Two-dimensional TQFTs, Jones polynomial and its categorification

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n-dimensional TQFT (Topological quantum field theory)
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A monoidal functor from the category of n-dimensional (oriented) cobordisms to some algebraic category

$$\mathrm{Cob}_n \xrightarrow{F} \Bbbk\text{-Vect}, \quad \Bbbk \text{ a field} \quad \left(\begin{array}{l}\text{additive} \\ \text{monoidal} \\ \text{category}\end{array}\right)$$

$n=1$ $\quad F(\overset{+}{\cdot}) = V, \quad F(\overset{-}{\cdot}) = V^*$ a vect. space

$\Bbbk \to V \otimes V^*$

$1 \mapsto \sum v_i \otimes v_i^*$

$n=2$ $\quad F(\bigcirc) = A$ commutative Frobenius algebra

$n=3$ $\quad$ WRT (Witten-Reshetikhin-Turaev) TQFT, its relatives

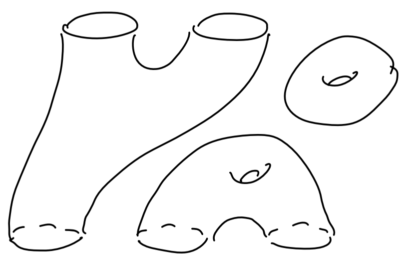
$n=4$ $\quad$ Donaldson-Floer theory, Heegaard-Floer homology
(restricted to link cobordisms: various link homology theories)

$n>4$ $\quad$ stable range, mostly algebraic topology?

$N=2$ Category of oriented 2-cobordisms Cob_2

Objects - oriented 1-manifolds S^1, $(S^1)^{\amalg n} := \underbrace{S^1 \amalg .. \amalg S^1}_{n}$

Morphisms $(S^1)^{\amalg 0} = \emptyset_1$



$(S^1)^{\amalg 2}$

$\uparrow c$

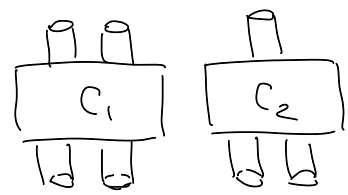
$(S^1)^{\amalg 3}$

composition - concatenation

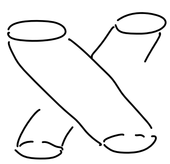
$\otimes$ is disjoint union - put manifolds in parallel

$(S^1)^{\otimes n} = (S^1)^{\amalg n}$

$C_1 \otimes C_2$



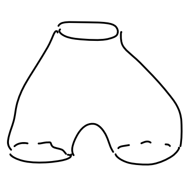
symmetric tensor category



permutation

2D TQFT a tensor functor $F: Cob_2 \longrightarrow \mathbb{k}\text{-Vect}$

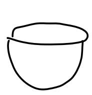
$F(\emptyset_1) = \mathbb{k}$ $F(S^1) = A$ $\mathbb{k}$-vect space



$F(\underbrace{S^1 \amalg .. \amalg S^1}_{k}) = F(S^1)^{\otimes k} = A^{\otimes k}$

$$\begin{array}{c} F(S^1) = A \\ \uparrow m \\ F(S^1 \amalg S^1) = A^{\otimes 2} \end{array}$$

multiplication

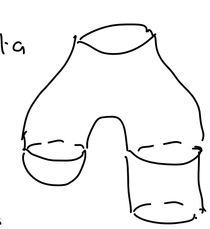


$$\begin{array}{c} F(S^1) = A \\ \uparrow i \\ F(\emptyset_1) = \mathbb{k} \end{array}$$

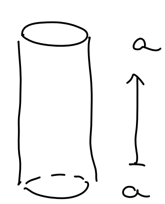
1 or $1_A := i(1)$

$$\begin{array}{c} m(i(1) \otimes a) = i(1)a = 1a \\ \uparrow \\ i(1) \otimes a \\ \uparrow \\ a \in A \end{array}$$

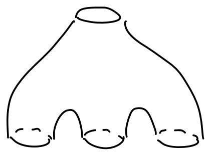
$1a = a = a1$

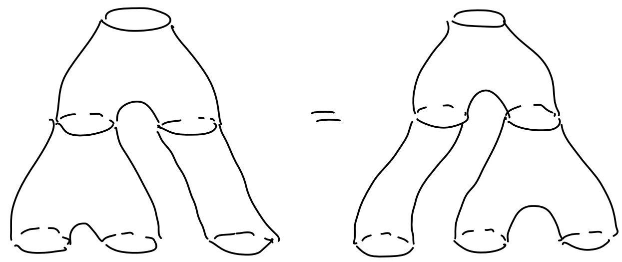


$\simeq$



m is associative





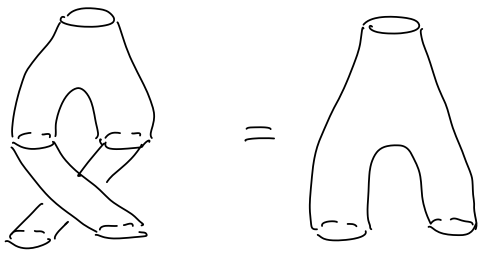
$A^{\otimes 3} \xrightarrow{m\otimes id} A^{\otimes 2} \longrightarrow A$ =

$A^{\otimes 3} \xrightarrow{id\otimes m} A^{\otimes 2} \longrightarrow A$

m is commutative

$a\otimes b \xrightarrow{P} b\otimes a \xrightarrow{m} ba = m(b\otimes a)$

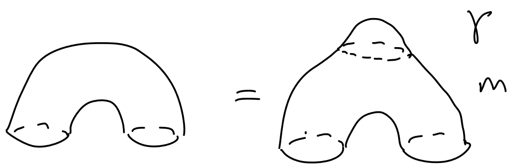
$a\otimes b \xrightarrow{m} ab = m(a\otimes b)$



diffeomorphic (or homeomorphic) rel Boundary

$\Bbbk \xleftarrow{\gamma} A$

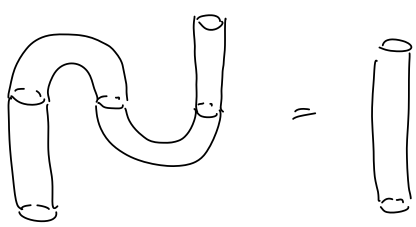
trace map. composition



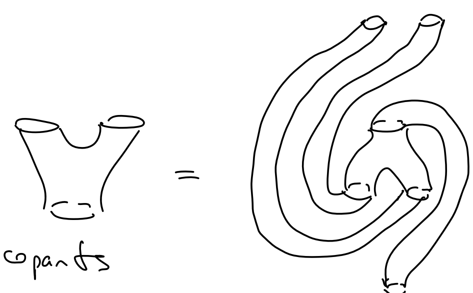
$A^{\otimes 2} \xrightarrow{m} A \xrightarrow{\gamma} \Bbbk$

is a non-degenerate pairing

$A \to A^{\otimes 3} \to A$



$A \xrightarrow{id} A$



copants

pants

$\Delta: A \to A^{\otimes 2}$ is dual to

$m: A^{\otimes 2} \to A$

(A, γ) a commutative Frobenius algebra

unital, commutative, associative, $\gamma : A \to \mathbb{k}$

nondegenerate $\forall a \in A, a \neq 0 \ \exists b \ \ \gamma(ab) \neq 0$

$A^* \cong A$ as A-modules $\quad A^* = \mathrm{Hom}_{\mathbb{k}}(A, \mathbb{k})$

$\gamma \longleftarrow\!\shortmid 1$

$\gamma(a\,*) \longleftarrow\!\shortmid a$ (projective A-modules = injective A-modules)

Examples: M^{2n} oriented closed $2n$-manifold

$A = H^{even}(M) = \bigoplus_{k=0}^{n} H^{2k}(M, \mathbb{k})$ cohomology algebra

$\gamma : H^{even}(M) \longrightarrow \mathbb{k} \qquad \gamma : H^{2n}(M, \mathbb{k}) \xrightarrow{\simeq} \mathbb{k}$

$\gamma|_{H^{2k}(M,\mathbb{k})} = 0 \quad k < n.$

additional grading

$M = \mathbb{CP}^n \qquad A = H^*(\mathbb{CP}^n, \mathbb{k}) \simeq \mathbb{k}[x]/(x^n)$

$\{1, x, x^2, \ldots, x^{n-1}\}$

$\gamma(x^{n-1}) = 1, \ \ \gamma(x^i) = 0 \quad i < n-1.$

$\Delta(1) = x^{n-1} \otimes 1 + x^{n-2} \otimes x + \ldots + x \otimes x^{n-2} + 1 \otimes x^{n-1}.$

$\underline{n=2}$ $\qquad M = \mathbb{CP}^1 = S^2$

$A = H^*(S^2, \mathbb{Z}) = \mathbb{Z}[x]/(x^2)$

$\{1, x\} \qquad x^2 = 0 \qquad \gamma(x) = 1, \ \gamma(1) = 0$

$\Delta(1) = x \otimes 1 + 1 \otimes x.$

graded case:

basis element	degree
x	2
1	0

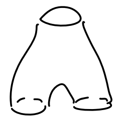
balance grading

	degree
x	1
1	-1

$$A \simeq \begin{array}{cc} \mathbb{Z}\cdot x & 1 \\ \oplus & \\ \mathbb{Z}\cdot 1 & -1 \\ \text{basis} & \deg \end{array}$$

$\begin{matrix} A \\ \uparrow i \\ \mathbb{Z} \end{matrix}$ $\quad x \quad 1 \mapsto 1 \quad$ 1, $0 \to -1$

$\deg i = -1$

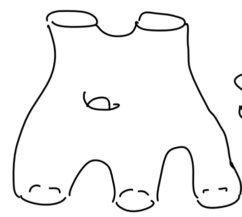


$\begin{matrix} A \\ \uparrow m \\ A^{\otimes 2} \end{matrix}$ $\quad \begin{matrix} x\otimes x \longrightarrow 0 \\ x\otimes 1 \longrightarrow x \\ 1\otimes 1 \longrightarrow 1 \end{matrix}$ $\quad \begin{matrix} 2 & \\ 0 & \to 1 \\ -2 & \to -1 \end{matrix}$

$\deg m = 1$

$\begin{matrix} \mathbb{Z} \\ \uparrow \gamma \\ A \end{matrix}$ $\quad x \to 1$, 1

$\deg \gamma = -1$

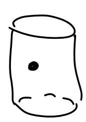


S

$\begin{matrix} A^{\otimes k} \\ \uparrow F(S) \\ A^{\otimes n} \end{matrix}$

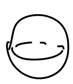
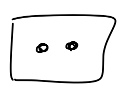
$$\boxed{\deg F(S) = -\chi(S)}$$

Convenient to add observables to cobordisms

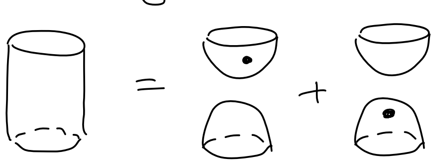


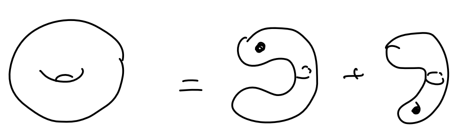
$\begin{matrix} A \\ \uparrow x \\ A \end{matrix}$ $\quad \begin{matrix} xa \\ \uparrow \\ a \end{matrix}$

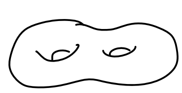
 freely floats on a component of a cobordism. Cannot jump from a component to a component.

 $\gamma(1) = 0$  $\gamma(x) = 1$  $= 0$ $\quad x^2 = 0$

Surgery formula



 $= 2 = 2k\,A$

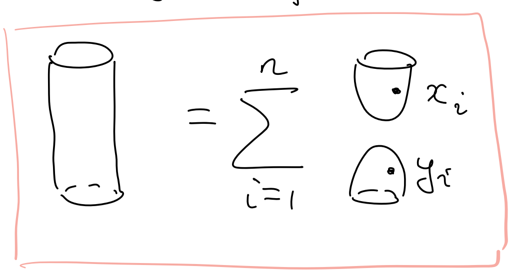
 $= 0$ (genus > 1)

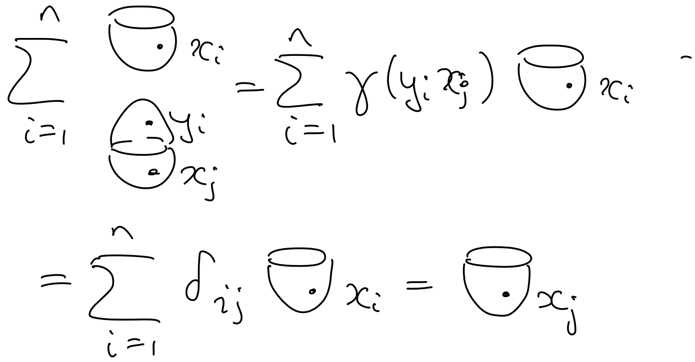
In a general commutative Frobenius algebra $A/\Bbbk$

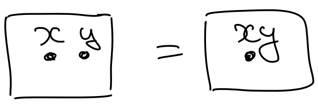
$\gamma : A \to \Bbbk$ $\quad \{x_1, \dots, x_n\}$ basis of A

$\qquad\qquad\quad \{y_1, \dots, y_n\}$ dual basis

$\gamma(x_i y_j) = \delta_{ij}$



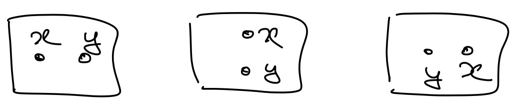




observables or defects

0-dimensional defects on 2-dim manifolds:

commutativity



Later: 0-dimensional defects on 1-manifolds.

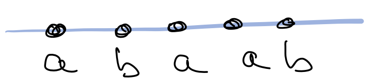
commutativity fails, lots of possibilities

1) over $\Bbbk$ – get noncommutative power series

2) over $\mathbb{B} = \{0, 1 \mid 1+1=1\}$ Boolean semiring (no –)

enhance familiar theory of regular languages

and finite state automata (complete to rigid tensor categories)



0-dim defects on 1-manifolds

Jones polynomial

oriented links in $\mathbb{R}^3$ $\qquad L \xrightarrow{J} J(L) \in \mathbb{Z}[q,q^{-1}]$

Skein relation:

$$q^2 J(\;) - q^{-2} J(\;) = (q-q^{-1})\, J(\;)$$

$$q^2 \; - q^{-2} \; = (q-q^{-1})$$

$$= \frac{q^2-q^{-2}}{q-q^{-1}} \; = (q+q^{-1})$$

Normalization $\qquad J(\bigcirc) = q+q^{-1}, \quad J(\emptyset) = 1$

$$J(\underbrace{\bigcirc \dots \bigcirc}_{k}) = (q+q^{-1})^k$$

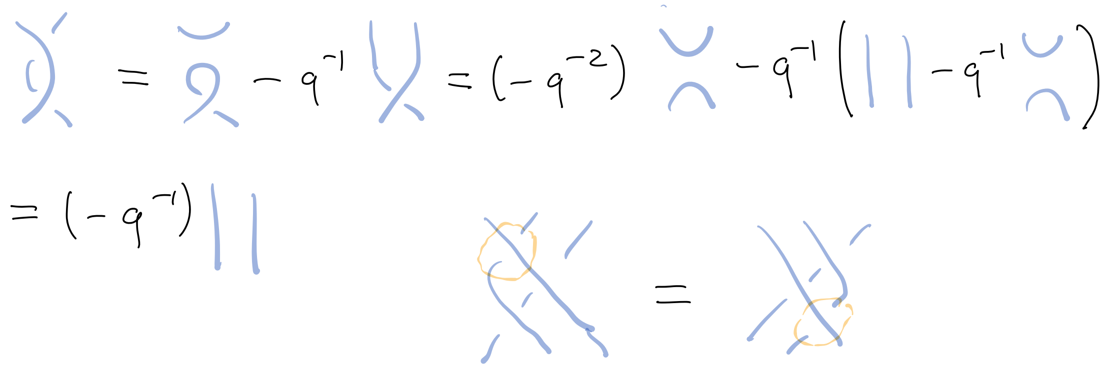
Easy way to show existence due to Louis Kauffman

$$\langle \; \rangle = \langle \; \rangle - q^{-1} \langle \;)(\; \rangle$$

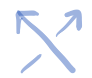
Kauffman's definition more symmetric, uses $q^{\pm 1/2}$

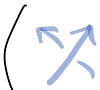
Start with an n-crossing diagram D of L. Decompose $\langle D \rangle$ into sum of 2^n terms, each $\pm q^{\cdot} \langle \underbrace{\bigcirc \bigcirc \bigcirc \bigcirc}_{k} \rangle = \pm q^{\cdot} (q+q^{-1})^k$

$$= \; - q^{-1} \;)\bigcirc = |\, -q^{-1}(q+q^{-1})\,| = (-q^{-2})\,|$$



Normalize to get rid of multiplicative factors

$x(D) = \#\Big($  $\Big)$ negative

$y(D) = \#\Big($  $\Big)$ positive

$$K(D) = (-1)^{x(D)}\, q^{2x(D)-y(D)} \langle D \rangle$$

Kauffman bracket of L (does not depend on choice of D)

$$\boxed{K(L) = J(L)}$$

On rep. theory side  are intertwiners

$$\begin{array}{c} V^{\otimes 2} \\ \uparrow \\ V^{\otimes 2} \end{array}$$

V - fund. representation of $U_q(sl_2) \xleftarrow[\text{deform}]{q-} U(sl_2)$

$\mathrm{Hom}_{U_q(sl_2)}(V^{\otimes 2}, V^{\otimes 2})$ is 2-dimensional

$$V^{\otimes 2} \simeq \underbrace{\Lambda^2_q V}_{} \oplus \underbrace{S^2_q V}_{}$$

irreducible reps

$\Rightarrow$ $\forall$ 3 intertwiners satisfy a linear relation

Likewise for Kauffman skein relation

(hides isomorphism $V \simeq V^*$)

$$\langle \times \rangle = \langle \asymp \rangle - q^{-1} \langle) (\rangle$$

$J(L) = K(L) \in \mathbb{Z}[q, q^{-1}]$ integral coefficients

Back in late 90's – many indications that some structures in quantum topology can be lifted one dimension up

1) Beilinson - Lusztig - MacPherson geometric realization of $\mathcal{U}_q(sl(N))$ via correspondences between flag varieties

2) L. Crane - I. B. Frenkel conjecture (still open) on categorification of finite $\mathcal{U}_q(sl(2))$ at a root of unity

3) Categorification of $TL_n \circlearrowright V^{\otimes n} \circlearrowleft \mathcal{U}(sl(2))$

algebra

highest weight categories (category $\mathcal{O}$) — projective & Zuckerman functors (Bernstein)

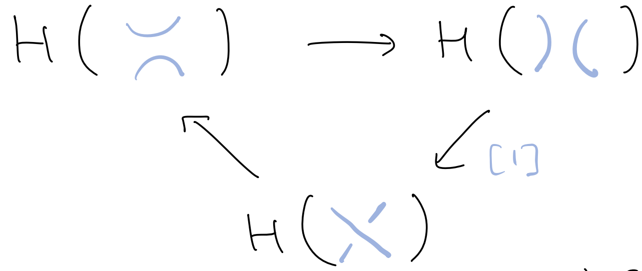
(J. Bernstein, I. Frenkel, M.K.)

4) Blossoming geometric representation theory

want to convert

$$\langle \times \rangle = \langle \asymp \rangle - q^{-1} \langle) (\rangle$$

to a distinguished triangle in the category of complexes (to a long exact sequence)

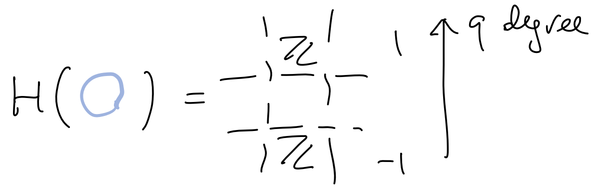


$J(L) \in \mathbb{Z}_+[q,q^{-1}]$

$\langle \bigcirc \rangle = q+q^{-1}$

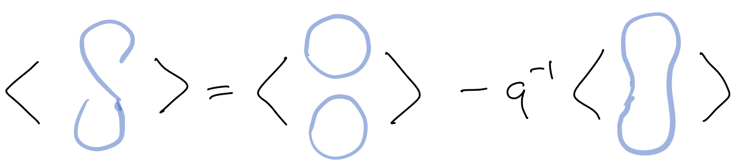
want bigraded homology groups

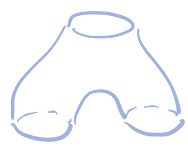
$\langle \underbrace{\bigcirc\ \bigcirc\ \bigcirc}_{k} \rangle = (q+q^{-1})^k$



$H(\bigcirc) = H^*(S^2, \mathbb{Z})$, balanced degrees

$\xrightarrow{0}$ homological degree





m

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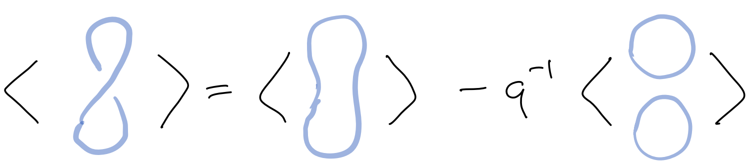
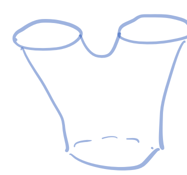
degree shift

$0 \to A^{\otimes 2} \xrightarrow{m} A\{-1\} \to 0$

complex of LHS. 2-dim homology, relative degrees match





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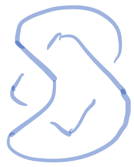
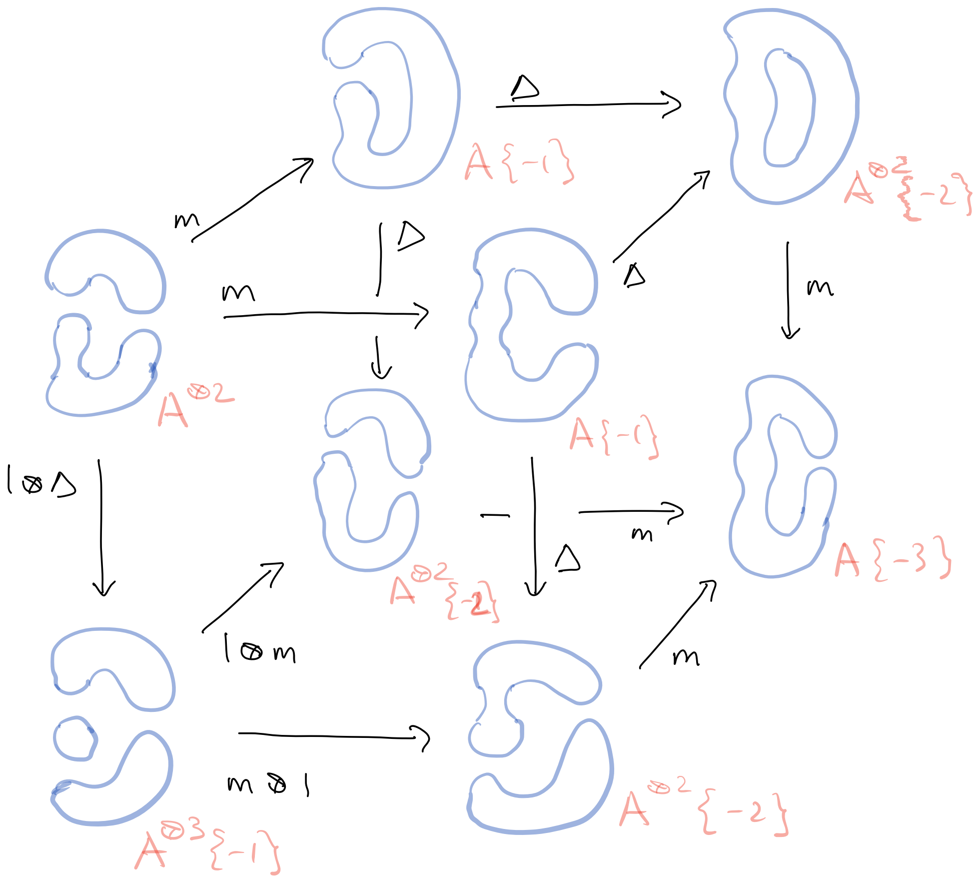
$0 \to A \xrightarrow{\Delta} A^{\otimes 2}\{-1\} \to 0$

In general, diagram D with n crossings

D 

n-dim commutative cube



$$
\begin{tikzcd}
 & A\{-1\} \arrow[rr, "\Delta"] \arrow[dd] & & A^{\otimes 2}\{-2\} \arrow[dd, "m"] \\
A^{\otimes 2} \arrow[ur, "m"] \arrow[rr, "m"] \arrow[dd] & & A\{-1\} \arrow[ur, "\Delta"] \arrow[dd, "\Delta"] & \\
 & A^{\otimes 2}\{-2\} \arrow[rr, "m"] & & A\{-3\} \\
A^{\otimes 3}\{-1\} \arrow[ur] \arrow[rr] & & A^{\otimes 2}\{-2\} \arrow[ur, "m"] &
\end{tikzcd}
$$

$$
0 \to A^{\otimes 2} \longrightarrow \begin{matrix} A\{-1\} \\ \oplus \\ A\{-1\} \\ \oplus \\ A^{\otimes 3}\{-1\} \end{matrix} \longrightarrow \begin{matrix} A^{\otimes 2}\{-2\} \\ \oplus \\ A^{\otimes 2}\{-2\} \\ \oplus \\ A^{\otimes 2}\{-2\} \end{matrix} \longrightarrow A\{-3\} \to 0
$$

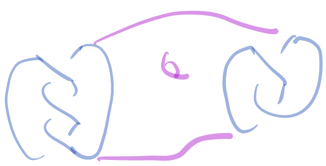
$(-1)^{x(D)} q^{2x(D)-y(D)}$

+ overall bigrading shift $[x(D)]\{2x(D)-y(D)\}$

get complex $C(D) \longrightarrow H(D) = \bigoplus_{i,j} H^{i,j}(D)$

Thm 1) $H(L)$ does not depend on a choice of diagram

2) Functorial under cobordisms



L_0 S L_1

$S \subset \mathbb{R}^3 \times [0,1]$

$$H(L_0) \xrightarrow{H(S)} H(L_1)$$

$H(S)$ is well-defined up to overall sign
(M. Jacobsson, D. Bar-Natan, MK)

can add decorations to hide sign
(D. Clark, S. Morrison, K. Walker; C. Caprau, P. Vogel)

recent – no decorations needed (T. Sano)

Jones polynomial → extend to tangles

Jones polynomial → $U_q(sl_2)$ representations

Jones polynomial → quantum invariants for $U_q(\mathfrak{g})$, $\mathfrak{g}$ - simple L.A

link homology → extend to tangles and tangle cobordisms

link homology → categorified $U_q(sl_2)$

link homology → categorifications of other quantum link invariants, for $U_q(\mathfrak{g})$

leads to foams, universal construction

$$q^{2} J(\nwarrow\!\!\!\!\nearrow) - q^{-2} J(\nwarrow\!\!\!\!\nearrow) = (q-q^{-1})\, J(\uparrow\,\uparrow)$$

$$q^{2} \rightsquigarrow q^{N}$$

$$q^{N} J(\nwarrow\!\!\!\!\nearrow) - q^{-N} J(\nwarrow\!\!\!\!\nearrow) = (q-q^{-1})\, J(\uparrow\,\uparrow)$$

$$J(\bigcirc) = [n] = \frac{q^{n}-q^{-n}}{q-q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{1-n}$$

$n=0$: $J(\bigcirc)=1$

$N=0$ Alexander polynomial

$N=1$ trivial

$N=2$ Jones polynomial

$N=3$ Kuperberg bracket

$a=q^{N}$ $\quad a^{-1}=q^{-N}$

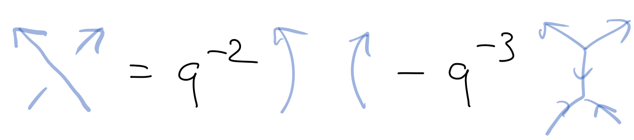
$b=q-q^{-1}$

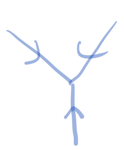
HOMFLYPT polynomial
2 variables

$$q^{3} P_3(\nwarrow\!\!\!\!\nearrow) - q^{-3} P_3(\nwarrow\!\!\!\!\nearrow) = (q-q^{-1})\, P_3(\uparrow\,\uparrow)$$

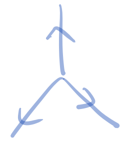
$$P_3(\bigcirc) = [3] = q^{2}+1+q^{-2}$$

Introduce trivalent graphs





"In" vertex



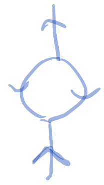
"out" vertex

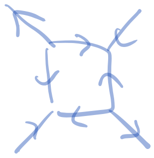
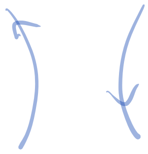
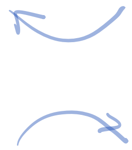
$V^{\otimes 3}$

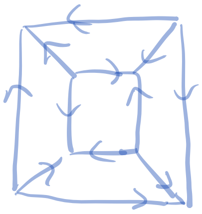
$\mathrm{Inv}_{U_q(\mathfrak{sl}(3))}(V^{\otimes 3})$

quantum skew-symm. element

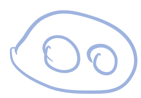
 $= q^2+1+q^{-2}$
$[3]$

 $=(q+q^{-1})$ 
$[2]$

 $=$  $+$ 



Γ planar trivalent oriented (bipartite) graph. $P(\Gamma) \in \mathbb{Z}_+[q,q^{-1}]$

analogue of  for $sl(2)$

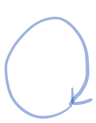
want to categorify $P_3(L)$. First categorify $P_3(\Gamma)$

$\rightsquigarrow H(\Gamma)$

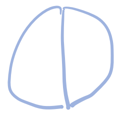
$H(\Gamma)$ $\mathbb{Z}$-graded rather than $\mathbb{Z}\oplus\mathbb{Z}$-graded

$$H(\Gamma) = \bigoplus_{j\in\mathbb{Z}} H^j(\Gamma)$$

$$P_3(\Gamma) = \sum_{j\in\mathbb{Z}} \operatorname{rk} H^j(\Gamma)\cdot q^j$$

Γ	$P_3(\Gamma)$	$H(\Gamma)$
$\emptyset$	1	$\mathbb{Z} = H^*(\cdot, \mathbb{Z})$
	$[3]=q^2+1+q^{-2}$	$H^*(\mathbb{CP}^2, \mathbb{Z})$: shift degree 2: $\mathbb{Z}x^2$; 0: $\mathbb{Z}x$; -2: $\mathbb{Z}\cdot 1$

$x^3=0$



$[3][2] =$
$= (q^2+1+q^{-2})(q+q^{-1})$

$H^*(Fl_3, \mathbb{Z})$

$Fl_3 = \{0 \subset U_1 \subset U_2 \subset \mathbb{C}^3\}$
$\dim U_i = i$

$\mathbb{Z}[x_1, x_2, x_3]/I$

$I = (E_1, E_2, E_3)$

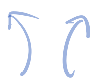
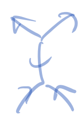
$E_1 = x_1 + x_2 + x_3$
$E_2 = x_1x_2 + x_1x_3 + x_2x_3$
$E_3 = x_1 x_2 x_3$

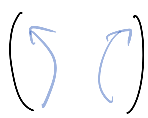
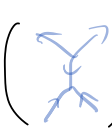
elementary symmetric functions

$A = H^*(\mathbb{CP}^2, \mathbb{Z})$ $\qquad [3]$
$B = H^*(Fl_3, \mathbb{Z})$ $\qquad [3][2]$

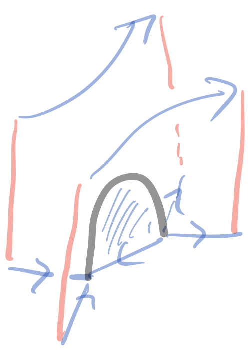
$$\begin{array}{c} Fl_3 \\ \downarrow{\scriptstyle \mathbb{CP}^1} \\ \mathbb{CP}^2 \end{array}$$

$H^*(Fl_3) = H^*(\mathbb{CP}^2) \otimes H^*(\mathbb{CP}^2)$
with shifts

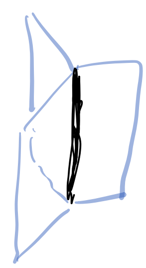
 $= q^{-2}$  $- q^{-3}$ 

to form cones, need maps $H($  $)$ $\rightleftarrows$ $H($  $)$

come from cobordisms

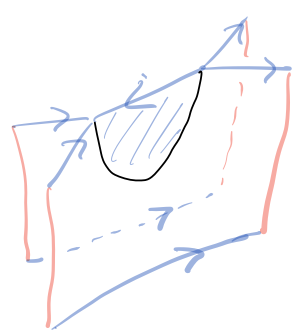


"singular"
saddle



foams

foam with
a singular
circle



foams F in $\mathbb{R}^2\times[0,1]$ - cobordisms between planar triv. graphs Γ_0, Γ_1

$$H(\Gamma_0) \xrightarrow{H(F)} H(\Gamma_1)$$

$$Fl_3 \subset \mathbb{CP}^2 \times \mathbb{CP}^2$$

$$\{0\subset U_1 \subset U_2 \subset \mathbb{C}^3\} \quad \begin{matrix}\Downarrow \\ U_1\end{matrix} \quad \begin{matrix}\Downarrow \\ U_2\end{matrix}$$

$$\begin{array}{ccc} H^*(\mathbb{CP}^2\times\mathbb{CP}^2) & \longrightarrow & H^*(Fl_3) \\ \wr\! \| & & \\ H^*(\mathbb{CP}^2)^{\otimes 2} & \longrightarrow & H^*(Fl_3) \\ \wr\! \| & & \wr\! \| \\ A^{\otimes 2} & \longrightarrow & B \end{array}$$

To define $sl(3)$ link homology, need to build a functor from the category of foams in $\mathbb{R}^3$ to the category of graded abelian groups.

TO BE CONTINUED TOMORROW

THANK YOU!