Foam evaluation, universal construction, and the Four-Color Theorem

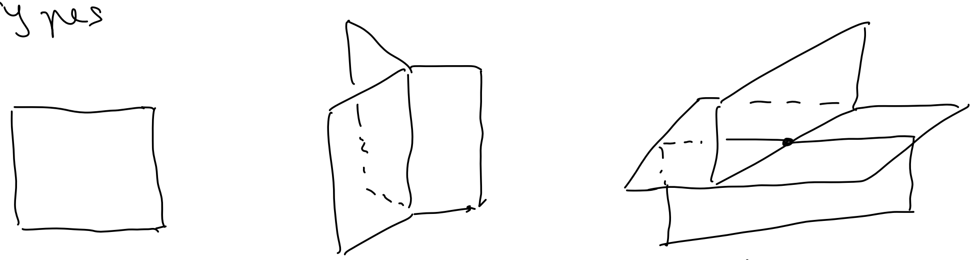
M. Khovanov

Virginia Mathematics Lectures

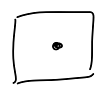
University of Virginia

April 2022

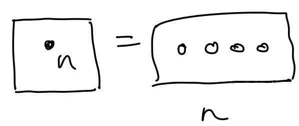
Foam F (unoriented $\mathfrak{sl}(3)$ foam) is a 2-dim CW-complex embedded in $\mathbb{R}^3$, with points of 3 possible types



smooth   seams   vertices



Decorations (observables): dots on facets

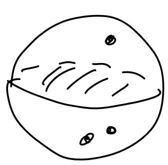


<u>Examples</u> 1)

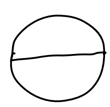
S surface

$s(F)=\emptyset$

2)

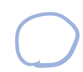


Θ - foam

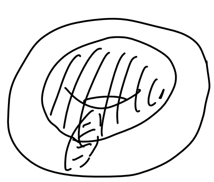


cross-section Θ-graph

$s(F)=$

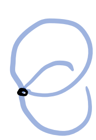


3)



2-torus $\cup$ 2 disks.

$s(F)=$



$s(F)$ - singular graph of F, union of seams & facets.

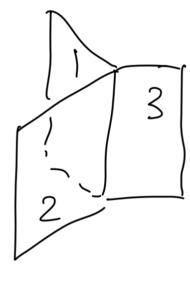
$s(F)$ - four-valent graph with verticeless loops

$f(T)$ - set of facets of F: connected components of $F \setminus s(F)$

Tait (admissible) coloring of F is a map

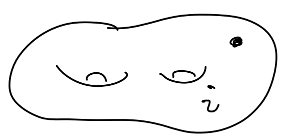
$f(T) \longrightarrow \{1,2,3\}$

such that along each seam the 3 colors are distinct



$adm(F)$ set of admissible colorings

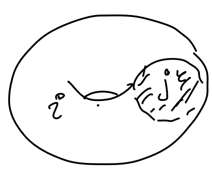
Examples

1) 

$i \in \{1,2,3\}$ 3 colorings

2) 

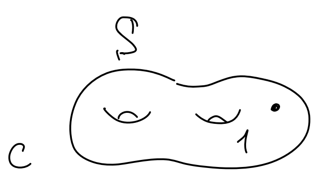
$\{i,j,k\} = \{1,2,3\}$ 6 colorings

3) 

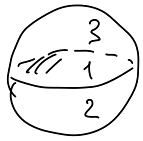
no colorings

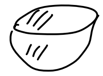
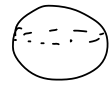
S_3 acts on $adm(F)$ by permuting colors

Bicolored surfaces $F_{ij}(c)$ = union of closures of i-faces & j-facets.



$F_{12}(c) = S$, $F_{13}(c) = S$, $F_{23}(c) = \emptyset$

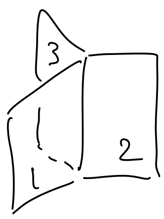
c 

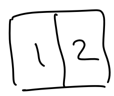
$F_{12}(c) =$  $F_{13}(c) =$  $F_{23}(c) =$ 

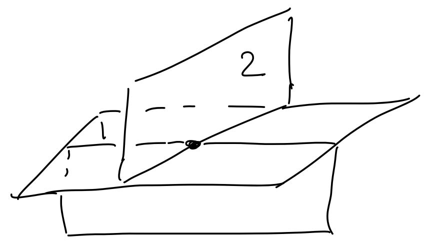
2-spheres S^2

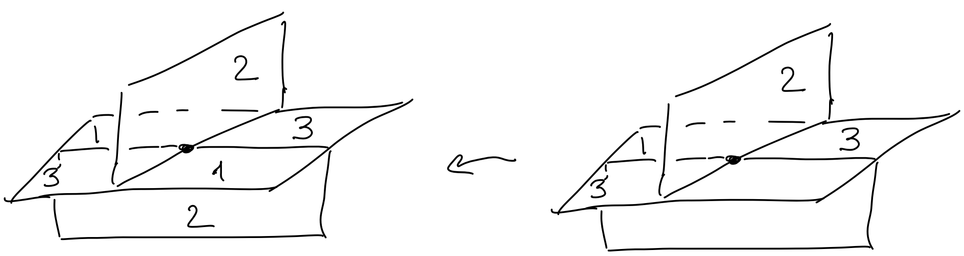
Thm $F_{ij}(c)$ is a closed surface embedded in $\mathbb{R}^3$

along a seam $F_{12}(c)$ at a vertex



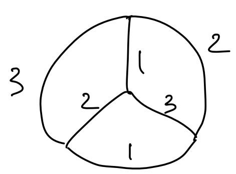


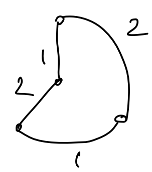




opposite corners carry the same color.
unique coloring up to S_3 action.

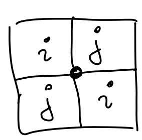
Link





$\simeq S^1$

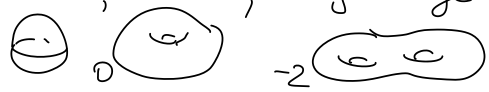
suspension

$F_{ij}(c) \cap U =$ 

Corollary $F_{ij}(c)$ is orientable and has even Euler characteristic.

connected components: S^2, T^2, higher genus $g \geq 2$

χ 2 0 −2



<u>Algebraic side</u>: $\Bbbk$ - field, $\operatorname{char} \Bbbk = 2$ $\quad \mathbb{F}_2$

$$R' = \Bbbk[x_1, x_2, x_3] \supset R = (R')^{S_3} = \Bbbk(E_1, E_2, E_3)$$

$$R \subset R'$$

$$E_1 = x_1 + x_2 + x_3$$
$$E_2 = x_1x_2 + x_1x_3 + x_2x_3$$
$$E_3 = x_1x_2x_3$$

Want $\boxed{\langle F \rangle = \sum_{c \in \operatorname{adm}(F)} \langle F, c \rangle}$ $\quad \langle F, c \rangle \in R'\left[\frac{1}{x_i + x_j}\right]_{1 \leq i < j \leq 3}$

$$\langle F \rangle \in R$$

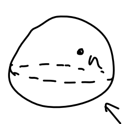
Define
$$\langle F, c \rangle = \frac{x_1^{d_1(c)} x_2^{d_2(c)} x_3^{d_3(c)}}{(x_1 + x_2)^{\chi_{12}(c)/2} (x_1 + x_3)^{\chi_{13}(c)/2} (x_2 + x_3)^{\chi_{23}(c)/2}}$$

$$\chi_{ij}(c) = \chi(F_{ij}(c)) \in 2\mathbb{Z}$$

$d_i(c)$ − # of dots on facets colored i

<u>Example</u>

$F = S^2$



c_1 − color 1

	$\chi/2$
$F_{12}(c) = S^2$	1
$F_{13}(c) = S^2$	1
$F_{23}(c) = \emptyset$	0

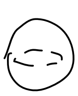
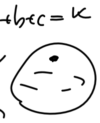
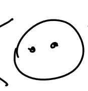
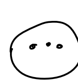
$$\frac{x_1^n}{(x_1 + x_2)(x_1 + x_3)}$$

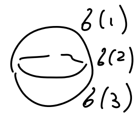
$$\langle F \rangle = \frac{x_1^n}{(x_1 + x_2)(x_1 + x_3)} + \frac{x_2^n}{(x_2 + x_1)(x_2 + x_3)} + \frac{x_3^n}{(x_1 + x_3)(x_2 + x_3)} =$$

$$= \frac{x_1^n(x_2 + x_3) + x_2^n(x_1 + x_3) + x_3^n(x_1 + x_2)}{(x_1 + x_2)(x_1 + x_3)(x_2 + x_3)} = h_{n-2}(x_1, x_2, x_3)$$

complete symmetric f'n

$$h_k(x_1, x_2, x_3) = \sum_{a+b+c=k} x_1^a x_2^b x_3^c$$

$\langle$  $\rangle = 0$ $\quad \langle$  $\rangle = 0$ $\quad \langle$  $\rangle = 1$ $\quad \langle$  $\rangle = x_1 + x_2 + x_3$ ...

$F =$  colorings $\leftrightarrow$ el's of S_3 (σ)  $f_{ij}(\sigma) = S^2$

$$\langle F, \sigma \rangle = \frac{x_{\sigma(1)}^{n_1} x_{\sigma(2)}^{n_2} x_{\sigma(3)}^{n_3}}{(x_1+x_2)(x_1+x_3)(x_2+x_3)}$$

$n_1 \geq n_2 \geq n_3$

$$\langle F \rangle = \sum_{\sigma \in S_3} \langle F, \sigma \rangle = s_\lambda(x_1, x_2, x_3) \in R$$

$s_\lambda = s_\lambda(n_1-2, n_1-1, n_3)$

Schur function (mod 2)

$\boxed{\underline{\text{Thm}} \quad \langle F \rangle \in R}$ F closed foam   instead of $R'\left[\frac{1}{x_i+x_j}\right]_{i<j}$

Universal construction

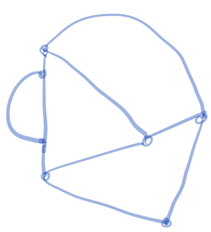
Have a multiplicative invariant $\langle F \rangle$ of closed n-dimensional objects $\quad \langle F_1 \sqcup F_2 \rangle = \langle F_1 \rangle \langle F_2 \rangle$, $\langle \emptyset \rangle = 1$

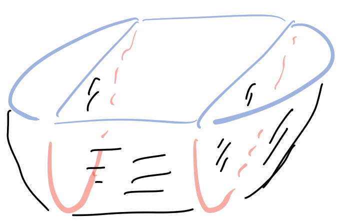
$\langle F \rangle \in R$ commutative ring $\qquad F \sqcup \emptyset = F$

Build an R-module (state space) for objects in one dimension less ("cross-sections" of F)

F

Cut F by a plane. Generic cross-section – planar trivalent graph Γ.

Γ 



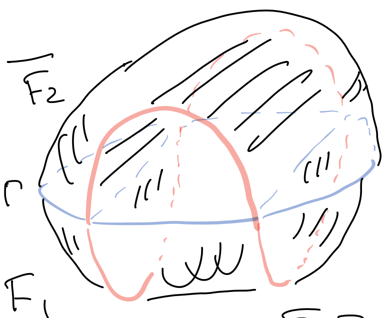
$\Gamma \in \mathbb{R}^2$

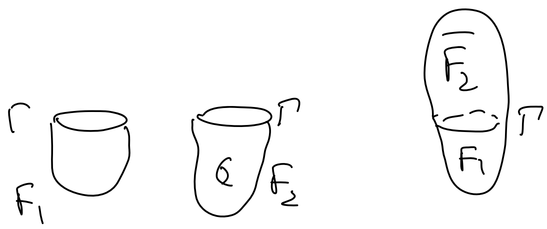
$F \in \mathbb{R}^3_-$

$\partial F = \Gamma$

foam with boundary Γ.

Given F_1, F_2 $\partial F_1 = \partial F_2 = \Gamma$ can pair up F_1, F_2 into a closed foam





$\overline{F_2}F_1$ - closed foam, can be evaluated

$\langle \overline{F_2} F_1 \rangle \in R$.

Fix Γ. Consider all $[F]$, $\partial F = \Gamma$. Form free R-module $Fr(\Gamma)$ on these generators $[F]$.

Define symmetric bilinear form (over R)

$$Fr(\Gamma) \times Fr(\Gamma) \longrightarrow R$$

$$\big([F_1], [F_2]\big)_\Gamma = \langle \overline{F_2}\, F_1 \rangle \in R$$

The state space $\langle \Gamma \rangle := Fr(\Gamma) / \ker((,)_\Gamma)$.

$\sum_i a_i F_i = 0$ in $\langle \Gamma \rangle$

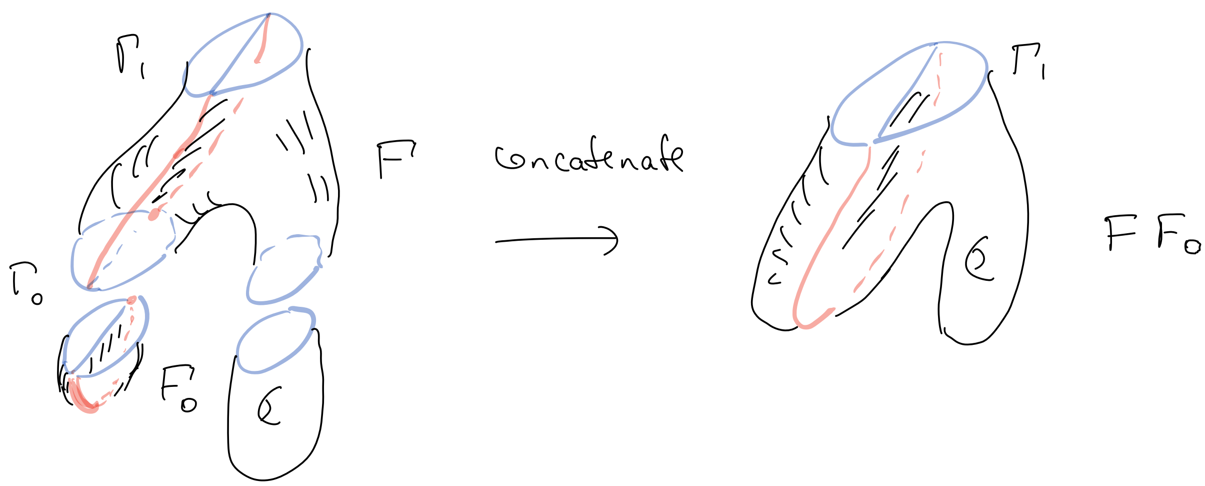
iff $\forall F$, $\partial F = \Gamma$

$$\sum_i a_i \langle F F_i \rangle = 0 \in R$$

Functoriality: given a foam F in $\mathbb{R}^2 \times [0,1]$ with $\partial_0 F = \Gamma_0$, $\partial_1 F = \Gamma_1$, it induces an R-module map

$$\langle \Gamma_0 \rangle \xrightarrow{[F]} \langle \Gamma_1 \rangle$$

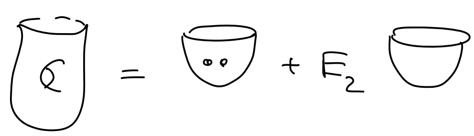
$$[F_0] \longmapsto [F F_0]$$



Get a topological theory, a functor from the category of foams (cobordisms between planar triv. graphs Γ) to the category of R-modules (+ additional grading)

deg $x_i = 2, \ldots$

Some skein relations

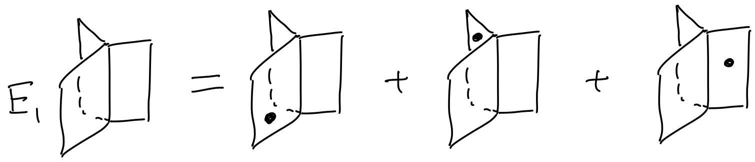


$$\boxed{\bullet 3} = E_1 \boxed{\bullet 2} + E_2 \boxed{\bullet} + E_3 \boxed{}$$

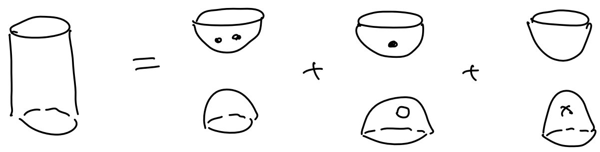
$x^3 + E_1 x^2 + E_2 x + E = 0$

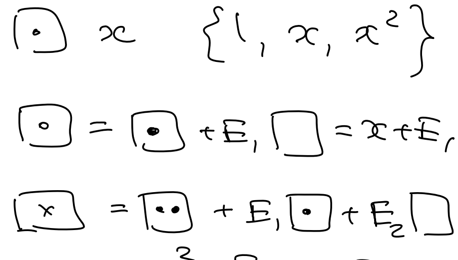
$(x + x_1)(x + x_2)(x + x_3) = 0$

coloring c



$E_1 = x_1 + x_2 + x_3$



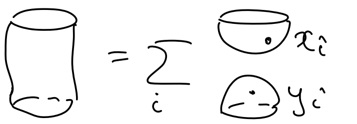


$x \qquad \{1, x, x^2\}$

$= x + E_1$

$= x^2 + E_1 x + E_2$

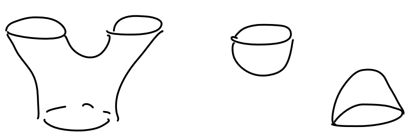
$\{x^2 + E_1 x + E_2,\ x + E_1,\ 1\}$



$\{x_i\}, \{y_i\}$ dual bases

$$H(\emptyset) = R = \Bbbk[E_1, E_2, E_3] = \Bbbk[x_1, x_2, x_3]^{S_3}$$

$H(\bigcirc)$ - commutative Frob. algebra$/H(\emptyset)$



special cases of foams.

$(x+x_1)(x+x_2)(x+x_3)$

$$H(\bigcirc) = R[x]/(x^3 + E_1 x^2 + E_2 x + E_3)$$

generic deformation of $\Bbbk[x]/(x^3)$

$H^*(\mathbb{CP}^2, \Bbbk)$.

$$R = H(\emptyset) = H^*_{U(3)}(\cdot, \Bbbk)$$ equivariant cohomology of a point

$$A = H(\bigcirc) = H^*_{U(3)}(\mathbb{CP}^2, \Bbbk)$$

$\dim V_i = i$

$$B = H(\Theta) = H^*_{U(3)}(Fl_3, \Bbbk)$$

$Fl_3 = \{0 \subset V_1 \subset V_2 \subset \mathbb{C}^3\}$

$Fl_3 \subset \mathbb{CP}^2 \times \mathbb{CP}^2$

$A^{\otimes 2} = H^*(\mathbb{CP}^2)^{\otimes 2} \longrightarrow H^*(Fl_3) = B$

Some of foam maps may be described via cohomology & Poincaré duality

Γ_1 $\quad B$

$\uparrow$

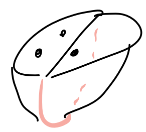
Γ_0 $\quad A$

Oriented graphs and foams version of this story gives Kuperberg bracket invariant of links

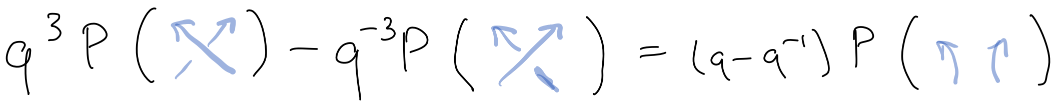
$1 \in B = H^*_{U(3)}(\mathrm{Fl}_3, \Bbbk)$

polynomials in x_1, x_2, x_3

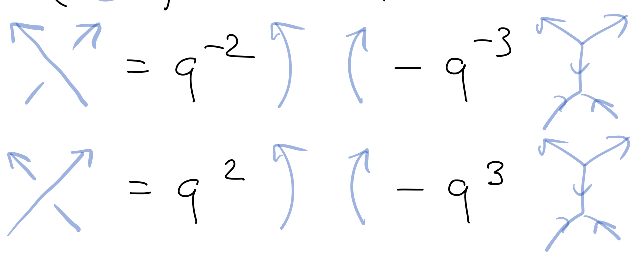
(slight simplification)



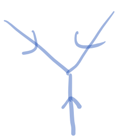
$x_1^2 x_2$



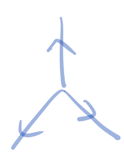
$P(\bigcirc) = [3] = q^2 + 1 + q^{-2}$



Introduce trivalent graphs



"In" vertex



"out" vertex

$V^{\otimes 3}$

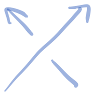
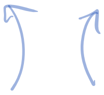
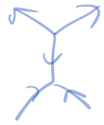
$\mathrm{Inv}_{U_q(\mathfrak{sl}(3))}(V^{\otimes 3})$

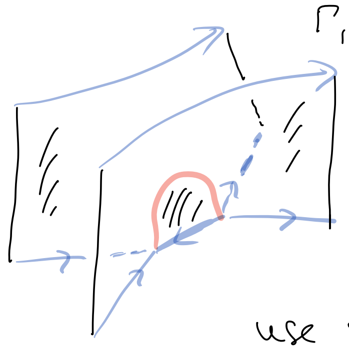
quantum skew-symm. element

Oriented version of foams and graphs.

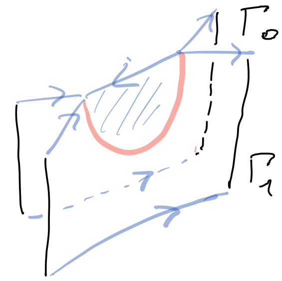
just need R, A, B above. Foams don't have singular vertices. Works over $\mathbb{Z}$ (not just $\mathbb{F}_2$). Get state spaces for (bipartite) <u>oriented</u> trivalent graphs. $\Gamma \rightsquigarrow H(\Gamma)$

$\langle \Gamma \rangle$

 $= q^2$  $- q^3$ 



$$H(\Gamma_1)$$
$$H(F) \uparrow$$
$$H(\Gamma_0)$$

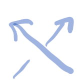
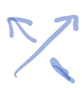


use these maps to build a commutative cube, form total complex & its homology.

get $\quad H(L) = \bigoplus\limits_{i,j\in\mathbb{Z}} H^{i,j}(L)$

$$P(L) = \sum_{i,j} (-1)^i \operatorname{rk} H^{i,j}(L)\, q^j$$

Kuperberg bracket

$q^N J($  $) - q^{-N} J($  $) = (q-q^{-1})\, J($  $)$

$J($  $) = [n] = \dfrac{q^n - q^{-n}}{q-q^{-1}} = q^{n-1} + q^{n-3} + \dots + q^{1-n}$

$n=0$

$J($  $) = 1$

$N=0$ Alexander polynomial

$N=1$ trivial

$N=2$ Jones polynomial

$N=3$ Kuperberg bracket

$a = q^N \quad a^{-1} = q^{-N}$

$b = q - q^{-1}$

HOMFLYPT polynomial
2 variables

$N>3$ quantum $sl(N)$ invariants

need foam evaluation for combinatorial categorification

$N \geq 4$ L.-H. Robert, E. Wagner 2017

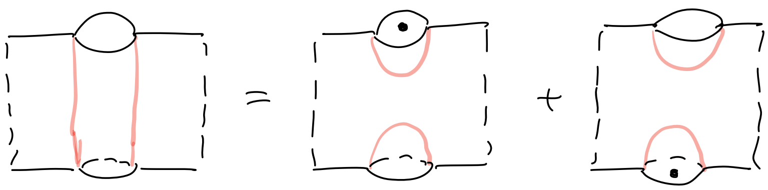
$N=3$, unoriented L.-H. Robert, MK 2018

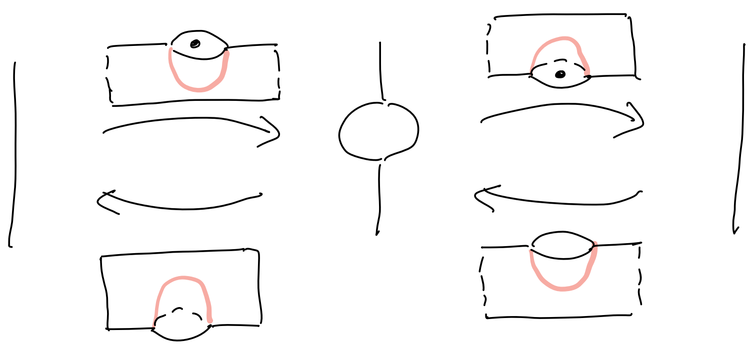
$N=3$ oriented M.K (non-equiv) P. Vaz – M. Mackaay (2008) equivariant

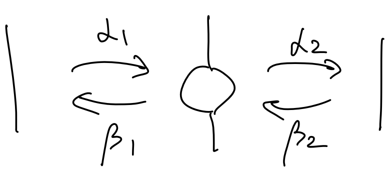
---

Back to skein relations in $sl(3)$ unoriented case. They lead to inductive direct sum decomposition for state spaces.

Prop $\langle$  $\rangle \simeq \langle \,|\, \rangle\{1\} \oplus \langle \,|\, \rangle\{-1\}$



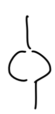




$\alpha_2\alpha_1 = 0$ $\qquad \beta_1\alpha_1 = \mathrm{id}_|$

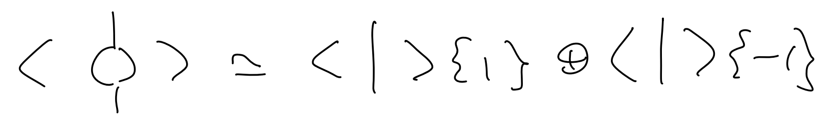
$\beta_1\beta_2 = 0$ $\qquad \alpha_2\beta_2 = \mathrm{id}_|$

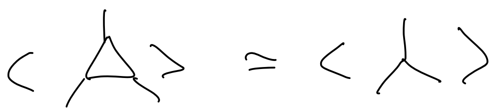
$\alpha_1\beta_1 + \beta_2\alpha_2 = \mathrm{id}$ 

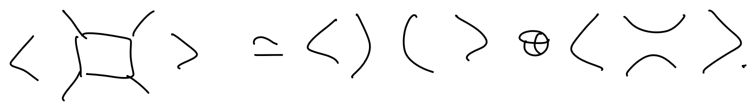
$\Rightarrow$  $\simeq \,|\, \oplus \,|\,$.

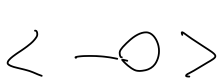
Thm Can inductively decompose state spaces

$$\langle \bigcirc \sqcup \Gamma \rangle \simeq \langle \Gamma \rangle\{-2\} \oplus \langle \Gamma \rangle \oplus \langle \Gamma \rangle\{2\}$$





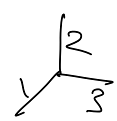


 $= 0$ (no Tait coloring)

Γ is called <u>reducible</u> if can be reduced to empty graphs this way (or get 0 state spaces at ends of some paths)

Thm If Γ is reducible, $\langle \Gamma \rangle$ is a free $\mathbb{R}$-module of rank $|\mathrm{Tait}(\Gamma)|$.

$\mathrm{Tait}(\Gamma)$ = set of Tait colorings of edges of Γ

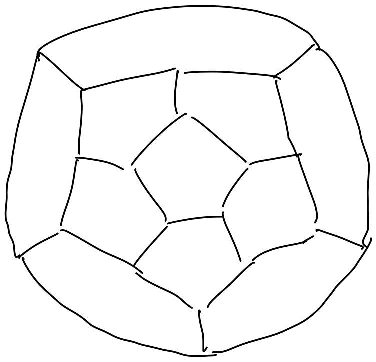


get q-deformation of $|\mathrm{Tait}(\Gamma)|$-colorings, different from Yamada polynomial ($U_q(sl_2)$, 3-dim rep

JW proj )

<u>Naive conjecture</u>: True for all Γ, not just reducible.

Smallest non-reducible graph - planar skeleton of dodecahedron



Γ_d

$\langle \Gamma_d \rangle$ is unknown

partial results (D. Boozer)

applications to link homology

Robert-Wagner eval of oriented $GL(N)$ foams

Motivation for unoriented $sl(3)$ foam eval

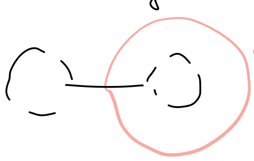
(L.-H. Robert, M.K.)

Kronheimer-Mrowka homology of knotted trivalent graphs

(Instanton Floer homology for 3-orbifolds)

$H_{KM}(\Gamma)$

Kronheimer-Mrowka prove nonvanishing result for their homology



If Γ is not of this form $H_{KM}(\Gamma) \neq 0$

<u>Conj</u> $\dim H_{KM}(\Gamma) = |\mathrm{Tait}(\Gamma)|$

(this would give a conceptual proof of 4CT)

Known:

$$\dim_{\mathbb{F}_2} H_0(\Gamma) \overset{\text{for reducible graphs}}{\underset{=}{\leq}} |\mathrm{Tait}(\Gamma)| \leq \dim_{\mathbb{F}_2} H_{KM}(\Gamma)$$

from foam eval

$$\mathbb{F}_2[E_1, E_2, E_3] \circlearrowleft R \longrightarrow \mathbb{F}_2$$

downsize

TO BE CONTINUED TOMORROW

THANK YOU!

foams F in $\mathbb{R}^2\times[0,1]$ - cobordisms between planar triv. graphs Γ_0, Γ_1

$$H(\Gamma_0) \xrightarrow{H(F)} H(\Gamma_1)$$

Γ_1 $\quad B$

$\uparrow$

Γ_0 $\quad A^{\otimes 2}$

$$Fl_3 \subset \mathbb{CP}^2 \times \mathbb{CP}^2$$

$$\{0\subset U_1\subset U_2\subset\mathbb{C}^3\} \quad \underset{U_1}{\Psi} \quad \underset{U_2}{\Psi}$$

$$\begin{tikzcd}
H^*(\mathbb{CP}^2\times\mathbb{CP}^2) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^*(Fl_3) \\
H^*(\mathbb{CP}^2)^{\otimes 2} \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & H^*(Fl_3) \arrow[d, phantom, "\cong" rotate=90] \\
A^{\otimes 2} \arrow[r] & B
\end{tikzcd}$$

To define $sl(3)$ link homology, need to build a functor from the category of foams in $\mathbb{R}^3$ to the category of graded abelian groups.