\alpha_m) < 0$ by a previous lemma, contradiction. $\square$

Note that we have established the following 1-1 correspondence:

$$\{\text{Bases of a root system}\} \xleftrightarrow{\text{canonical}} \{\text{Weyl chambers}\} \xleftrightarrow{\text{fix } \Delta} W = \text{Weyl group}$$

$$\Delta \longmapsto C(\Delta)\ ;\ \sigma \cdot C(\Delta) \longleftarrow\!\!| \ \sigma$$

Lemma. $n(\sigma) = \ell(\sigma)$

Pf: Write $\sigma = S_1 \cdots S_m$ a shortest length representation. Then $n(\sigma) \leq \ell(\sigma)$ is obvious since each simple reflection sends only one positive element to Φ^-.

If $n(\sigma) < \ell(\sigma)$, then σ necessarily sends some simple root to a negative simple root then back to a positive simple root again, this would imply that such a representation can be shortened, contradiction. $\square$

Note that $-\Delta$ is also a base $\Rightarrow \exists! \sigma \in W$, $\sigma(\Delta) = -\Delta$, then $\sigma(\Phi^+) = \Phi^-$. and this element has maximal length $|\Phi^+|$. Thus we can call it σ_{max}. However σ_{max} is not necessarily $-Id$, i.e. $\sigma_{max}(\alpha_i) = -\alpha_i$ is not necessarily true for all simple roots $\alpha_i \in \Delta$. For instance $\sigma = -Id$ for A_1 but $\sigma \neq -Id$ for A_2 ($-Id$ is not in the Weyl group).

Cor. (1). $\ell(\sigma\sigma') \leq \ell(\sigma) + \ell(\sigma')$

(2). $W \xrightarrow{\ell} \mathbb{Z}_{\geq 0} \xrightarrow{\text{parity}} \mathbb{Z}/2$ is the sign homomorphism.

Pf: (1) Take the shortest representations of σ, σ', then their product is a representation of $\sigma\sigma'$, not necessarily the shortest.

(2). Write $\sigma = S_{i_1} \cdots S_{i_{\ell(\sigma)}} \Rightarrow \det \sigma = (-1)^{\ell(\sigma)}$. $\square$

Structure of a root system

The structure of a base determines the root system:

$(E, \Phi, \Delta) = \oplus$ Irreducible root systems $= \oplus(E_i, \Phi_i, \Delta_i)$, $(\Delta_i \perp \Delta_j$ if $i \neq j)$

$\Delta = \{\alpha_1, \cdots, \alpha_n\}$, and we have a set of Cartan integers $\{\langle \alpha_i, \alpha_j \rangle\}$

Construct a graph $\Gamma(\Phi)$ as follows:

(1). Each vertex stands for a simple root

(2). For each pair of simple roots α_i, α_j, assuming $|\alpha_i| > |\alpha_j|$

If α_i, α_j span: $A_1 \times A_1$ $\quad A_2 \quad B_2 \quad G_2$

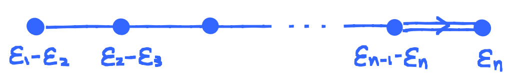
Then to each root system Φ, we have associated it with a graphical invariant $\Gamma(\Phi)$. If Φ is reducible, then the connected components of $\Gamma(\Phi)$ will correspond to irreducible summands of Φ.

E.g.

The classical case:

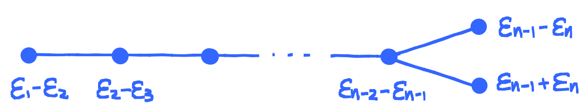
A_n:

B_n:



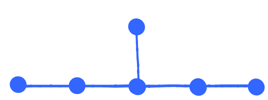
C_n:

D_n:



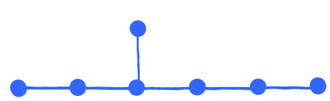
The exceptional case:

E_6:



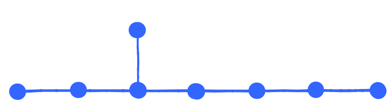
F_4:

E_7:



G_2:

E_8:



Note that if Φ is irreducible, the E is an irrep of W. Indeed, otherwise, if $E = E' \oplus E''$ as W-rep's, $E' \perp E''$, then any $\alpha \in \Phi$ lies either in E' or E'', since S_α preserves E' and E''. This is impossible if Φ is irreducible.

In particular, if Φ is irreducible, $\alpha, \beta \in \Phi$, then we can find $\sigma \in W$ s.t. $(\sigma(\alpha), \beta) \neq 0$ since $W \cdot \alpha$ must span E. It follows that $\sigma(\alpha)$, β span a non-trivial rank 2 root system. $\Rightarrow \frac{|\alpha|^2}{|\beta|^2} \in \{1, 2, \frac{1}{2}, 3, \frac{1}{3}\}$. It follows that there are at most 2 root length in an irreducible root system, for otherwise $\frac{|\alpha|^2}{|\beta|^2} = 2$ $\frac{|\beta|^2}{|\gamma|^2} = 3 \Rightarrow \frac{|\alpha|^2}{|\beta|^2} = 6$.

Def. A root system is called simply-laced if all roots have the same length

Note that rescaling of $(\ ,\ )$ preserves the Cartan integers, and thus preserves the graph. Usually, we rescale $(\ ,\ )$ so that $(\alpha, \alpha) = 2$ for a shortest root.

Thm. Any irreducible root system is one of the following:

A_n $(n \geq 1)$ B_n $(n \geq 2)$ C_n $(n \geq 3)$ D_n $(n \geq 4)$

E_6, E_7, E_8, F_4, G_2.

Idea of proof:

As we did for MaKay correspondence, we associate Φ with:

$\Phi \rightsquigarrow \Delta \rightsquigarrow \Gamma(\Phi) \rightsquigarrow \mathbb{R}^n$, with an inner product $(\ ,\ )$ satisfying a shortest root α has $(\alpha, \alpha) = 2$ and $\frac{2(\alpha_1, \alpha_2)}{(\alpha_2, \alpha_2)} = \langle \alpha_1, \alpha_2 \rangle$ $\forall\ \alpha_1, \alpha_2 \in \Delta$. $\langle \alpha_i, \alpha_j \rangle \langle \alpha_j, \alpha_i \rangle =$ # of edges connecting α_i, α_j.

If more generally we do this for any graph, then the associated inner product will be:

(1). Indefinite, i.e. $\exists\ v \in \mathbb{R}^n$, $(v, v) < 0$.

(2). positive semi-definite (affine)

(3). positive definite (Dynkin)

and the only possible positive definite case will be $A_n - G_2$ cases above.

(C.f. Humphreys) $\square$

More on Weyl groups

For each $\alpha_i, \alpha_j \in \Delta$, define $m(i,j) \triangleq$ order of $s_i s_j$ in W.

$m(i,j)=2$ $\quad$ $m(i,j)=3$ $\quad$ $m(i,j)=4$ $\quad$ $m(i,j)=6$

Thm. W has generaters s_i, $i \in \Delta$ and defining relations $s_i^2=1$, $(s_i s_j)^{m_{ij}}=1$ over all $i,j \in \Delta$.

For a proof, see Humphreys, Reflection groups and Coxter groups.

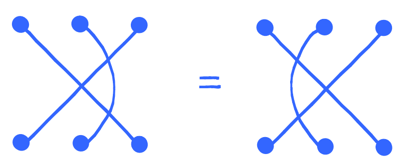
Note that since $s_i^2=1$

$(s_i s_j)^2=1 \iff s_i s_j = s_j s_i$ $\qquad$ $(s_i s_j)^4=1 \iff s_i s_j s_i s_j = s_j s_i s_j s_i$

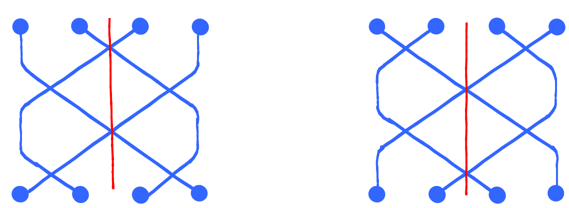
$(s_i s_j)^3=1 \iff s_i s_j s_i = s_j s_i s_j$ $\qquad$ $(s_i s_j)^6=1 \iff s_i s_j s_i s_j s_i s_j = s_j s_i s_j s_i s_j s_i$

Thus any 2 minimal representation of $\sigma \in W$ as products of simple reflections can be related by the above relations. (These relations keep length).

E.g.



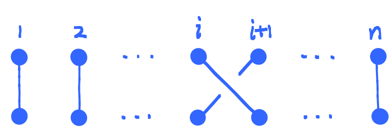
$(s_i s_j s_i = s_j s_i s_j$: α_i, α_j span an $A_2)$



$(s_i s_j s_i s_j = s_j s_i s_j s_i$: α_i, α_j span a $B_2)$

If we remove the restriction $s_i^2=1$, we obtain the so called Artin (braid) group. For example, for A_n, we obtain the braid group on $n+1$ strands.

s_i :



Given W, we may consider the connected components of $E \setminus \bigcup_{\alpha\in\Phi} P_\alpha$, i.e. $\pi_0(E\setminus \bigcup_{\alpha\in\Phi} P_\alpha)$. Moreover, if we consider $E^{\mathbb{C}} \setminus \bigcup_{\alpha\in\Phi} P_\alpha^{\mathbb{C}}$, which is then connected, we obtain:

$$1 \longrightarrow \pi_1(E^{\mathbb{C}} \setminus \bigcup_{\alpha\in\Phi} P_\alpha^{\mathbb{C}}) \longrightarrow Br(\Phi) \longrightarrow W \longrightarrow 1$$

where $Br(\Phi)$ is the Artin braid group of Φ. $\pi_1(E^{\mathbb{C}} \setminus \bigcup_{\alpha\in\Phi} P_\alpha^{\mathbb{C}})$ is called the pure braid group. A nontrivial fact is that $E^{\mathbb{C}} \setminus \bigcup_{\alpha\in\Phi} P_\alpha^{\mathbb{C}}$ is $K(\pi_1, 1)$!

Generators and relations.

Recall that from a simple L.A., we obtain an irreducible root system: $(L, H) \rightsquigarrow (E, \Phi, \Delta)$, and $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$. For each $\alpha_i \in \Delta$, we can form a copy of $\mathfrak{sl}(2)_{\alpha_i}$ by choosing $x_i \in L_{\alpha_i}$, $y_i \in L_{-\alpha_i}$, $h_i \in H$.

Prop. L is generated by $\{x_i, y_i, h_i\}_{i=1}^n$ as a LA.

Pf: Recall that if $\alpha, \beta \in \Phi$, not propotional and non-zero, then $[L_\alpha, L_\beta] = L_{\alpha+\beta}$ ($L_{\alpha+\beta} = 0$ if $\alpha+\beta \notin \Phi$). Moreover, any $\alpha \in \Phi^+$ can be written as a sum of simple roots $\alpha = \alpha_1' + \cdots + \alpha_r'$ such that each partial sum $\alpha_1' + \cdots + \alpha_s'$ $(1 \le s \le r)$ is a root $\Rightarrow L_\alpha = [L_{\alpha_r'} \cdots [L_{\alpha_2'}, L_{\alpha_1'}]]$. Similarly for $\alpha \in \Phi^-$. $\square$

Relations satisfied by $\{x_i, y_i, h_i\}$: define $C_{ij} \triangleq \langle \alpha_i, \alpha_j \rangle = \frac{2(\alpha_i, \alpha_j)}{(\alpha_j, \alpha_j)} \in \{\pm 3, \pm 2, \pm 1, 0\}$.

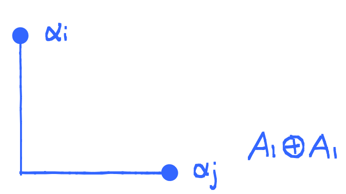
Then: (i). $[h_i\ x_j] = C_{ji} x_j$

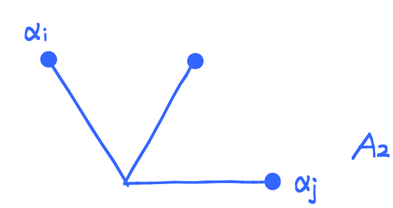
(ii). $[h_i, y_j] = -C_{ji} y_j$

(iii). $[h_i, h_j] = 0$

(iv). $[x_i, y_j] = \delta_{ij} h_i$ (since if $i \neq j$, $[x_i, y_j]$ lies in $L_{\alpha_i - \alpha_j} = 0$, as $\alpha_i - \alpha_j \notin \Phi$).

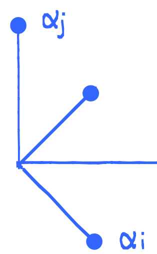
Moreover, we have the commutation relations between $\{x_i, x_j\}$, $\{y_i, y_j\}$'s.



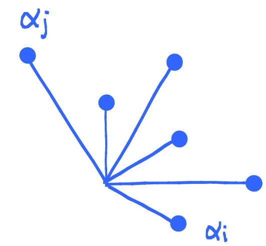


$\langle \alpha_j, \alpha_i \rangle = 0$, $[x_i, x_j] = 0$

$\langle \alpha_j, \alpha_i \rangle = -1$, $[x_i, [x_i, x_j]] = 0$



B_2



G_2

$\langle \alpha_j, \alpha_i \rangle = -2 \quad [x_i,[x_i,[x_i,x_j]]]=0$ $\qquad \langle \alpha_j, \alpha_i \rangle = -3, \ [x_i,[x_i,[x_i,[x_i,x_j]]]]=0$

Hence if $i \neq j$:

(v). $(\operatorname{ad} x_i)^{-c_{ji}+1}(x_j)=0$

(vi). $(\operatorname{ad} y_i)^{-c_{ji}+1}(y_j)=0$

Thm. (Serre). Given an irreducible root system Φ, $\Delta=\{\alpha_1,\cdots,\alpha_n\}$, the Lie algebra with generators $\{x_i, y_i, h_i\}_{i=1}^n$ satisfying the relations (i) – (vi) exists and is finite dimensional and simple.

Free Lie algebras

Given generators $x_1,\cdots,x_n$, then all possible $\mathbb{C}$-linear combination of iterated commutators $[x_i,x_j]$, $[x_i,[x_j,x_k]]$, $[[x_i,x_j],[x_k,x_\ell]]$, $\cdots\cdots$ modulo the relations:

(1). Anti-symmetry: $[a,b]=-[b,a]$

(2). Jacobi's identity: $[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0$

is the free Lie algebra generated by $x_1,\cdots,x_n$, denoted $FLA(x_1,\cdots,x_n)$.

Compare with the free associative algebra $\mathbb{C}\langle x_1,\cdots,x_n\rangle$

$\mathcal{U}FLA(x_1,\cdots,x_n) \cong \mathbb{C}\langle x_1,\cdots,x_n\rangle$ ($\triangleq$ tensor algebra over $x_1,\cdots,x_n$).

Proof of Serre's thm.

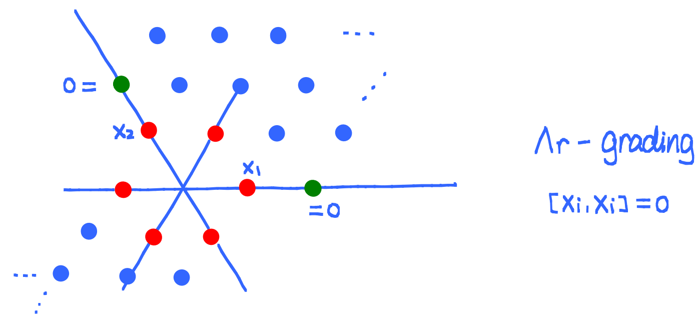
Form the Lie algebra L' with generators $\{x_i,y_i,h_i\}_{i=1}^n$, subject to relation (i) – (iv). Since $[h,[a,b]]=[[h,a],b]+[a,[h,b]]$, we see that, by relations (i), (ii), (iii) terms involving h can always be reduced to those only involving x_i, y_j's. Further, $[x_i,y_j]$ can always be reduced into $\bigoplus_{i=1}^n \mathbb{C}h_i \triangleq H$. Thus we see that:

$$L' = L'_+ + H + L'_-$$

where L'_+ is the subalgebra generated by $\{x_i\}$, L'_- generated by $\{y_i\}$.

Lemma. $L' = L'_+ \oplus H \oplus L'_-$

Pf: This follows since we have a multi-grading on L', by assigning $\deg x_i = \alpha_i$, $\deg y_j = -\alpha_j$, $\deg h_k = 0$, $1 \le i, j, k \le n$. This gives a grading on L' by the root lattice Λ_r, since the ideal generated by the relations preserves the grading.



$\square$

Cor. Any ideal I of L' respects the grading: $I = \oplus_{\lambda \in \Lambda_r} I \cap L'_\lambda$

Pf: Look at the H action, this grading is just the H-wgt decomposition. $\square$

Idea: take the simple quotient of L', i.e. take the sum of all such ideals of L' that doesn't intersect H, and by the cor, it's still an ideal. But firstly we need to know the size of L', in particular, if it's sufficiently large (non-0). For this purpose, we should consider a sufficiently large rep of L'.

Consider $\mathbb{C}_0 = \mathbb{C} \cdot v$, the trivial rep of $L'_+ \oplus H$. $hv = 0$, $x_i v = 0$, $\forall i$. Then $\mathrm{Ind}_{\mathcal{U}(L'_+ \oplus H)}^{\mathcal{U}(L')} \mathbb{C}_0 = \mathbb{C}_0 \otimes_{\mathcal{U}(L'_+ \oplus H)} \mathcal{U}(L')$ has the same size as $\mathcal{U}(L'_-) \cong \mathbb{C}\langle y_1, \cdots, y_n \rangle$. (Since there are no relations among x_i's at the moment). In other words, we may build a $\mathcal{U}(L')$-module out of $\mathbb{C}\langle y_1, \cdots, y_n \rangle \cdot v \triangleq M_0$, as follows:

$M_0 = \mathbb{C}\{ y_{i_1} \cdots y_{i_m} v \mid 1 \le i_1, \cdots, i_m \le n,\ m \in \mathbb{Z}_{\ge 0} \}$.

y_i acts by left multiplication: $y_i(y_{i_1} \cdots y_{i_m} v) = y_i y_{i_1} \cdots y_{i_m} v$.

h_i acts by $h_i v = 0$; $h_i y_{i_1} \cdots y_{i_m} v = \sum_{k=1}^{m} (-C_{i_k, i}) y_{i_1} \cdots y_{i_m} v$

x_i acts by $x_i v = 0$; $x_i y_{i_1} \cdots y_{i_m} v = \sum_{k=1}^{m} \delta_{i, i_k} y_{i_1} \cdots y_{i_{k-1}} \cdot h_i \cdot (y_{i_{k+1}} \cdots y_{i_m} v)$

$$= \sum_{k=1}^{m-1} \delta_{i, i_k} \sum_{\ell = k+1}^{m} (-C_{i_\ell, i}) y_{i_1} \cdots \hat{y}_{i_k} \cdots y_{i_m} v.$$

We can check that the relations (i) – (iv) hold for this action. In particular, this argument shows that $x_i, h_i, y_i \ne 0$ in L'.

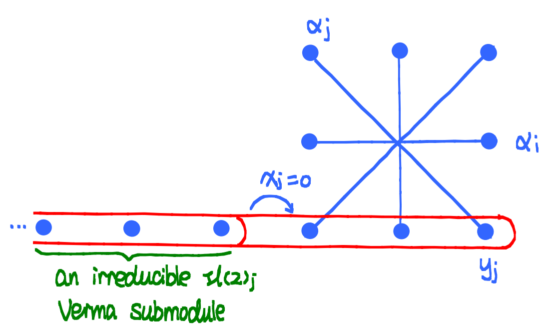
(Actually, $L_\pm'$ are free LA's on generators $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ resp.).

Next, consider the ideal D_- of L_-' generated by all $y_{ij} \triangleq (\mathrm{ad}\, y_i)^{-c_{ji}+1}(y_j)$, $i \neq j$.
Claim: D_- is an ideal in L'.
It suffices to check : $[x_k, D_-] \subseteq D_-$, $[h_k, D_-] \subseteq D_-$,
$[h_k, (\mathrm{ad}\, y_i)^{-c_{ji}+1}(y_j)]$ just rescales the element.
$[x_k, (\mathrm{ad}\, y_i)^{-c_{ji}+1}(y_j)]$: there are 3 cases:
If $k \neq i, j$, it's 0.
If $k = j$, $[x_j, (\mathrm{ad}\, y_i)^{-c_{ji}+1}(y_j)] = (\mathrm{ad}\, y_i)^{-c_{ji}+1}([x_j, y_j]) = (\mathrm{ad}\, y_i)^{-c_{ji}+1}(h_j)$

$$= \begin{cases} (\text{if } c_{ji} = \langle \alpha_j, \alpha_i \rangle = 0)\ [y_i, h_j] = c_{ji} y_i = 0 \\ (\text{if } c_{ji} < 0) = [\cdots [y_i, [y_i, h_j]] \cdots] = [\cdots [y_i, c_{ji} y_i] \cdots] = 0. \end{cases}$$

If $k = i$. We have a copy of $\mathfrak{sl}(2) \cong \mathbb{C}\{x_i, y_i, h_i\} \subseteq L'$. As an $\mathfrak{sl}(2)$-module, h_i is diagonalizable on L' and y_j generates a copy of $\mathfrak{sl}(2)$-Verma module. Indeed, $[x_i, y_j] = 0$ and $y_j \xmapsto{\mathrm{ad}\, y_i} [y_i, y_j] \xmapsto{\mathrm{ad}\, y_i} [y_i [y_i, y_j]] \longmapsto \cdots$ y_j has h.w. $-c_{ji}$, and thus from our knowledge on $\mathfrak{sl}(2)$-modules we know that

$$[x_i, (\mathrm{ad}\, y_i)^{-c_{ji}+1}(y_j)] = 0$$



Likewise $\{(\mathrm{ad}\, x_i)^{-c_{ij}+1}(x_j)\}$ generates an ideal D_+ of L'. Thus taking their sum, we obtain an ideal $D = D_+ \oplus D_-$ of L'. Let $L \triangleq L'/D$, which is still graded by L' since D is.

Claim: L is simple and has wgt decomposition given by the root system Φ.
Indeed, note that $\mathrm{ad}\, x_i$, $\mathrm{ad}\, y_i$ are locally nilpotent operators on L (not on L', that's the difference), i.e. they act nilpotently on all generators $\{x_k, y_k, h_k\}_{k=1}^n$;

$\mathrm{ad}\, h_i$ acts semisimply on L' (thus on L). Hence L can be decomposed into $L = \bigoplus$ finite dim'l irrep's of $\mathfrak{sl}(2)_i$.

By our construction, $H \subseteq L$, $L_{\alpha_i} = \mathbb{C}x_i$, $L_{-\alpha_i} = \mathbb{C}y_i$. $\Rightarrow \dim L_\alpha = \dim L_{s_i\alpha}$ from our knowledge of finite dim'l $\mathfrak{sl}(2)$-modules. It follows that $\dim L_\beta = 1$, $\forall \beta \in \Phi$ since $W\cdot\Delta = \Phi$. Furthermore, $L_{\pm k\beta} = 0$, $\forall k > 1$, since this is true for the simple roots, and again $W\cdot\Delta = \Phi$. Finally, if $\beta \in \Lambda_r$ is not a multiple of a root, then for some $\sigma \in W$, $\sigma(\beta)$ will have both possitive and negative coefficients w.r.t. α, and hence $L_{\sigma(\beta)} = 0 \Rightarrow L_\beta = 0$. It follows that L is simple since H is a CSA $L = H \oplus \bigoplus_{\alpha\in\Phi} L_\alpha$ is the root decomposition, and Φ is an irreducible root system.

Altogether, the above arguments prove Serre's thm.

Cor. D is the maximal ideal disjoint from H.

Pf: Indeed, if D' is another such ideal. $L \twoheadrightarrow L'/(D+D') \neq 0$. L simple $\Rightarrow D + D' = D \Rightarrow D' \subseteq D$. $\square$

Cor. (Uniqueness of L). Isomorphic irreducible root systems give rise to isomorphic simple Lie algebras.

Pf: Our construction only used the integers $c_{ij} = \langle \alpha_i, \alpha_j \rangle$. $\square$

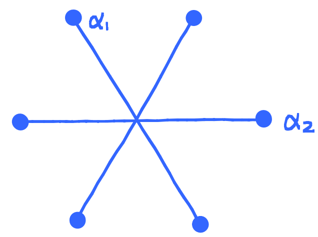
Cor. Inclusion of (reducible) root systems $(E', \Phi') \hookrightarrow (E, \Phi)$ give rise to inclusion of (semi) simple LA's. $\square$

Cor. (Existence of exceptional LA's). $\exists\, E_6, E_7, E_8, F_4, G_2$ simple LA's. $\square$

Rmk: By our construction, $L \cong L_+ \oplus H \oplus L_-$. $L_+ \oplus H$ is solvable (actually a maximal solvable subalgebra in L), called the positive Borel subalgebra in L; $L_- \oplus H$ the negative Borel subalgebra, (which is positive w.r.t. $-\Delta$).

Fact: Any maximal solvable subalgebra of $L \cong b_+$. For a proof, see Humphreys or Sternberg.

E.g. $\mathfrak{sl}(3)$



x_1, x_2 generates L_+, subject to relations $[x_1,[x_1,x_2]]=0$ and $[x_2,[x_2,x_1]]=0$

$\Rightarrow$ Any double commutator is 0, by Jacobi's identity.

$\Rightarrow$ L_+ is generated by $x_1, x_2, [x_1,x_2]$.

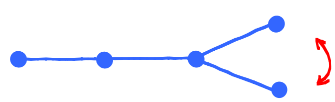
Automorphism of Dykin diagram.

Now that we have an exact sequence:

$$1 \longrightarrow W(\Phi) \longrightarrow Aut(\Phi) \longrightarrow Aut(\Phi)/W(\Phi) \longrightarrow 1$$

Since $Aut(\Phi)$ acts transitively on the set of bases Δ of Φ, and $W(\Phi)$ acts simply transitively on the set of bases $\Rightarrow Aut(\Phi)/W(\Phi) \cong Stab_{Aut(\Phi)}(\Delta)$, and thus the sequence splits, since $Stab_{Aut(\Phi)}(\Delta)$ is a subgroup of $Aut(\Phi)$. Hence $Aut(\Phi)/W(\Phi) \cong Stab_{Aut(\Phi)}(\Delta)$ can be identified as the automorphism group of the Dynkin diagram, denoted $Aut(\Gamma)$

E.g. For D_5, $Aut(\Gamma) \cong \mathbb{Z}/2$.



In general,

$$Aut(\Gamma) \cong \begin{cases} \{1\} : & A_1,\ B_n\ (n\geq 2),\ C_n\ (n\geq 3),\ F_4, E_7, E_8, G_2 \\ \mathbb{Z}/2 : & A_n\ (n\geq 2),\ D_n\ (n\geq 5) \\ S_3 : & D_4. \end{cases}$$

Moreover, fixing a C.S.A. H of L, $Aut(L,H)$ satisfies:

$$1 \longrightarrow (\mathbb{C}_*)^n \longrightarrow Aut(L,H) \longrightarrow Aut(\Phi) \longrightarrow 1$$

Indeed, the kernel comes from rescaling: $x_{\alpha_i} \mapsto \lambda x_{\alpha_i}$, $h_{\alpha_i} \mapsto h_{\alpha_i}$, $y_{\alpha_i} \mapsto \lambda^{-1} y_{\alpha_i}$, and thus one $\mathbb{C}_*$ for each $\alpha_i \in \Delta$.

Rmk: $\mathrm{Aut}(L) \cong \widetilde{G}_L / Z(\widetilde{G}_L)$ where $\widetilde{G}_L$ is the simply connected Lie group with L.A. L, and $Z(\widetilde{G}_L)$ its center.

(to be shown in the next semester).

§8. Finite Dimensional L-Modules

Recall that we know already, for finite dimensional $\mathfrak{sl}(n)$-modules, we have

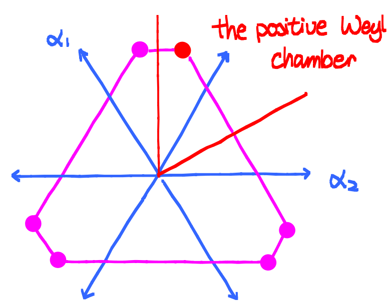
$$\{\text{Irrep's of } \mathfrak{sl}(n)\} \xleftrightarrow{1:1} \{\text{Positive integral wgts } \lambda\in\Lambda^+\}$$

The story is more generally true, for (semi) simple Lie algebras L.

Let $\Delta=\{\alpha_1,\cdots,\alpha_n\}$ be a base. Λ_r = root lattice $\cong \mathbb{Z}\cdot\Delta$. Λ = wgt lattice $\Lambda^+\subseteq\Lambda$ the abelian semigroup of integral wgts s.t. $\langle\lambda,\alpha_i\rangle\geq 0$, $\forall\alpha_i\in\Delta$.
The fundamental wgts $\{\lambda_1,\cdots,\lambda_n\}$ are defined by $\langle\lambda_i,\alpha_j\rangle=\delta_{ij}$. Then $\Lambda=\oplus_{i=1}^n\mathbb{Z}\lambda_i$ and $\lambda\in\Lambda^+$ iff $\lambda=\sum a_i\lambda_i$, $a_i\in\mathbb{Z}_{\geq 0}$.

Take V a finite dimensional L-module, $V=\oplus_{\mu\in\Lambda}V(\mu)$. Recall that we say $v\in V$ is a h.w. vector if $L_+\cdot v=0$, or equivalently, $x_i\cdot v=0$ for each simple root vector.
Furthermore, if V is irreducible, V has a h.w. vector $v\in V(\lambda)$, unique up to a non-zero scalar, and $\lambda\in\Lambda^+$:

Recall that as an $\mathfrak{sl}(2)_{\alpha_i}$-module, V decomposes as direct sums of irrep's and $\dim V(\mu)=\dim V(S_{\alpha_i}\mu)$ $\Rightarrow$ the wgt diagram of V has W-symmetry: $ch(V)\in\mathbb{Z}[\Lambda]^W$
In particular, this shows that the h.w. $\lambda\in\Lambda^+$, since otherwise, $\langle\lambda,\alpha_i\rangle<0\Rightarrow$ $\lambda-\langle\lambda,\alpha_i\rangle\alpha_i>\lambda$ is another wgt of V. Note that the wgt diagram lies in the convex hall formed by $W\cdot\lambda$.

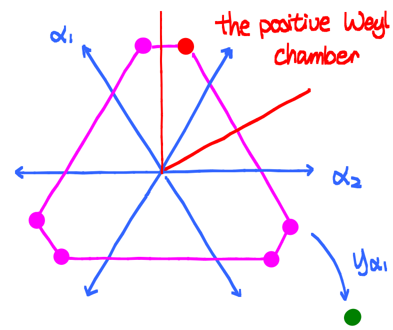


Thus given V an irrep, we can find $v\in V(\lambda)$ a h.w. vector.
$\Rightarrow \exists\, M_\lambda\xrightarrow{f}V\longrightarrow 0$, $v_\lambda\mapsto v$, where M_λ is the Verma module of h.w. λ.
($M_\lambda\cong\mathfrak{U}(L_-)\cdot v_\lambda$, $h\cdot v_\lambda=\lambda(h)v_\lambda$, $x_i v_\lambda=0$, $\forall i=1,\cdots,n$). Furthermore, V irrep $\Rightarrow f$ is surjective. Now we analyze $\ker f$.

Now that $S_{\alpha_i}(\lambda)=\lambda-\langle\lambda,\alpha_i\rangle\alpha_i=\lambda-a_i\alpha_i$, $a_i\triangleq\langle\lambda,\alpha_i\rangle$. Thus $\dim V(\lambda-a_i\alpha_i)=1$ and $V(\lambda-a_i\alpha_i)\cong\mathbb{C}y_i^{a_i}v_\lambda$. By the convexity of the wgt diagram of V, $y_i^{a_i+1}v=0$

$\Rightarrow y_i^{a_i+1} v_\lambda \in \ker f$. Furthermore, we claim that $y_i^{a_i} v_\lambda$ is a h.w. vector of wgt $\lambda-(a_i+1)\alpha_i$

Indeed, if $j\neq i$, $x_j y_i^{a_i} v_\lambda = y_i^{a_i} x_j v_\lambda = 0$.
If $j=i$, this follows from the fact that the $\mathfrak{sl}(2)$ Verma module M_m contains a unique maximal Verma submodule M_{-m-2}. $(m\geq 0)$



This is true for all $\alpha_i \in \Delta \Rightarrow \exists$ homomorphism τ:

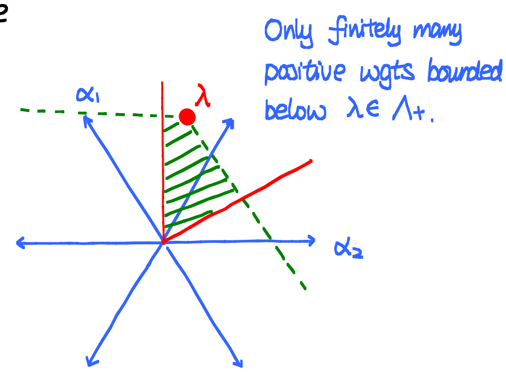
$$\bigoplus_{i=1}^{n} M_{\lambda-(a_i+1)\alpha_i} \xrightarrow{\tau} M_\lambda \longrightarrow 0$$

$$v_{\lambda-(a_i+1)\alpha_i} \mapsto y_i^{a_i+1} v_\lambda$$

Prop: $V_\lambda \cong M_\lambda/\operatorname{Im}\tau$ is a finite dimensional irrep of h.w. λ.

Pf: We will first show that, the quotient is finite dimensional. Fix i, and look at $\mathfrak{sl}(2)_i \cong \mathbb{C}\{x_i, h_i, y_i\}$ and its action on $M_\lambda/\operatorname{Im}\tau$. Our first observation is that the action of x_i, y_i are locally nilpotent, i.e. $x_i^{N_i} v = 0$, $y_i^{N_i} v = 0$, $\forall v \in M_\lambda/\operatorname{Im}\tau$, and $N_i \gg 0$. This is automatic for $x_i^{N_i} v = 0$ since all wgts are bounded above. For the second equality, note that it's true for v_λ by our def. Furthermore, any $v \in M_\lambda$ is of the form $\sum a_{j_1\cdots j_n} y_1^{j_1}\cdots y_n^{j_n} v_\lambda$ (P.B.W. Thm). The second equality follows now since y_i acts ad-nilpotently on each y_j and nilpotently on v_λ.

It follows that $M_\lambda/\operatorname{Im}\tau$ decomposes into (possiblely infinitely many) direct sums of finite $\mathfrak{sl}(2)_i$-modules. $\Rightarrow$ The wgts of $M_\lambda/\operatorname{Im}\tau$ are invariant under S_{α_i}. Since this holds for all $\alpha_i \in \Delta$, the wgts of $M_\lambda/\operatorname{Im}\tau$ are invariant under W. Moreover, the positive wgts bounded by λ are finite
$\Rightarrow \dim M_\lambda/\operatorname{Im}\tau$ is finite

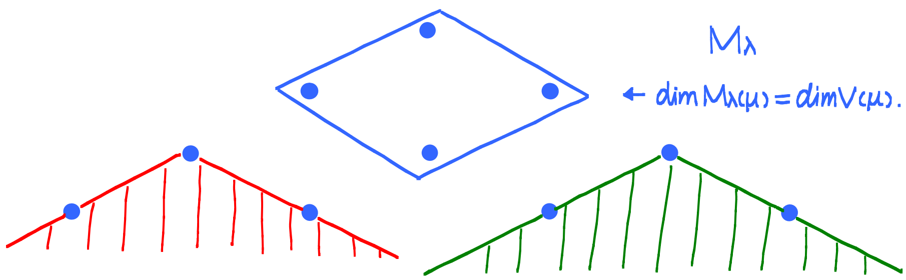


Recall that every Verma module has a unique maximal proper submodule M_λ', s.t. M_λ/M_λ' is irreducible. Now $M_\lambda/\operatorname{Im}\tau$ is finite dimensional cyclic, thus must be irreducible by reducibility of finite dim'l L-modules $\Rightarrow \operatorname{Im}\tau = M_\lambda'$, thus V_λ is irreducible. $\square$

Note that $M_\lambda \cong \mathcal{U}(L_-)v_\lambda \cong \mathbb{C}\{y_1^{b_1}\cdots y_n^{b_n} v_\lambda,\ b_i \in \mathbb{Z}_{\geq 0}\}$. Thus

$\dim M_\lambda(\mu) = \#\{$presentations of $\lambda-\mu$ as sums of positive roots$\}$

In particular, we can count the dimension of wgt spaces of V_λ lying above $\mathrm{Im}\,\tau$, which are: $\dim V(\mu) = \dim M_\lambda(\mu)$

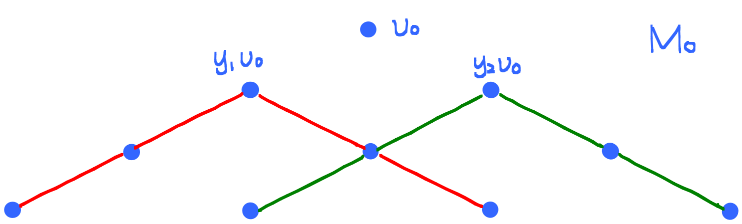


Constructions (existence)

$\forall \lambda \in \Lambda_+$, $\lambda = \sum_{i=1}^n a_i\lambda_i$, $a_i \in \mathbb{Z}_{\geq 0}$. In particular, if $\lambda = \lambda_i$, a fundamental wgt, we have, by our construction, $V_{\lambda_i} = M_{\lambda_i}/\mathrm{Im}\,\tau = M_{\lambda_i}/\bigoplus_{j\neq i} M_{\lambda_i-\alpha_j} \oplus M_{\lambda_i - 2\alpha_i}$.

E.g. (Trivial rep as $M_\lambda/\mathrm{Im}\,\tau$)

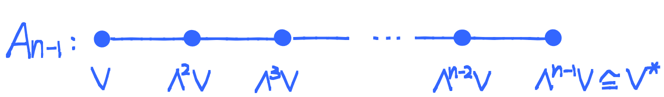
$$V_0 \cong M_0/\oplus_{i=1}^n M_{-\alpha_i} \cong \mathbb{C}v_0$$



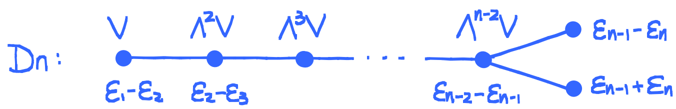
In general, similar as for $\mathfrak{sl}(n)$, if we have V_{λ_i}, then V_λ would be contained in $S^{a_1}V_{\lambda_1} \otimes \cdots \otimes S^{a_n}V_{\lambda_n}$.

Now, each fundamental representation is labeled by the vertices of the Dykin diagram, $\alpha_i \longleftrightarrow V_{\lambda_i}$, we look at cases.

$\mathfrak{sl}(n)$: Let V be the defining representation of $\mathfrak{sl}(n)$ on $\mathbb{C}^n$.

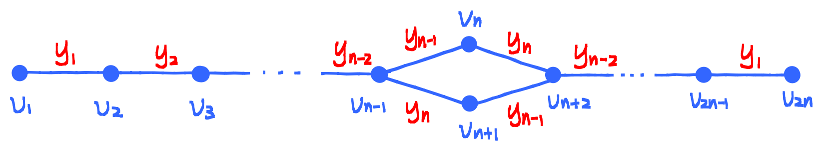


$\mathfrak{so}(2n)$: Let V be the defining representation of $\mathfrak{so}(2n)$ on $\mathbb{C}^{2n}$.



We can check that, $V, \Lambda^2 V, \cdots, \Lambda^{n-2}V$ are irrep's of $\mathfrak{so}(2n)$ of h.w. $\lambda_1,\cdots,\lambda_{n-2}$, h.w. vector v_1, $v_1\wedge v_2, \cdots, v_1\wedge\cdots\wedge v_{n-2}$ resp. by applying $\mathcal{U}(L_-)$ on the h.w. vector and counting dimensions. For instance:

$V = V_{\lambda_1}$:



Here recall that V is the defining representation of $\mathfrak{so}(2n) = \left\{ \begin{array}{cc} \begin{array}{cc} n & n \end{array} & \\ \left(\begin{array}{c|c} * & * \\ \hline * & * \end{array}\right) & \begin{array}{c} n \\ n \end{array} \end{array} \right\}$, the C.S.A $H = \mathrm{Span}\{h_i \mid h_i = e_{ii} - e_{n+i,n+i},\ 1\le i\le n\}$, V has as basis $v_1,\cdots,v_{2n}$, $v_i = (0,\cdots,\underset{i\text{-th}}{1},\cdots 0, \underbrace{0,\cdots,0}_{n})$ $(1\le i\le n)$, $v_{i+n} = (\underbrace{0,\cdots,0}_{n},0,\cdots,\underset{n+i\text{ th}}{1},\cdots 0)$. They are respectively of wgts:

h.w. $v_1 : (1,0,\cdots,0)$ $\qquad$ $v_{n+1} : (0,0,\cdots,-1)$

$v_2 : (0,1,\cdots,0)$ $\qquad$ $v_{n+2} : (0,0,\cdots -1,0)$

$\cdots$ $\qquad$ $\cdots$

$v_n : (0,0,\cdots,1)$ $\qquad$ $v_{2n} : (-1,0,\cdots,0)$

$\Lambda^2 V = V_{\lambda_2}$. Since v_1 is of wgt $(1,0,\cdots,0)$, v_2 of wgt $(0,1,\cdots,0)$, $v_1\wedge v_2$ is of wgt $(1,1,0,\cdots,0) \Rightarrow V_{\lambda_2} \subseteq \Lambda^2 V$. Moreover, $\{y_i \mid i=1,\cdots,n\}$ carry $v_1\wedge v_2$ to all $v_i\wedge v_j$ $(i<j)$ thus $\Lambda^2 V$ is cyclic and $\Lambda^2 V = V_{\lambda_2}$, and $v_1\wedge v_2$ is of h. wgt.

In general, $V, \Lambda^2 V, \cdots, \Lambda^{n-2} V$ form the fundamental irrep's $V_{\lambda_1},\cdots, V_{\lambda_{n-2}}$, by a similar argument as above.

The fundamental rep's corresponding to $\varepsilon_{n-1}-\varepsilon_n$ and $\varepsilon_{n-1}+\varepsilon_n$ are called half spin rep's, which are not rep's of the compact Lie group $SO(2n)$. Recall that $\pi_1(SO(n)) \cong \mathbb{Z}/2$ for $n\ge 3$: $SO(3) \cong SU(2)/(\pm I) \cong S^3/\mathbb{Z}_2 \cong \mathbb{RP}^3$; $SO(n-1) \hookrightarrow SO(n) \longrightarrow S^{n-1}$

$\Rightarrow \{1\} = \pi_2(S^{n-1}) \to \pi_1(SO(n-1)) \longrightarrow \pi_1(SO(n)) \longrightarrow \pi_1(S^{n-1}) = \{1\}$ $(n\ge 3) \Rightarrow \pi_1(SO(n)) \cong \pi_1(SO(n-1))$

Thus the universal cover $\widetilde{SO}(n)$ form a double cover of $SO(n)$. This is called the spin group $Spin(n)$.

Fact: The fundamental rep $V_{\lambda_{n-1}}$, V_{λ_n} are rep's of $Spin(2n)$ which don't descend to $SO(2n)$, thus are not built from V.

$$\alpha_{n-1} = (0,\cdots,1,-1) \Rightarrow \lambda_{n-1} = (\tfrac{1}{2},\tfrac{1}{2},\cdots,\tfrac{1}{2})$$

$$\alpha_n = (0,\cdots 1, 1) \Rightarrow \lambda_n = (\tfrac{1}{2},\tfrac{1}{2},\cdots,-\tfrac{1}{2})$$

The wgts of $V_{\lambda_{n-1}}$ (V_{λ_n}) are within the convex hall formed by $W\cdot\lambda_{n-1}$. ($W\cdot\lambda_n$). Since W consists of interchanging coordinates and even number of sign changes,

$$W\cdot\lambda_{n-1} = \{ (\pm\tfrac{1}{2},\pm\tfrac{1}{2},\cdots,\pm\tfrac{1}{2}) \mid \text{even \# of "}-\text{"} \}$$

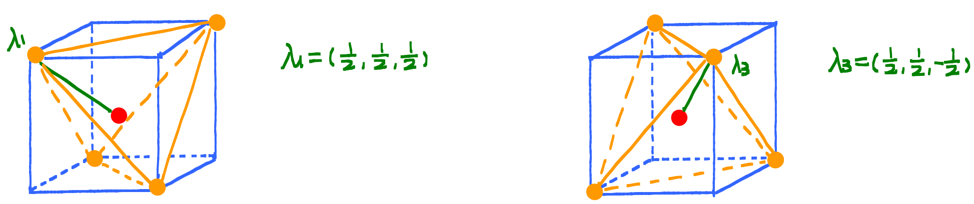
(odd # for λ_n). Note that the only wgt within the convex hall is 0, which is not a wgt of $V_{\lambda_{n-1}}$ (V_{λ_n}) since its difference with any member of $W\lambda_{n-1}$ ($W\lambda_n$) is not an integral combination of roots. It follows that $\dim V_{\lambda_{n-1}} = \text{size of } W\lambda_n = 2^{n-1}$, and:

$$V_{\lambda_{n-1}} = \bigoplus V_{\lambda_{n-1}}(\pm\tfrac{1}{2},\cdots,\pm\tfrac{1}{2})$$

$$V_{\lambda_n} = \bigoplus V_{\lambda_n}(\pm\tfrac{1}{2},\cdots,\pm\tfrac{1}{2})$$

with even number of negative signs, and each wgt space is 1-dim'l, since they are on the Weyl group orbit of the h.w.

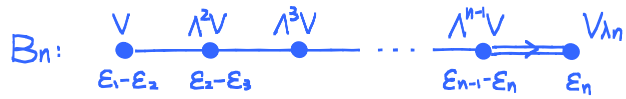
E.g. $\mathfrak{sl}(4) \cong \mathfrak{so}(6)$, since their Dynkin diagrams are the same:



Over $\mathbb{R}$. $Spin(6) \cong SU(3)$, these 2 fundamental rep's are isomorphic to the defining rep $\mathbb{C}^3$ of $SU(3)$ and $\Lambda^2\mathbb{C}^2 \cong (\mathbb{C}^3)^*$.

Note in particular that if n is even, $-\lambda_{n-1} = (-\frac{1}{2},-\frac{1}{2},\cdots,-\frac{1}{2}) \in W\lambda_{n-1}$ and $-\lambda_n = (-\frac{1}{2},-\frac{1}{2},\cdots,\frac{1}{2}) \in W\lambda_n$, and thus $V_{\lambda_{n-1}}$ and V_{λ_n} are self-dual. On the other hand, if n is odd, $V_{\lambda_n} \cong V^*_{\lambda_{n-1}}$ and $V^*_{\lambda_{n-1}} \cong V_n$ since now $-\lambda_{n-1} \in W\lambda_n$, $-\lambda_n \in W\lambda_{n-1}$.

$\mathfrak{so}(2n+1)$: Again let V be the defining representation, $\dim V = 2n+1$.
The difference here from $\mathfrak{so}(2n)$ case is that now 0 is a wgt besides $\{(0,\cdots,\pm1,\cdots 0)\}$.
(In particular, V is not miniscule now since $0 \notin W\cdot\lambda_1$.) Indeed, recall our construction about $\mathfrak{so}(2n+1) = \left\{ \left(\begin{array}{c|c|c} 0 & * & * \\ \hline * & * & * \\ * & * & * \end{array}\right) \right\}$ with CSA $\left\{ \begin{pmatrix} 0 & & \\ & * & \\ & & \ddots & \\ & & & * \end{pmatrix} \right\}$, and the CSA kills $(1,0,\cdots 0)$.

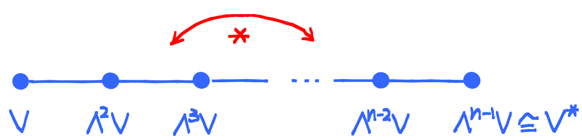


The fundamental rep's $V_{\lambda_1}, \cdots, V_{\lambda_{n-1}}$ are constructed similar as above, while there is only 1 spin representation V_{λ_n}. $V_{\lambda_n} \notin \otimes V^{\otimes m}$ for any m (since $-I$ of $Spin(n)$ would act as Id on $V^{\otimes m}$). Furthermore, V_{λ_n} is miniscule with h.w. $\lambda_n = (\frac{1}{2}, \cdots, \frac{1}{2})$ and wgts $W\lambda_n = \{(\pm\frac{1}{2}, \pm\frac{1}{2}, \cdots, \pm\frac{1}{2})\}$ (recall that $W(B_n) \cong (\mathbb{Z}/2)^n \ltimes S_n$, all sign changes are there!)
$\Rightarrow \dim V_{\lambda_n} = 2^n$.

Dual fundamental representations.

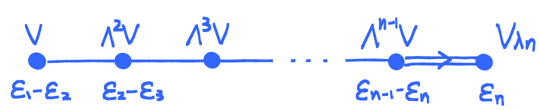
Note that $V \mapsto V^*$ always gives an automorphism of the Dynkin diagram.

$\mathfrak{sl}(n)$:



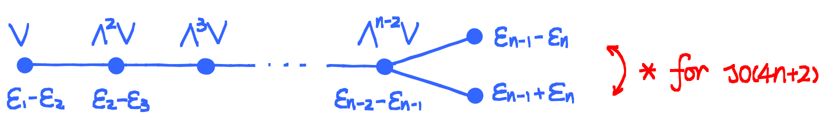
$$\Lambda^k V \cong (\Lambda^{n-k} V)^*$$

$\mathfrak{so}(2n+1)$:



The only automorphism of the Dykin diagram is trivial. thus all fundamental rep's are self-dual.

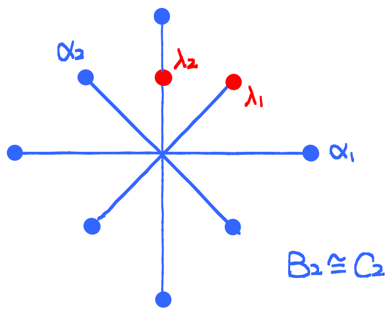
$\mathfrak{so}(2n)$:

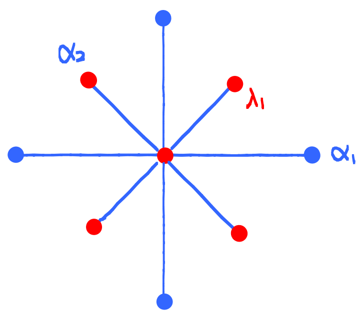


For $\mathfrak{so}(4n)$, all fundamental reps are self-dual, since in this case, $V_{\lambda_{2n-1}}^* \cong V_{\lambda_{2n-1}}$ $V_{\lambda_{2n}}^* \cong V_{\lambda_{2n}}$. On the other hand, for $\mathfrak{so}(4n+2)$, $V_{\lambda_{2n}}^* \cong V_{\lambda_{2n+1}}$, $V_{\lambda_{2n+1}}^* \cong V_{\lambda_{2n}}$, this corresponds to the non-trivial flip of the Dynkin diagram.

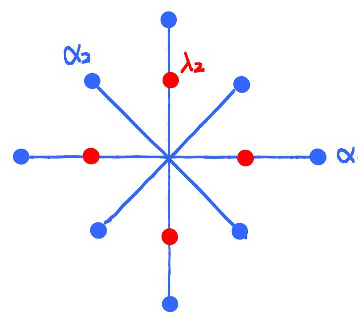
To summarize, any rep V of $\mathfrak{so}(m)$ is self-dual if $m \neq 2 \pmod 4$. ($m=2$, $SO(2) \cong U(1)$ and irrep's are parametrized by the winding number $z \longmapsto z^n$, $V(n)^* \cong V(-n)$).

E.g. $\mathfrak{so}(5) \cong \mathfrak{sp}(2)$





$\dim V_{\lambda_1} = 5$, which is the 5-dim'l defining rep of $\mathfrak{so}(5)$ (note that there is only one way to reach 0 by $y_{\alpha_1}, y_{\alpha_2}$'s, and thus $\dim V_{\lambda_1}(0) = 1$).



$\dim V_{\lambda_2} = 4$, which is the defining rep of $\mathfrak{sp}(2)$ on $\mathbb{H}^2 \cong \mathbb{C}^4$

Note that both rep's are self-dual.

- Summary of Lie algebras and their representations.
  - L: any Lie algebra over $\mathbb{C}$. Then:

$$0 \longrightarrow \mathrm{Rad}L \longrightarrow L \longrightarrow L^{s.s.} \longrightarrow 0$$

and $L^{s.s.} \cong \oplus$ simple LA's, where $\mathrm{Rad}L$ = maximal solvable ideal.

  - Simple LA's $/\mathbb{C} \xleftrightarrow{1:1}$ Dynkin diagrams : $A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4, G_2$.
  - There is no way to classify solvable LA's except in low dimensions, and there is no classification of representations of a given solvable Lie algebra. (Yet any finite dimensional irrep is 1 dim'l.). Solvable subalgebras of $\mathfrak{gl}(n)$ are conjugate to some subalgebra contained in the upper triangular matrices.
  - Classification of rep's of a s.s. LA : $L \cong L_1 \oplus \cdots \oplus L_n$, L_i simple. Then any finite dim'l rep U of L is completely reducible : $U \cong U_1 \oplus \cdots \oplus U_n$, and each U_i is of the form $V_1 \otimes \cdots \otimes V_n$ where each V_i is an irrep of L_i (only $/\mathbb{C}$!)
  - Finally, if L is simple, then upon choosing a CSA, we obtain a root system (all CSA's of L are conjugate, thus give isomorphic root systems), and by choosing a base, we obtain a positive root lattice and positive wgt lattice Λ^+

$$\{\text{Irrep's of } L\} \xleftrightarrow{1:1} \{\text{positive integral (dominant) wgts}\} = \Lambda^+$$

The basic building blocks are the fundamental wgts $\lambda_1, \cdots, \lambda_n$, in the sense that if $\lambda \in \Lambda^+$, $\lambda = a_1\lambda_1 + \cdots + a_n\lambda_n$, then $V_\lambda \subseteq S^{a_1}V_{\lambda_1} \otimes \cdots \otimes S^{a_n}V_{\lambda_n}$.