Hypersurface Singularities and Matrix Factorization

- A report on Orlov's paper

Triangulated category of singularities

Notation: $A = \oplus_{i\geq 0} A_i$: noetherian graded algebra / $\Bbbk$.

$\mathrm{mod} A$: the category of finitely generated right A-modules.

$\mathrm{gr} A$: the category of finitely generated, graded right A-modules.

$\mathrm{Mod} A$: the category of all right A-modules.

$\mathrm{Gr} A$: the category of all graded right A-modules.

$(p) : \mathrm{Gr}A \longrightarrow \mathrm{Gr}A$, $\forall p \in \mathbb{Z}$. $M(p)_i = M_{p+i}$.

Def. The triangulated category of singularities $D_{Sg}(A)$ / $D^{gr}_{Sg}(A)$ is defined as the quotient

$$D^b(\mathrm{mod}A)/D^b(\mathrm{proj}\text{-}A)$$
$$D^b(\mathrm{gr}A)/D^b(\mathrm{gr.proj}\text{-}A)$$

Rmk: $D^{gr}_{Sg}(A)/D_{Sg}(A)$ is trivial if A has finite homological dimension.

To compute the morphisms in $D^{gr}_{Sg}(A)$ ($D_{Sg}(A)$), we have the following:

Lemma. (i). $\forall\, T \in D^{gr}_{Sg}(A)$ ($D_{Sg}(A)$), $\exists\, M \in \mathrm{gr}A$ ($\mathrm{mod}A$) depending on T s.t. $T \cong M[k]$.

(ii). T as above, if A has finite injective dimension, then $\forall k \gg 0$, $\exists\, M$ as above with $\mathrm{Ext}^i_A(M,A) = 0$, $\forall i > 0$.

(iii). Let $M \in \mathrm{gr}\text{-}A$ s.t. $\mathrm{Ext}^i_A(M,A) = 0$, $\forall i > 0$. Then, $\forall N \in \mathrm{gr}A$, we have

$$\mathrm{Hom}_{D_{Sg}(A)}(M,N) \cong \mathrm{Hom}_{\mathrm{gr}A}(M,N)/R$$

where R is the ideal of morphisms that factor through a projective.

Sketch of pf: Take a bounded above projective resolution:

$$P^\cdot \longrightarrow T^\cdot$$

The stupid truncation gives us

$$0 \longrightarrow \sigma^{\geq -k+1} P^\cdot \longrightarrow P^\cdot \longrightarrow \sigma^{\leq -k} P^\cdot \longrightarrow 0$$

which leads to a Δ in $D^b(grA)$:

$$\sigma^{\geq -k+1} P^\cdot \longrightarrow P^\cdot \longrightarrow \sigma^{\leq -k} P^\cdot \xrightarrow{[1]} \sigma^{\geq -k+1} P^\cdot [1]$$

For $k >> 0$, we get $\sigma^{\leq -k} P^\cdot \cong H^{-k+1}(\sigma^{\geq -k+1} P^\cdot) \triangleq M$, since $T^\cdot$ is bounded.
Then in $D^{gr}_{sg}(A)$, we have

$$T^\cdot \cong P^\cdot \cong M[k].$$

(ii) follows from $T^\cdot$ being bounded. $\square$

Quotient category of graded modules

$A = \oplus_{i \geq 0} A_i$: noetherian graded algebra / $\Bbbk$. $A_0 = \Bbbk$ (connected).
$torA \subseteq grA$: finite dimensional modules.
$TorA \subseteq GrA$: torsion modules in the sense that $\forall\, m \in M$, $\exists\, N \geq 0$ s.t.

$$m \cdot (\oplus_{k \geq N} A_k) = 0.$$

It's easy to check that $torA$ (resp. $TorA$) are thick subcategories and we can take the quotients.

Def. $qgr(A) \triangleq grA / torA$. The objects are taken to be the same as grA (denoted $\pi(M)$, $\forall\, M \in grA$), and morphisms

$$Hom_{qgrA}(\pi(M), \pi(N)) = \varinjlim_{M'} (M', M)$$

where M/M' is finite dim'l.

Similarly define $QGr(A)$, and $qgr(A) \subseteq QGr(A)$. We can identify these categories with $Coh(X)$ / $QCoh(X)$, where

$$X = \mathbb{P}roj(A) = [(Spec A \setminus \{0\}) / \mathbb{G}_m]$$

More explicitly, we state this as:

Prop. A as above, commutative. The category of (quasi) coherent sheaves on $\mathbb{P}\mathrm{roj}(A)$ is equivalent to the category $\mathrm{qgr}A$ ($\mathrm{QGr}A$). $\square$

As a cor. if A is generated by deg 1 elements, $\mathbb{P}\mathrm{roj}A \cong \mathrm{Proj}A$ (its coarse moduli space) and we obtain the classical Serre's thm.

Now if we denote the canonical projection

$$\pi : \mathrm{gr}A \longrightarrow \mathrm{qgr}A$$
$$\Pi : \mathrm{Gr}A \longrightarrow \mathrm{QGr}A$$

we have their right adjoints:

$$\left.\begin{array}{l} \omega : \mathrm{qgr}A \longrightarrow \mathrm{gr}A \\ \Omega : \mathrm{QGr}A \longrightarrow \mathrm{Gr}A \end{array}\right\} \quad \Omega\Pi N \triangleq \bigoplus_{n=-\infty}^{\infty} \mathrm{Hom}_{\mathrm{QGr}}(\Pi A, \Pi N(n))$$

Thus Ω is left exact, and since $\mathrm{QGr}A$ has enough injectives, we have its right derived functor

$$R\Omega : D^{+}(\mathrm{QGr}A) \longrightarrow D^{+}(\mathrm{Gr}A)$$

It's proved by Artin and Zhang that under mild assumptions

- "χ" : $\mathrm{Ext}_A^i(\mathbb{k}, M)$ has its grading right bounded for all i, $\forall M \in \mathrm{gr}A$

$R\Omega$ restricts to

$$R\omega : D^b(\mathrm{qgr}A) \longrightarrow D^b(\mathrm{gr}A)$$

Note that $R\omega$ is fully faithful since $\pi R\omega = \mathrm{Id}$.

Gorenstein algebras

Let A be a graded algebra with finite injective dimension both as a left and right module. We have two functors:

$$\mathcal{D} = R\mathrm{Hom}_A(-, A) : D^b(\mathrm{gr}A)^\circ \longrightarrow D^b(\mathrm{gr}A^\circ),$$
$$\mathcal{D}^\circ = R\mathrm{Hom}_{A^\circ}(-, A) : D^b(\mathrm{gr}A^\circ)^\circ \longrightarrow D^b(\mathrm{gr}A)$$

Def. We say a connected, graded Noetherian algebra A is Gorenstein if it has:

(i). finite injective dimension n;

(ii). $\mathcal{D}(\Bbbk) = R\mathrm{Hom}_A(\Bbbk, A) \cong \Bbbk(a)[-n]$ for some integer a.

(a is called the Gorenstein parameter of A).

(ii) $\Rightarrow$ A satisfies condition "χ" so that we have a fully faithful
$$R\omega : D^b(\mathrm{qgr}A) \longrightarrow D^b(\mathrm{gr}A)$$

Thm. Let A be Gorenstein with Gorenstein parameter a. Then $D^{gr}_{Sg}(A)$ and $D^b(\mathrm{qgr}A)$ are related as follows:

(i). If $a > 0$, there is a semiorthogonal decomposition:
$$D^b(\mathrm{qgr}A) = \langle \pi A(-a+1), \cdots, \pi A, D^{gr}_{Sg}(A) \rangle$$
(ii). If $a < 0$, there is a semiorthogonal decomposition:
$$D^{gr}_{Sg}(A) = \langle q\Bbbk, \cdots, q\Bbbk(a+1), D^b(\mathrm{qgr}A) \rangle$$
where $q : D^b(\mathrm{gr}A) \longrightarrow D^{gr}_{Sg}(A)$ is the natural projection.

(iii). If $a = 0$, there is an equivalence:
$$D^{gr}_{Sg}(A) \xrightarrow{\sim} D^b(\mathrm{qgr}A).$$
Pf omitted. The "discrepancy" comes about when applying duality:
$$\mathcal{D} : D^b(\mathrm{gr}A_{\geq i}) \longrightarrow D^b(\mathrm{gr}A^\circ_{\leq -i-a-1}) \qquad \square$$

E.g. Let A be Gorenstein with Gorenstein parameter a. Suppose A has finite homological dimension, then $D^{gr}_{Sg}(A) = 0$. Thm. $\Rightarrow$ $a \geq 0$ and $D^b(\mathrm{qgr}A)$ has a full strong exceptional collection

$$\sigma = \langle \pi A(0), \cdots, \pi A(a-1) \rangle$$

Thus $D^b(qgrA)$ is equivalent to the derived category

$$D^b(\operatorname{mod} Q(A))$$

where

$$Q(A) = \operatorname{End}_{qgrA}(\oplus_{i=0}^{a-1} \pi A(i)) = \operatorname{End}_{grA}(\oplus_{i=0}^{a-1} A(i)).$$

As an even more special case, take $A = \Bbbk[x_0, \cdots, x_n]$, we recover (Beilinson): $\mathbb{P}^n$ admits a full exceptional collection $\langle \mathcal{O}, \cdots, \mathcal{O}(n) \rangle$.

More generally, $\mathbb{P}^n(a_0, \cdots, a_n)$ has a full exceptional collection $\langle \mathcal{O}, \cdots, \mathcal{O}(\Sigma a_i - 1) \rangle$

E.g. As another e.g. let A be finite dim'l Gorenstein (a.k.a. Frobenius algebra). In this case $qgrA$ is trivial so that by the thm, $a \leq 0$ and $D^{gr}_{sg}(A)$ admits a full exceptional sequence (NOT strong though!): $\langle q\Bbbk(0), \cdots, q\Bbbk(a+1) \rangle$

However, one can check that the modules $A(i+a+1)/A(i+a+1)_{\geq a}$ form a strong exceptional sequence.

Gorenstein schemes

Let X be a connected projective Gorenstein scheme of dim n and $\mathcal{L}$ a very ample line bundle on X. $A = \oplus_{i=0}^{\infty} H^0(X, \mathcal{L}^i)$. Assume furthermore that $H^j(X, \mathcal{L}^k) = 0 \quad \forall k \in \mathbb{Z}$, $j \neq 0, n$. (This holds, for instance, when X is a complete intersection in $\mathbb{P}^N$).

Lemma. Let X be as above with $\omega_X \cong \mathcal{L}^{-r}$ for some $r \in \mathbb{Z}$. Then A is a Gorenstein algebra with Gorenstein parameter $a = r$.

Pf: It suffices to show that Spec$A$ is a Gorenstein scheme. This is a local question, and our assumptions reduce us to check this only at 0

By assumption,

$$R\Omega(\pi(A)) = R\mathrm{Hom}(\mathcal{O}, \oplus_{j\in\mathbb{Z}} \mathcal{O}(j))$$

$$\Rightarrow \begin{cases} R^0\Omega(\pi(A)) \cong \oplus_{j\in\mathbb{Z}} H^0(\mathcal{O}(j)) = A \\ R^n\Omega(\pi(A)) \cong \oplus_{j\in\mathbb{Z}} H^n(\mathcal{O}(j)) \cong \oplus_{j\in\mathbb{Z}} H^0(\mathcal{O}(-j-r))^* \cong A^*(r) \\ R^i\Omega(\pi(A)) = 0 \quad \text{if} \quad i \neq 0, n \end{cases}$$

where $A^* = \mathrm{Hom}_{\mathbb{k}}(A, \mathbb{k})$ is the restricted dual of A.

Now adjunction gives: $\forall s \in \mathbb{Z}$

$$R\mathrm{Hom}_{Gr}(\mathbb{k}(s), R\Omega(\pi(A))) \cong R\mathrm{Hom}_{QGr}(\pi(\mathbb{k}(s)), \pi(A)) = 0$$

The above computation shows that the spectral sequence

$$E_2^{p,q} = R^p\mathrm{Hom}_{GrA}(\mathbb{k}, R^q\Omega(\pi(A)))$$

degenerates at $E_2, \cdots, E_n$ and

$$\begin{aligned} d_n: R^{n+1}\mathrm{Hom}_{GrA}(\mathbb{k}, A) \xrightarrow{\cong} & R^0\mathrm{Hom}_{GrA}(\mathbb{k}, A^*(r)) \\ &= \mathrm{Hom}_{GrA}(\mathbb{k}, \mathrm{Hom}_{\mathbb{k}}(A, \mathbb{k}))(r) \\ &= A(r) \end{aligned}$$

Similarly, $R^j\mathrm{Hom}_{GrA}(\mathbb{k}, A) = 0$ if $j \neq n+1$. $\Rightarrow$

$$R\mathrm{Hom}_{GrA}(\mathbb{k}, A) \cong \mathbb{k}(r)[-n-1]$$

Lastly, A has finite Krull dim $= n+1$ finishes the argument. □

Combining the lemma with the algebraic thm, we obtain:

Thm. X as in the lemma. Then $\mathcal{D}^b(\mathrm{Coh}(X))$ and $\mathcal{D}^{gr}_{sg}(A)$ are related as follows:

(i). If $r>0$, i.e. X is Fano, then there is a semiorthogonal decomposition:

$$D^b(\mathrm{Coh}(X)) \cong \langle \mathcal{L}^{-r+1}, \cdots, \mathcal{O}_X, D^{gr}_{Sg}(A) \rangle$$

(ii) If $r<0$, i.e. X is of general type, then there is a semi-orthogonal decomposition:

$$D^{gr}_{Sg}(A) \cong \langle q\Bbbk(r+1), \cdots, q\Bbbk, D^b(\mathrm{Coh}(X)) \rangle$$

(iii) If $r=0$, i.e. X is Calabi-Yau, then $\exists$ equivalence:

$$D^{gr}_{Sg}(A) \xrightarrow{\sim} D^b(\mathrm{Coh}(X)) \qquad \square$$

In the following, we will try to understand $D^{gr}_{Sg}(A)$ more concretely.

Matrix factorization

Let $B = \oplus_{i\geq 0} B_i$ be a finitely generated connected graded algebra$/\Bbbk$. $W \in B$ a central element of deg n which is not a zero divisor. $A \triangleq B/W\cdot B$.

Def. (i). The exact category $MF(W)$ consists of:

objects: ordered pairs $\bar{P} = (P_0 \xrightarrow{p_0} P_1 \xrightarrow{p_1} P_0)$ s.t. P_0, P_1 are free B-modules

$\deg p_0 = n$, $\deg p_1 = 0$ and $p_0 p_1 = W \cdot \mathrm{Id}_{P_1}$ $p_1(n) p_0 = W \cdot \mathrm{Id}_{P_0}$.

morphisms:

$$\mathrm{Hom}(\bar{P}, \bar{Q}) = \left\{ f = (f_0, f_1) \;\middle|\; \deg f_i = 0, \text{ and } \begin{array}{ccccc} P_0 & \xrightarrow{p_0} & P_1 & \xrightarrow{p_1} & P_0 \\ \downarrow f_0 & \circlearrowright & \downarrow f_1 & \circlearrowright & \downarrow f_0 \\ Q_0 & \xrightarrow{q_0} & Q_1 & \xrightarrow{q_1} & Q_0 \end{array} \right\}$$

Rmk: We also identify objects of $MF(W)$ with 2-periodic sequences:

$$K^\bullet = (\cdots \longrightarrow K^i \xrightarrow{k^i} K^{i+1} \xrightarrow{k^{i+1}} K^{i+2} \xrightarrow{k^{i+2}} \cdots)$$

s.t. $k^{i+1} k^i = W$, and $K^\bullet[2] \cong K^\bullet(n)$.

(ii). $f: \bar{P} \longrightarrow \bar{Q} \in MF(W)$ is called null-homotopic if $\exists s: P_0 \longrightarrow Q_1$, $t: P_1 \longrightarrow Q_0$ s.t.

$$\begin{tikzcd} P_0 \arrow[r, "p_0"] \arrow[d, "f_0"] & P_1 \arrow[r, "p_1"] \arrow[d, "f_1"] \arrow[dl, "t"'] & P_0 \arrow[r, "p_0"] \arrow[d, "f_0"] \arrow[dl, "s"'] & P_1 \arrow[d, "f_1"] \arrow[dl, "t"'] \\ Q_0 \arrow[r, "q_0"] & Q_1 \arrow[r, "q_1"] & Q_0 \arrow[r, "q_0"] & Q_1 \end{tikzcd} \quad \& \quad \begin{cases} f_1 = q_0(n)t + sp_1 \\ f_0 = t(n)p_0 + q_1 s \end{cases}$$

Rmk: In terms of 2-periodic sequences, this is the usual notion of homotopy equivalence.

(iii). The homotopy category $HMF(w)$ is the quotient $MF(w)$ by null-homotopies.

It's easy to check that $HMF(w)$ is triangulated with:

Translation: $K[1]^\bullet = (K[1]^i = K^{i+1},\ d^i_{K[1]} = -d^{i+1})$.

Cone: $f: K^\bullet \longrightarrow L^\bullet$, the cone as if they are complexes.

Triangle: standard triangles:

$$K^\bullet \xrightarrow{f} L^\bullet \longrightarrow C_f^\bullet \longrightarrow K^\bullet[1]$$

A Δ in $HMF(w)$ is one that's isomorphic to the image of a standard one as above.

Matrix factorization v.s. hypersurface singularities

Main Thm. If B has finite homological dimension, then there is an equivalence of categories:

$$F: HMF(w) \longrightarrow D^{gr}_{Sg}(A).$$

The proof will be completed in a sequence of lemmas. Before doing that, we conbine it with our previous thm to obtain:

Thm. Let $X = \mathbb{A}^N$ and w a homogeneous polynomial of deg d. Let Y be the hypersurface of degree d given by $w=0$. Then $D^b(Coh(Y))$ and $HMF(w)$ are related as follows:

(i). If $d<N$, i.e. Y is Fano, then there is a semiorthogonal decomposition

$$D^b(\mathrm{Coh}\,Y) = \langle \mathcal{O}_Y(d-N+1), \cdots, \mathcal{O}_Y, \mathrm{HMF}(\omega)\rangle$$

(ii). If $d>N$, i.e. Y is of general type, then there is a semiorthogonal decomposition:

$$\mathrm{HMF}(\omega) = \langle F^{-1}q(k(r+1)), \cdots, F^{-1}q(k), D^b(\mathrm{Coh}(Y))\rangle$$

where $q: D^b(\mathrm{gr}A) \longrightarrow D^{gr}_{Sg}(A)$ is the natural projection, and F is the equivalence of the previous thm.

(iii). If $d=N$, i.e. Y is CY, then we have an equivalence:

$$\mathrm{HMF}(\omega) \xrightarrow{\sim} D^b(\mathrm{Coh}(Y)) \qquad \square$$

Rmk: Similar results hold for hypersurfaces in weighted projective spaces.

Before starting the proof of the thm, we prove an algebraic lemma about reduction of $\mathrm{MF}(\omega)$ to $\mathrm{Comp}(A)$:

Lemma. $\forall\, K^\bullet \in \mathrm{MF}(\omega)$, $K^\bullet \otimes_B A$ is an acyclic complex in $\mathrm{Comp}(A)$.

Pf: Consider the diagram:

$$\begin{tikzcd}
0 \arrow[r] & K^{i-2} \arrow[r, "k^{i-2}"] \arrow[d, "\omega"] & K^{i-1} \arrow[r] \arrow[d, "\omega"] & \mathrm{Cok}(k^{i-2}) \arrow[r] \arrow[d, "\omega"] & 0 \\
0 \arrow[r] & K^{i} \arrow[r, "k^{i}"] & K^{i+1} \arrow[r] & \mathrm{Cok}(k^{i}) \arrow[r] & 0
\end{tikzcd}$$

Notice that $\omega|_{\mathrm{Cok}(k^i)} = 0$ since $\forall\, x \in \mathrm{Cok}(k^i)$ $x\omega = k^i k^{i-1}(x) \in \mathrm{Im}\,k^i$. Snake lemma $\Rightarrow$

$$0 \longrightarrow \mathrm{Cok}(k^{i-2}) \longrightarrow K^i \otimes_B A \xrightarrow{k^i} K^{i+1} \otimes_B A \longrightarrow \mathrm{Cok}(k^i) \longrightarrow 0$$

This implies that $K^\bullet \otimes_B A$ is acyclic. $\square$

Now we prove the main thm. We first define F:

$K^\bullet \in \mathrm{MF}(\omega) \Rightarrow \exists$ s.e.s.

$$0 \longrightarrow K^{-1} \xrightarrow{k^{-1}} K^0 \longrightarrow \mathrm{Cok}\,k^{-1} \longrightarrow 0,$$

and $\forall\, x \in K^0$, $x\cdot\omega = k^{-1}k^0(x) \equiv 0 \mod \mathrm{Im}\,k^{-1} \Rightarrow \mathrm{Cok}\,k^{-1}$ is an A-module.

Lemma (Def. of F). Cok extends to an exact functor:

$$\begin{tikzcd} MF(\omega) \arrow[r, "Cok"] \arrow[d] & grA \arrow[d] \\ HMF(\omega) \arrow[r, "F"] & D^{gr}_{Sg}(A) \end{tikzcd}$$

Pf: Let $\tilde{F}$ be the composition $MF(\omega) \xrightarrow{Cok} grA \longrightarrow D^{gr}_{Sg}(A)$. To show $\tilde{F}$ descends to F on $HMF(\omega)$, we need to check that if $f: K^\bullet \longrightarrow L^\bullet$ is null-homotopic, then it goes to 0 in $D^{gr}_{Sg}(A)$.

Consider the decomposition of f:

$$\begin{tikzcd} K^{-1} \arrow[r, "k^{-1}"] \arrow[d, "(s^{-1}, f^{-1})"'] & K^0 \arrow[r] \arrow[d, "(s^0, f^0)"'] & Cok\, k^{-1} \arrow[r] \arrow[d] & 0 \\ L^{-2} \oplus L^{-1} \arrow[r, "u^{-1}"] \arrow[d, "pr"'] & L^{-1} \oplus L^0 \arrow[r] \arrow[d, "pr"'] & L^0 \otimes_B A \arrow[r] \arrow[d] & 0 \\ L^{-1} \arrow[r, "\ell^{-1}"] & L^0 \arrow[r] & Cok\, \ell^{-1} \arrow[r] & 0 \end{tikzcd}$$

where $u^{-1} = \begin{pmatrix} -\ell^{-2} & id \\ 0 & \ell^{-1} \end{pmatrix}$

This yields a factorization of $Cok\, k^{-1} \longrightarrow L^0 \otimes_B A \longrightarrow Cok\, \ell^{-1}$ through a free module $\Rightarrow F(f) = 0$ in $D^{gr}_{Sg}(A)$.

Our previous lemma implies that we have a s.e.s.

$$0 \longrightarrow Cok\,(k^{-1}) \longrightarrow K^1 \otimes_B A \longrightarrow Cok\,(k^0) \longrightarrow 0$$

Since $K^1 \otimes_B A$ is free, $Cok(k^0) \cong Cok(k^{-1})[1]$ in $D^{gr}_{Sg}(A)$. But by our def., $Cok(k^0) = F(K^\bullet[1])$. Hence F commutes with $[1]$.

It follows by def. that F takes a Δ to Δ. This finishes the proof of the lemma. $\square$

Now we need to check that :

(i). F is fully-faithful.
(ii). F is essentially surjective on objects.

We show (ii) now and prove (i) in the next subsection:
Since B has finite homological dimension $\Rightarrow$ A has finite injective dim. By the first lemma, any $T^\bullet \in D^{gr}_{Sg}(A)$ can then be replaced by $M \in grA$ s.t. $\mathrm{Ext}^i_A(M, A) = 0$, $\forall i > 0$. Thus $D(M) = R\mathrm{Hom}_A(M, A)$ is a left A-module, so that we may choose a left projective A-module resolution $Q^\bullet \longrightarrow D(M)$. Dualizing again we get a right projective resolution:

$$0 \longrightarrow M \longrightarrow P^\bullet \triangleq D(Q^\bullet)$$

Now $0 \longrightarrow B \xrightarrow{w} B \longrightarrow A \longrightarrow 0$ implies that, any projective A-mod when regarded as a B-mod, has $\mathrm{Ext}^i_B(P, N) = 0$, $\forall i > 1$, $\forall N \in B$-mod (i.e. $\mathrm{pd}_B P \leq 1$). Therefore $R\mathrm{Hom}(M, N) = R\mathrm{Hom}(P^\bullet, N)$, and the s.s.

$$\mathrm{Ext}^p_B(P^{i-p}, N) \Rightarrow \mathrm{Ext}^i_B(M, N)$$

implies if $\mathrm{Ext}^i_B(M, N) = 0$ if $i > 1$. Thus by dim shifting, if we choose a free B-mod $K^0 \twoheadrightarrow M$, the kernel K^{-1} is a projective B-mod. Since B is connected graded, K^{-1} is free. Thus we obtain:

$$0 \longrightarrow K^{-1} \xrightarrow{k^{-1}} K^0 \longrightarrow M \longrightarrow 0$$

Now since multiplication by w is 0 on M, we have, as B-mod,

$$\begin{tikzcd}
0 \arrow[r] & K^{-1} \arrow[r, "k^{-1}"] \arrow[d, "w"] & K^0 \arrow[r] \arrow[d, "w"] \arrow[dl, "k^0"'] & M \arrow[r] \arrow[d, "0"] & 0 \\
0 \arrow[r] & K^{-1} \arrow[r, "k^{-1}"] & K^0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Hence we obtain a homotopy $k^0 : K^0 \longrightarrow K^1(n)$ s.t. $k^0 k^{-1} = w$, $k^{-1}(n) k^0 = w$. In this way we get the desired double periodic complex of free B-mod. $\square$

Fully-faithfulness

Before showing this, we make a simple observation:
Lemma. The A-modules $\mathrm{Cok}(k^{-1})$ for any $K^\bullet \in MF(w)$ satisfies:

$$\mathrm{Ext}_A^i(\mathrm{Cok}(k^{-1}), A) = 0, \quad \forall i > 0.$$

Pf: By a previous lemma,

$$\cdots \longrightarrow K^{-2}\otimes_B A \xrightarrow{k^{-2}} K^{-1}\otimes_B A \xrightarrow{k^{-1}} K^0\otimes_B A \longrightarrow \mathrm{Cok}(k^{-1}) \longrightarrow 0$$

is exact and each $K^{-i}\otimes_B A$ is a free A-mod. Now $\mathrm{Ext}^i(\mathrm{Cok}(k^{-1}), A)$ is computed as the homology of the dual complex of the above. The result follows. $\square$

This observation allows us to replace morphisms from $\mathrm{Cok}(k^{-1})$ in $D_{Sg}^{gr}(A)$ by A-module maps, via a previous lemma.

Lemma. $\mathrm{Cok} : MF(w) \longrightarrow D_{Sg}^{gr}(A)$ is full.

Pf: Any $\mathrm{Cok}(k^{-1}) \longrightarrow \mathrm{Cok}(\ell^{-1})$ can be lifted to a B-mod map

$$\begin{tikzcd} 0 \arrow[r] & K^{-1} \arrow[r, "k^{-1}"] \arrow[d, "f^{-1}"] & K^0 \arrow[r] \arrow[d, "f^0"] & \mathrm{Cok}(k^{-1}) \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & L^{-1} \arrow[r, "\ell^{-1}"] & L^0 \arrow[r] & \mathrm{Cok}(\ell^{-1}) \arrow[r] & 0 \end{tikzcd}$$

since K^0 is free. We extend this to $f^{2i-1} = f^{-1}(in)$, $f^{2i} = f^0(in)$, $\forall i \in \mathbb{Z}$. Then $f^\bullet : K^\bullet \longrightarrow L^\bullet$ is a morphism, since ℓ^1 is injective and:

$$\ell^1(f^1k^0 - \ell^0 f^0) = f^2k^1k^0 - \ell^1\ell^0 f^0 = f^2 w - w f^0 = 0 \qquad \square$$

Consequently, F is full as well. It now suffices to check that:

Lemma. F is faithful, i.e. $F(K^\bullet) = 0 \Rightarrow K^\bullet = 0$ in $HMF(w)$.

Pf: $F(K^\bullet) = 0 \Rightarrow \mathrm{Cok}(k^{-1})$ is a perfect complex of A-modules. In fact $\mathrm{Cok}(k^{-1})$ is a projective A-module. Indeed, $\exists m$ s.t. $\mathrm{Ext}_A^i(\mathrm{Cok}(k^{-1}), N) = 0$, $\forall i \geq m$, $\forall N \in A$-mod. The exact complex:

$$0 \longrightarrow \mathrm{Cok}(k^{-2m-1}) \longrightarrow K^{-2m}\otimes_B A \longrightarrow \cdots \longrightarrow K^{-1}\otimes_B A \xrightarrow{k^{-1}} K^0\otimes_B A \longrightarrow \mathrm{Cok}(k^{-1}) \longrightarrow 0$$

gives us, by dim shifting, that

$$\mathrm{Ext}^i_A(\mathrm{Cok}(k^{-2m-1}), N) = 0, \ \forall i>0, \ \forall N \in A\text{-mod}$$

$\Rightarrow$ $\mathrm{Cok}(k^{-2m-1})$ is projective $\Rightarrow$ so is $\mathrm{Cok}(k^{-1}) \cong \mathrm{Cok}(k^{-2m-1})(-mn)$.

Now we have a lift $\mathrm{Cok}(k^{-1}) \xrightarrow{f} K^0 \otimes_B A$ splitting the projection. We can use it to get a factorization of s.e.s. of B-modules (s^{-1}, u):

$$\begin{tikzcd}
0 \arrow[r] & K^{-1} \arrow[r, "k^{-1}"] \arrow[d, "s^{-1}"'] & K^0 \arrow[r] \arrow[d, "u"] \arrow[ddl, red, "s^0"'] & \mathrm{Cok}(k^{-1}) \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & K^{-2} \arrow[r, "w"] \arrow[d, "k^{-2}"'] & K^0 \arrow[r] \arrow[d, "\mathrm{id}"] & K^0 \otimes_B A \arrow[r] \arrow[d, "\mathrm{pr}"] & 0 \\
0 \arrow[r] & K^{-1} \arrow[r, "k^{-1}"] & K^0 \arrow[r] & \mathrm{Cok}(k^{-1}) \arrow[r] & 0
\end{tikzcd}$$

Since $\mathrm{pr} \circ f = \mathrm{id}_{\mathrm{Cok}(k^{-1})}$, we have a homotopy $s^0 : K^0 \longrightarrow K^{-1}$ s.t.

$$\begin{cases} s^0 k^{-1} = \mathrm{id}_{K^{-1}} - k^{-2} s^{-1} \\ k^{-1} s^0 = \mathrm{id}_{K^0} - u \end{cases}$$

Moreover, we have by commutativity of the diagram:

$$0 = u k^{-1} - w s^{-1} = u k^{-1} - s^{-1}(n) w = (u - s^{-1}(n) k^0) k^{-1}$$

$$\Rightarrow \quad u = s^{-1}(n) k^0 \quad (\because \nexists\ \mathrm{Cok}(k^{-1}) \longrightarrow K^0 \text{ nonzero})$$

In this way we get the homotopy (s^{-1}, s^0) s.t.

$$\begin{cases} s^0 k^{-1} = \mathrm{id}_{K^{-1}} - k^{-2} s^{-1} \\ k^{-1} s^0 = \mathrm{id}_{K^0} - s^1 k^0 \end{cases} \qquad (s^1 = s^{-1}(n))$$

□

Rmk: Such a homotopy gives us that $K^\bullet$ is isomorphic to the obviously contractible MF : $(\cdots \xrightarrow{1} K^0 \xrightarrow{w} K^0 \xrightarrow{1} K^0 \xrightarrow{w} K^0 \xrightarrow{1} \cdots)$.

Appendix: Semiorthogonal decomposition

Notation: $\mathcal{N} \subseteq \mathcal{D}$: full triangulated subcategory.

$$\mathcal{N}^{\perp} \triangleq \{ M \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(N, M) = 0, \ \forall N \in \mathcal{N} \} \quad : \quad \text{right orthogonal}$$

$$^{\perp}\mathcal{N} \triangleq \{ M \in \mathcal{D} \mid \mathrm{Hom}_{\mathcal{D}}(M, N) = 0, \ \forall N \in \mathcal{N} \} \quad : \quad \text{left orthogonal}$$

Def. $I: \mathcal{N} \hookrightarrow \mathcal{D}$: embedding of full triangulated subcategories. $\mathcal{N}$ is called right (resp. left) admissible if I has a right (resp. left) adjoint $Q: \mathcal{D} \to \mathcal{N}$. $\mathcal{N}$ is called admissible if it's both left and right admissible.

Lemma 1. $I: \mathcal{N} \hookrightarrow \mathcal{D}$ as in the def. If $\mathcal{N}$ is right (resp. left) admissible, then $\mathcal{D}/\mathcal{N} \cong \mathcal{N}^{\perp}$ (resp. $^{\perp}\mathcal{N}$). Conversely, if $Q: \mathcal{D} \to \mathcal{D}/\mathcal{N}$ has a left (resp. right) adjoint, then $\mathcal{D}/\mathcal{N} \cong \mathcal{N}^{\perp}$ (resp. $^{\perp}\mathcal{N}$). $\square$

Def. i). As in lemma 1, if $\mathcal{N}$ is right (resp. left) admissible, we say that $\mathcal{D}$ has a weak semiorthogonal decomposition $\langle \mathcal{N}^{\perp}, \mathcal{N} \rangle$ (resp. $\langle \mathcal{N}, {}^{\perp}\mathcal{N} \rangle$.)

ii). More generally, $\langle \mathcal{N}_1, \cdots, \mathcal{N}_n \rangle$ is called a weak semiorthogonal decomposition if there is a sequence of left admissible subcategories $\mathcal{N}_1 = \mathcal{D}_1 \subseteq \mathcal{D}_2 \subseteq \cdots \subseteq \mathcal{D}_n$ and $\mathcal{N}_p$ is left orthogonal to $\mathcal{D}_{p-1}$ in $\mathcal{D}_p$ ($\mathcal{D}_p = \langle \mathcal{D}_{p-1}, \mathcal{N}_p \rangle$).

iii). In ii), if all $\mathcal{N}_p$ are admissible, then the decomposition $\langle \mathcal{N}_1, \cdots, \mathcal{N}_n \rangle$ is called semiorthogonal.

In the above def, $\mathcal{D} = \langle \mathcal{N}_1, \cdots, \mathcal{N}_p \rangle$ is very simple when

(i). $\mathcal{N}_i$ is generated by one object E_i;

(ii). $\mathrm{Hom}(E_i, E_i[p]) = 0$ if $p \neq 0$.

(iii). $\mathrm{Hom}(E_i, E_j[p]) = 0$, $\forall p$ if $i > j$.

Def. i). An object E is called exceptional if $\mathrm{Hom}(E, E[p]) = 0$ when $p \neq 0$.

ii). An exceptional collection is a sequence of exceptional objects $(E_0, \cdots E_n)$ satisfying $\mathrm{Hom}(E_i, E_j[p]) = 0$, $\forall p$ when $i > j$.

iii). An exceptional collection is called full if it generates $\mathcal{D}$. In this case $\mathcal{D}$ has a semiorthogonal decomposition with $\mathcal{A}_p = \langle E_p \rangle$.

iv). An exceptional collection is called strong if in addition, $\forall i, j, p \neq 0$,

$$\mathrm{Hom}(E_i, E_j[p]) = 0 .$$