Topological theories and automata

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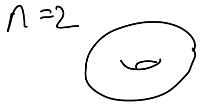
(joint work with Mee Seong Im, arXiv:2022.13398)

Universal construction:

Start with an invariant of closed n-manifolds



$M \longmapsto \alpha(M) \in R$ R - commutative ring

$\alpha(M_1 \sqcup M_2) = \alpha(M_1)\alpha(M_2)$ multiplicative

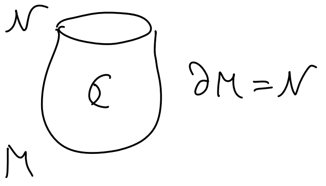
$\alpha(\emptyset_n) = 1$ $\emptyset_n$ empty n-manifold

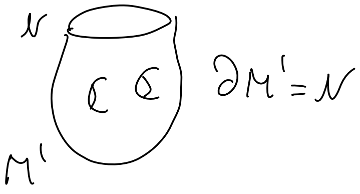
State spaces for cross-sections

$\dim N = n-1$ N closed $(n-1)$-manifold

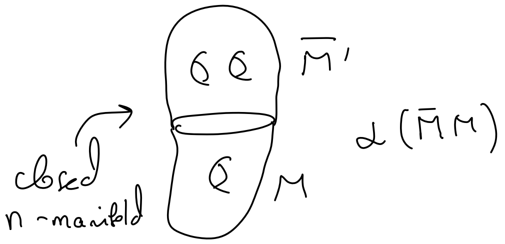
$Fr(N)$ - free R-module with basis $\{[M] \mid \partial M = N\}$

$Fr(N) \times Fr(N) \longrightarrow R$ bilinear pairing





$$([M],[M'])_N = \underset{\underset{R}{\cap}}{\alpha(\overline{M'}M)}$$



(If keeping track of orientations, may require $\alpha(-M) = \overline{\alpha(M)}$ for some involution $\overline{}$ on R)

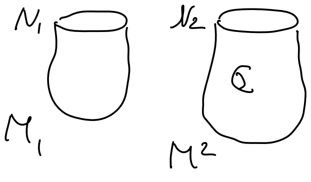
Define $\mathcal{L}(N) = \mathrm{Fr}(N) / \ker((,)_N)$

$x \in \ker((,)_N)$ if $\forall y$ $(x,y)_N = 0$

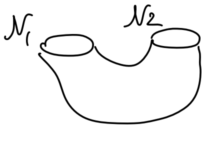
$\mathcal{L}(N)$ is the state space of N, an R-module

$$\mathcal{L}(N_1) \otimes \mathcal{L}(N_2) \xrightarrow{j} \mathcal{L}(N_1 \sqcup N_2)$$

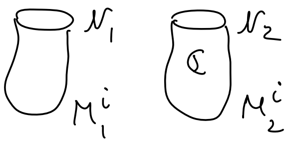
(lax tensor structure)



in general



$\neq \sum_i a_i$



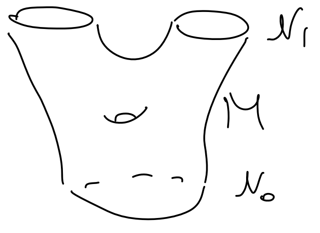
in $\mathcal{L}(N_1 \sqcup N_2)$

j is injective (if R is a field), usually <u>not</u> surjective

Weaker than Atiyah's TQFT, where j is an isomorphism.

Functoriality:

M induces an R-linear map



$\partial M = (-N_0) \sqcup N_1$

$[M_1]: \mathcal{L}(N_0) \longrightarrow \mathcal{L}(N_1)$

$[M_2]: \mathcal{L}(N_1) \longrightarrow \mathcal{L}(N_2)$

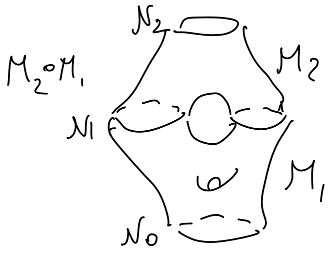
$$\begin{array}{rcl} [M]: \mathcal{L}(N_0) & \longrightarrow & \mathcal{L}(N_1) \\ \rotatebox{90}{$\in$} & & \\ {[M']} & \longrightarrow & [MM'] \end{array}$$

compose M' with M along N_0

well-defined, R-linear

respects composition

$$[M_2 \circ M_1] = [M_2] \circ [M_1]$$



get a functor

(n-1)-manifold $N \longrightarrow \mathcal{L}(N)$

n-manifold $M \longrightarrow [M]: \mathcal{L}(N_0) \to \mathcal{L}(N_1)$

$$\mathrm{Cob}_n \longrightarrow \text{R-mod}$$

Categ. of n-dim cobordisms

R-modules

call it a topological theory

Interesting case: $\alpha(N)$ is a finite-rank R-mod, $\forall N$.

Skein relations $\quad \lambda_i \in R$

$$\sum_i \lambda_i [M_i] = 0 \text{ in } \alpha(N) \text{ iff } \forall$$

$\partial M' = N$

$$\sum_i \lambda_i \, \alpha(M' M_i) = 0 \in R$$

Today, we discuss case $n=1$.

$\alpha(\bigcirc) = \lambda \in R$ single parameter λ. Get (oriented or unoriented) Brauer algebra or category and its quotient by negligible morphisms.

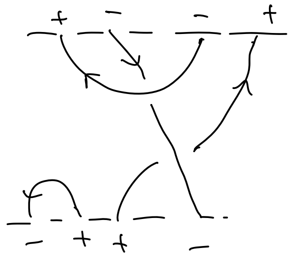
To spice it up, we make 3 changes:

I) Count interval as a closed 1-manifold

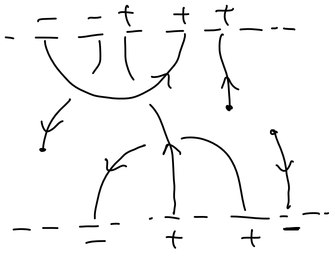
$\alpha(\bullet\!\leftarrow\!\bullet) = \mu$, $\alpha(\circlearrowleft) = \lambda$ (rook Brauer algebra & category & negligible quotients)

2 parameters, more general colordisms

Before:



Now



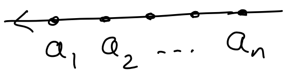
"outer" endpoints

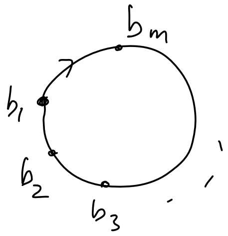
"inner" endpoints

(intersections virtual)

II Allow defects

Σ - finite set of defects (letters)





$a_i, b_j \in \Sigma$

III Change from commutative ring or field R to Boolean semiring

$\mathbb{B} = \{0, 1 \mid 1+1=1\}$
"false", "true"

addition, multiplication,
no subtraction

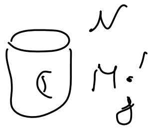
$0+0=0$
$0+1=1$
$1+1=1$
$0 \circ 0=0$
$0 \circ 1=0$
$1 \circ 1=1$

Need to change from linear ☺
to set-theoretic setup ☹ (harder)

when defining $\alpha(N)$, cannot just mod out by kernel of bilinear form. Instead, impose equivalence relation

$$\sum_i \lambda_i [M_i] \sim \sum_j \mu_j [M'_j]$$ iff for closure by any M

Cannot subtract!



over $\mathbb{B}$ can assume all $\lambda_i=1, \mu_j=1$

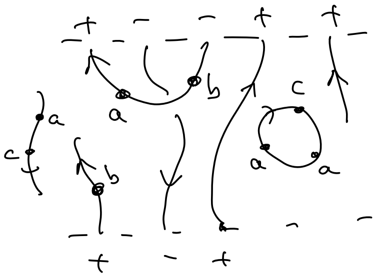
$$\sum_i \lambda_i \, \alpha(M M_i) = \sum_j \mu_j \, \alpha(M M'_j)$$

Cannot cancel
$[M_1]+[M_2]=[M_1]+[M_3]$
$\not\Rightarrow$
$[M_2]=[M_3]$

Recall finite set Σ of labels for dots (letters or defects)

1) Category C_Σ - objects are sequences of +, − (oriented 0-manifolds)

Morphisms are Σ-decorated oriented 1-cobordisms



morphism from $(+-+)$ to $(+--++)$

composition is concatenation.

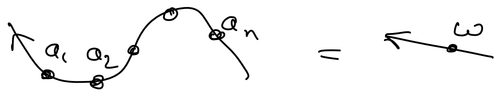
Cobordisms may 'end' in the middle ⇒ 2 types of boundary points: top/bottom (outer) and inner/floating.

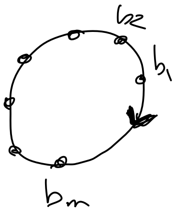
$a, b, c \in \Sigma$
(intersections are virtual)

Closed morphism is a hom from $\emptyset_0$ to $\emptyset_0$ (everything is floating)

Connected components:

$a_i \in \Sigma$





$\omega = a_1 a_2 .. a_n$ word

$\omega \in \Sigma^*$ free monoid on Σ

$b_1 .. b_m$ circular word

$\omega_1 \omega_2 \sim \omega_2 \omega_1$ rotational equivalence

$\omega_1, \omega_2 \in \Sigma^*$

each such must evaluate via α to 0 or 1 (el's of $\mathbb{B}$)

words

$\alpha_I : \Sigma^* \longrightarrow \mathbb{B}$

circular words

$\alpha_0 : \Sigma^* /_{\text{rotations}} \longrightarrow \mathbb{B}$

language L_I

$\omega \in L_I$ iff $\alpha_I(\omega) = 1$

circular language L_0

$\omega \in L_0$ iff $\alpha_0(\omega) = 1$

$$\alpha = (\alpha_I, \alpha_0) = (\underset{\substack{\uparrow \\ \text{language}}}{L_I}, \; L_0)$$

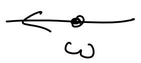
↖ rotationally-invariant language.

Language L is a subset of Σ^* (L is a set of words in alphabet Σ)

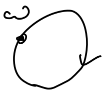
L_I, L_0 are, in general, unrelated.

2) Category C'_α (intermediate category)

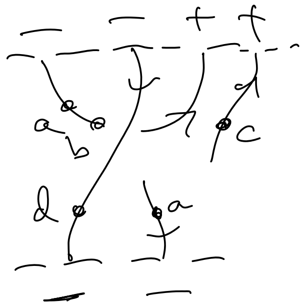
a) add relations to evaluate floating 1-manifolds



$= \alpha_I(\omega)$



$= \alpha_0(\omega)$

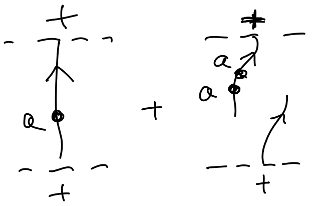


Morphisms reduce to those w/o floating components.

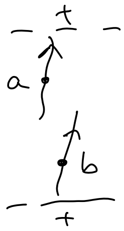
b) allow $\mathbb{B}$-(semi)linear combinations of morphisms.

$\mathbb{B} = \{0, 1 \mid 1+1=1\}$

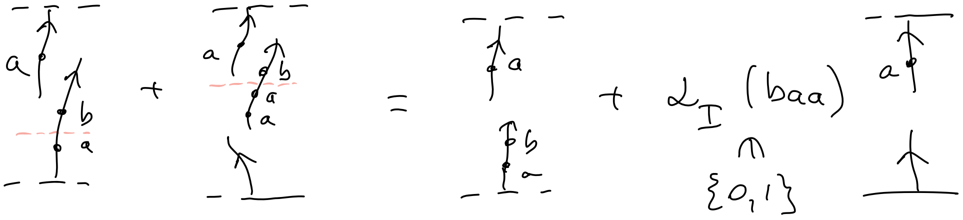
$x =$



$y =$



$yx =$

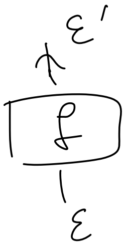


3) Mod out by the universal construction to get C_α

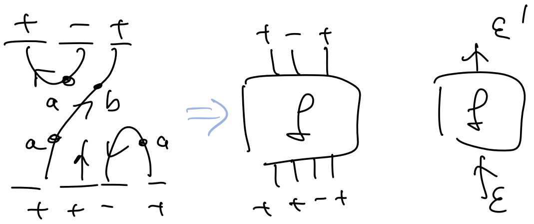
see R. Sazdanovic, M.K. arxiv 2020 for these categories in the linear case (+ extensions of the Deligne cat. Rep S_t)

$$\begin{tikzcd} C_\Sigma \arrow[r, "\text{evaluate floating components, } \mathbb{B}\text{-lin. combinations}"'] & C'_\alpha \arrow[r, "\text{equivalence relations (universal construction)}"'] & C_\alpha \end{tikzcd}$$

Σ-decorated 1-cobordisms



schematic notation for decorated 1-cobordism



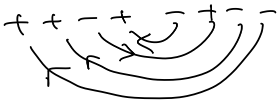
$\varepsilon' = (+ - +)$

$\varepsilon = (+ + - +)$

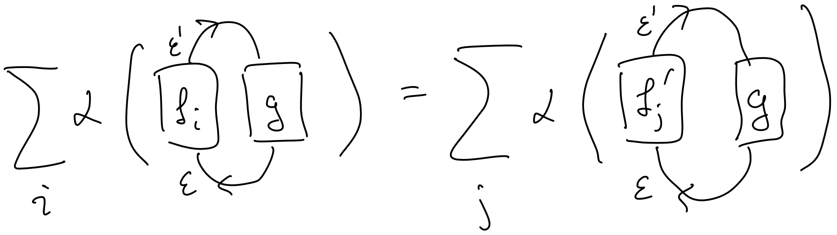
Quotient category C_α (equivalent to universal construction)

$$\sum_i \overset{\varepsilon'\uparrow}{\underset{\varepsilon\uparrow}{\boxed{f_i}}} = \sum_j \overset{\varepsilon'\uparrow}{\underset{\varepsilon\uparrow}{\boxed{f_j'}}} \quad \text{in } \mathrm{Hom}_{C_\alpha}(\varepsilon, \varepsilon') \quad \text{iff}$$

$\varepsilon = (+ + - +) \Rightarrow \varepsilon^* = (- + - -)$



$$\forall g \quad \overset{\varepsilon'^*\downarrow}{\underset{\varepsilon^*\downarrow}{\boxed{g}}} :$$

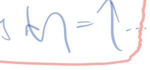


equality in $\mathbb{B}$.

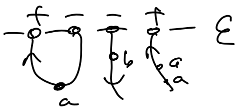
C_α : objects are ε, sequences of +, −

Morphisms : $\mathbb{B}$-semilinear combinations of decorated 1-cobordisms. Floating components evaluated via α + universal construction quotient.

Rigid: cups/caps $\curvearrowright$, $\curvearrowleft$ + isotopy



State space: $A(\varepsilon) := \mathrm{Hom}_{C_\alpha}(\emptyset_0, \varepsilon)$



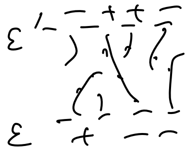
$\mathbb{B}$ - lin combinations / bilin form

- - - - - - $\emptyset_0$

C_α - <u>rigid</u> symmetric monoidal $\mathbb{B}$-semilinear category

$\alpha = (\alpha_I, \alpha_\circ) = (\angle_I, \angle_\circ)$

Call α <u>Cob-regular</u> if all hom spaces $\mathrm{Hom}_{C_\alpha}(\varepsilon, \varepsilon')$ are finite (= finitely-generated $\mathbb{B}$-modules)



(Finite) $\mathbb{B}$-semimodule M $\quad a+b$

$\mathbb{B} = \{0, 1\}$ $\quad 1+1=1 \Rightarrow a+a=a$ $\quad$ addition is idempotent

$0 \cdot a = 0$

M - idempotent abelian monoid under $+$

Partial order on M: $a \leq b \iff a+b=b$

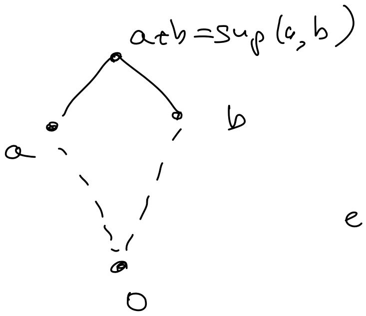
$a \vee b = \sup(a,b) = a+b$
$\uparrow$
least upper bound of a, b

$M \iff$ semilattice with 0
(poset)

$a \vee b = b \vee a$
$(a\vee b)\vee c = a \vee (b \vee c)$
$a \vee 0 = a = 0 \vee a$

$a \leq b \iff a \vee b = b$

$\mathbb{B}$-semimodules $\iff$ semilattices with 0.



Examples of semimodules
(from Boolean matrices)

embedding M into $\mathbb{B}^n$

$$\begin{array}{c|cccc} & x_1 & x_2 & x_3 & x_4 \\ \hline y_1 & 1 & 1 & 0 & 1 \\ y_2 & 1 & 0 & 1 & 1 \\ y_3 & 0 & 1 & 1 & 1 \end{array}$$

$x_1 + x_2 = x_1 + x_3 = x_2 + x_3 = x_4$

M, x_1, x_2, x_3 generators, relations

M is dual to the semimodule generated by rows

$$\begin{array}{c|cc} & x_1 & x_2 \\ \hline y_1 & 1 & 1 \\ y_2 & 1 & 0 \end{array}$$

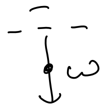
$x_1 + x_2 = x_1$ $\quad$ no cancellation

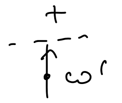
$M = \langle x_1, x_2 \mid x_1 + x_2 = x_1 \rangle$

$M = \{0, x_2, x_1\}$.

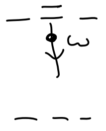
Given α evaluation, look at $A(+)$, $A(-)$.

$A(-) = \mathrm{Hom}_{C_\alpha}(\emptyset, -)$

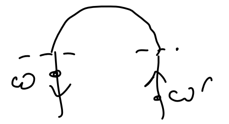




$A(+)$



$A(-) \times A(+) \longrightarrow \mathbb{B}$



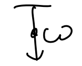
$\alpha_I(\omega\,\omega')$

$A(-)$ spanned by these diagrams

$\omega \times \omega' \longrightarrow \alpha_I(\omega\omega')$

$\mathcal{L}_I$ internal language

$\langle \omega | \in A(-)$, modulo relations

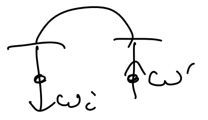
$\langle \omega | :=$ 

$\langle \emptyset | :=$  empty word

$A(-)$ = spanned by $\langle \omega |$ subject to relations

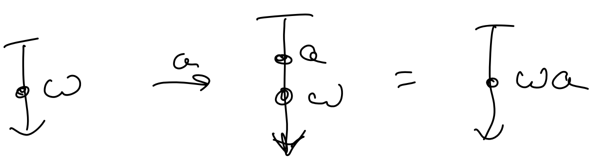
$\sum_i \langle \omega_i | = \sum_j \langle \omega_j |$ iff ∀ way to close up & evaluate ∀ word ω'

$\sum_i \alpha(\omega_i \omega') = \sum \alpha(\omega_j \omega')$



$\xrightarrow{\alpha} \alpha_I(\omega_i \omega')$

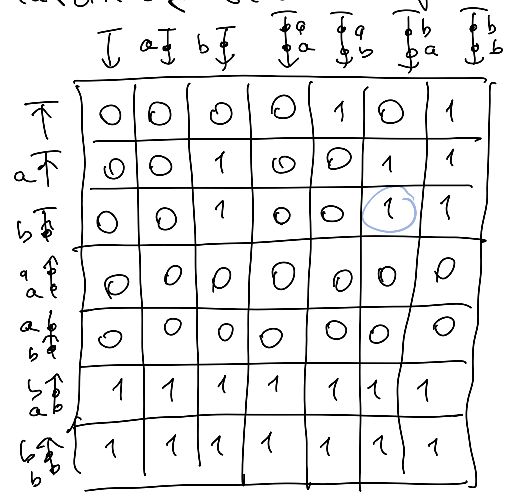
$A(-)$ is a $\mathbb{B}$-semimodule with an action of Σ^*



$$L_I = (a+b)^* b (a+b)$$

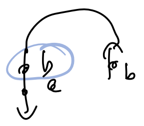
<u>Example</u> $L_I = \{ \omega \in \{a,b\}^* \mid$ 2nd to last letter is $b \}$

Matrix of bilinear pairing



0	0	0	0	1	0	1
0	0	1	0	0	1	1
0	0	1	0	0	1	1
0	0	0	0	0	0	0
0	0	0	0	0	0	0
1	1	1	1	1	1	1
1	1	1	1	1	1	1
x	x	y	x	z	y	w

$w = y+z$



abaaba
bbba $\in L_I$
aabb

$$A(-) = \left\langle x, y, z \;\middle|\; \begin{matrix} x+y=y \\ x+z=z \end{matrix} \right\rangle$$

$y+z$ not irreducible

$$A(-) = \{0, x, y, z, y+z\}$$

Let $Q = \{ \langle \omega|, \ \omega \in \Sigma^* \} \subset A(-)$

$\langle \omega| = \omega \downarrow$

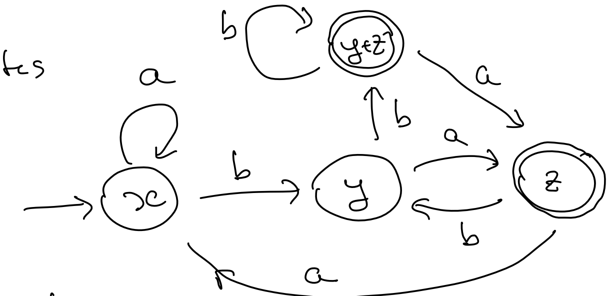
x, y, z are <u>irreducible</u> α's

$u \neq 0$, $u = a+b \Rightarrow a = u$ or $b = u$

$u = u+u$

Set Q gives rise to the minimal <u>deterministic</u> <u>finite</u> <u>automaton</u> that accepts language L_I

abba terminates at z



→○ initial state

◎ accepting states

$$\alpha\left(\downarrow \omega\right) = 1$$

irreducible α's of $A(-)$ are always in Q.

$0 \leftarrow$ not in automaton, but in $A(-)$

In general, $Q \subset A(-)$ can be a much smaller subset. In above example, $0 \notin Q$ since L_I does not have <u>unrecoverable</u> words ω : $\omega\omega\omega' \notin L_I \ \forall \omega'$

minimal DFA for L_I sits inside $A(-)$
as set $Q = \{\langle \omega | \mid \omega \in \Sigma^*\}$

<u>Review</u>: regular languages & finite state automata.

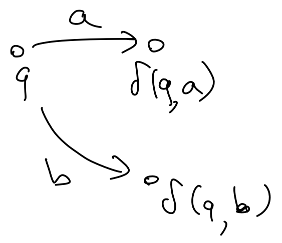
Deterministic FSA for alphabet Σ is a (Q, δ, q_{in}, Q_t)

Q - finite set of states

$q_{in} \in Q$ - initial state

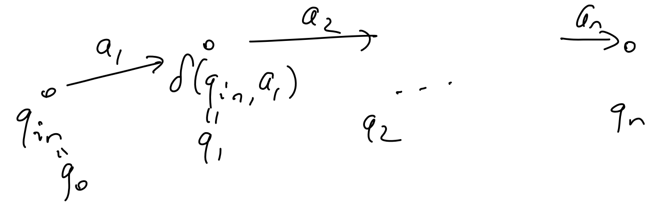
$\delta : Q \times \Sigma \longrightarrow Q$ in state a, read letter $a \in \Sigma$
transition function $\Rightarrow$ go to state $\delta(q, a)$

Q_t - set of accepting states $Q_t \subset Q$



given a word $\omega = a_1 a_2 \ldots a_n$,

start in q_{in}



$q_{i+1} = \delta(q_i, a_{i+1})$

if $q_n \in Q_t$, $\omega \in L$

if $q_n \notin Q_t$, $\omega \notin L$

L is regular if it's accepted by some finite state automaton.

Another characterization of regular languages:

Smallest set of languages that

(a) contains all finite languages

(b) closed under sum (union) $L_1 + L_2 = L_1 \cup L_2 = \{\omega \mid \omega \in L_1 \text{ or } \omega \in L_2\}$ and
product $L_1 L_2 = \{\omega_1 \omega_2 \mid \omega_1 \in L_1, \omega_2 \in L_2\}$

(c) with L contains $L^* = \emptyset + L + LL + \ldots$ - all concatenations of words in L.
(star closure of L) $L^* = \sum_{n=0}^{\infty} L^n$

$\emptyset$ empty sequence

Thm L_I is regular $\iff$ $\exists$ a deterministic FSA for L_I $\iff$ $A(-)$ (or $A(\varepsilon)$) is a finite $\mathbb{B}$-semimodule.

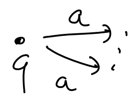
Nondeterministic FSA: $(\underset{\substack{\uparrow \\ \text{states}}}{Q}, \underset{\substack{\uparrow \\ \text{transition}}}{\delta}, Q_{in}, Q_t)$

$Q_{in} \subset Q$ subset init states

$Q_t \subset Q$ subset accepting states

$Q \times \Sigma \xrightarrow{\delta} P(Q)$ ← powerset of Q

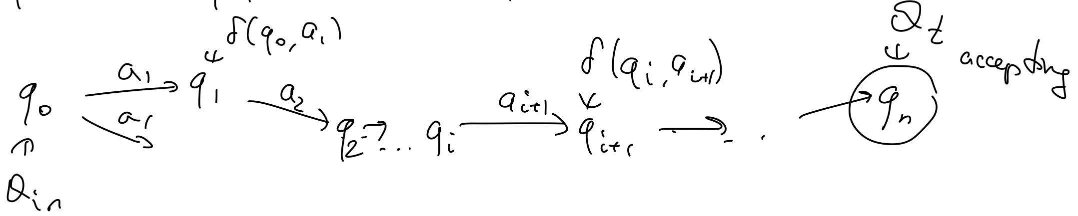
from a state can go to any state in $\delta(q,a) \subset Q$



$\omega \in L$ if for some

$\overset{\shortparallel}{a_1 \dots a_n}$

$q_0 \dots q_n$

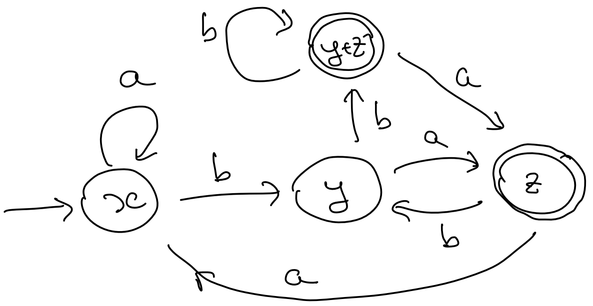


Thm (Classic)

L regular $\iff$ L accepted by some DFSA $\iff$ L accepted by some NFSA.

Nondeterministic FSA for a language L can have exponentially smaller states than the smallest deterministic FSA for L

$L_I = (a+b)^* b (a+b)$, see earlier



$$A(-) = \langle x, y, z \mid x+y=y,\ x+z=z \}$$

← deterministic FA

$m_a : A(-) \to A(-)$

$\langle w| \mapsto \langle wa|$

$a \in \Sigma$ $\quad \widetilde{\langle \emptyset|}$ lift initial state

$\cap$

$m_a \circlearrowright \mathbb{B}^J$ free $\mathbb{B}$-(semi)module, action of Σ^* free semimodule

$J \supset irr(A(-))$

$\downarrow$

$m_a \circlearrowright A(-)$ $\mathbb{B}$-semimodule, action of Σ^* semimodule

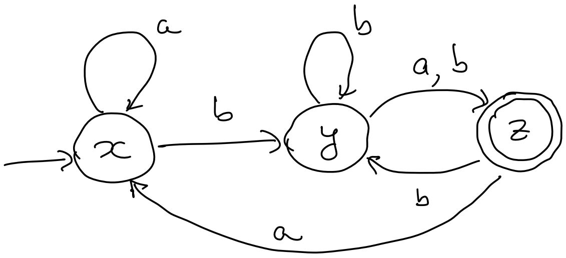
$\cup$

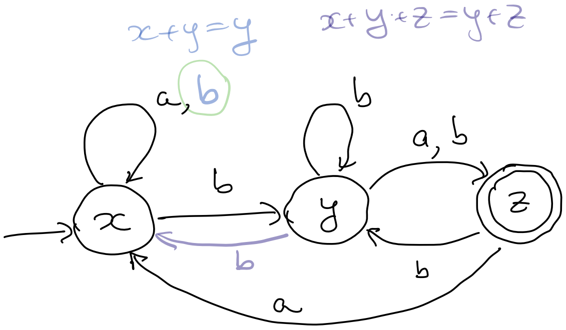
$m_a \circlearrowright Q$ states of minimal DFA for L_I set

$\in$

$\langle \emptyset|$

$J = \{x, y, z\}$





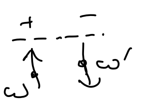
For min # of states, take $J = irr(A(-))$

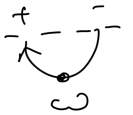
If want unique q_{in}, add $\langle\emptyset|$ to J if $\langle\emptyset|$ is not irreducible.

Minimal NFA is not unique, in general

Nonuniqueness is due to many ways of lifting action of Σ from $A(-)$ to $\mathbb{B}^J$.

Beyond $A(-)$, $A(+)$

$A(+-)$: 2 types of diagrams 



and their $\mathbb{B}$-linear combinations.

$A(+-) \supset A(+) \otimes A(-)$ usually a proper embedding

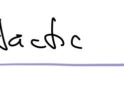
$\otimes$ is a bit tricky with semimodules (usually want an explicit realization, i.e. via embedding into free semimodules).

No neck-cutting, in general, and gets complicated.

We don't know how to write down a set of defining relations in C_α for some simple pairs (L_I, L_0) even for one-letter languages

<u>Example</u> Let $L_0 = \emptyset$ (empty language, no words) 

$\xrightarrow{\alpha} 0$ $\forall \omega$

Then $\{$ 

$\}_\omega$ is the <u>syndactic monoid</u> of L_I. (subset of $A(+-)$)

In standard Modern Algebra course we spend a semester studying (finite) groups, but rarely cover (finite) monoids. Possible reasons:

(Finite) groups have better structural theory, deep connections to number theory, topology, etc.

Symmetries vs non-invertible maps.

<u>Another reason</u>: they are secretly studied in CS courses

Each language L gives rise to its syntactic monoid:

$\Sigma^*/\sim$ $\quad w_1 \sim w_2$ iff $\forall$ words x, y

$x w_1 y$, $x w_2 y$ are either both in L or not in L.

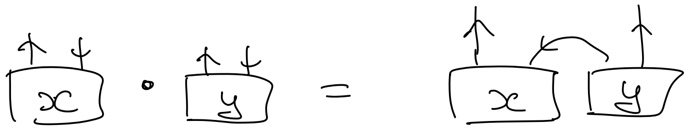
Prop L is regular $\Longleftrightarrow$ its syntactic monoid is finite.

Prop Syntactic monoid $(L) \subset A(+-)$ as $\{$  $\}_\omega$

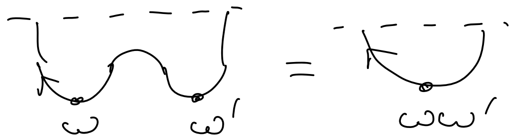
if $L_0 = \emptyset$

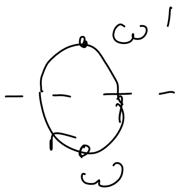
$A(+-)$ is a unital semiring.

$1 =$ 

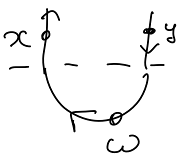


associative unital multiplication





$\xrightarrow{\alpha} 0$



$\xrightarrow{\alpha} \alpha_I(x \omega y)$

$A(+-) \supset$ semiring spanned by  $\supset$ syntactic monoid (L_I).

if $L_0 \neq \emptyset$, only get a surjective map onto syntactic monoid.

M finite $\mathbb{B}$-semimodule $\iff$ M finite semilattice with 0 $\iff$ M finite lattice

0, a+b

a+a=0

comm, assoc.

0, sup(a, b), = a+b
$a \vee b$

inf(a,b)
$a \wedge b$ $:= \sum_{\substack{c \leq a \\ c \leq b}} c$

$0 \substack{\leq a \\ \leq b}$

L(M) - lattice of M

L(M)=M as sets

$1 = \sum_{a \in M} a$

$a \wedge b = b \wedge a$

$(a \wedge b) \wedge c = a \wedge (b \wedge c)$

$a \wedge (a \vee b) = a$

$a \vee (a \wedge b) = a$

$\mathbb{B}$ – fmod. Category of finite $\mathbb{B}$-semimodules

Hom (M, N)

$f: M \to N$

$f(0)=0$, $f(a+b)=f(a)+f(b)$

A retract of a free module

$$M \xrightarrow{i} \mathbb{B}^n \xrightarrow{p} M \qquad pi = id_M$$

usually not a direct summand

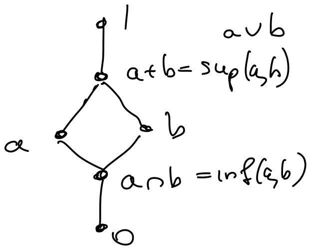
i: $x_1 \mapsto \binom{1}{0}$, $x_2 \mapsto \binom{1}{1}$

p: $\binom{1}{0} \mapsto x_1$

$\binom{0}{1} \mapsto x_2$

$M = \{x_1, x_2 \mid x_2 = x_1 + x_2\}$ $\quad M \xrightarrow{i} \mathbb{B}^2 \xrightarrow{p} M$

3 elements 4 elements



Thm (Hofmann, Mislove, Stralka 1974) TFAE:

1) M is projective in $\mathbb{B}$-fmod

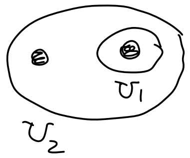
2) M is injective in $\mathbb{B}$-fmod

3) M is a retract of a free module

4) Lattice of M is distributive: $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

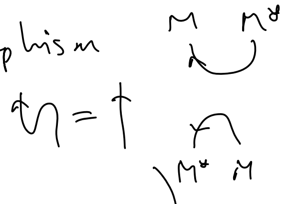
Birkhoff thm $\updownarrow$

5) M is the lattice of open sets of a finite top space X



$U_1 + U_2 = U_2$

Also, M-projective $\Rightarrow$ $\exists$ morphism

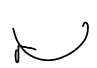


$\mathbb{B} \to M \otimes M^*$

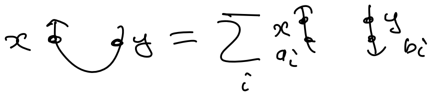
s.t

(usually, $M^* \otimes M \longrightarrow Hom(M, M)$ is not an isomorphism even for finite M)

Given a regular $\alpha = (L_I, L_o)$, assume $A(-)$ is a distributive lattice (a projective $\mathbb{B}$-module)

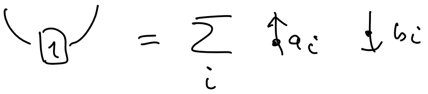
Then can write "" $:= \sum_{i=1}^{n} a_i \otimes b_i$ $\quad a_i \in A(+)$, $b_i \in A(-)$

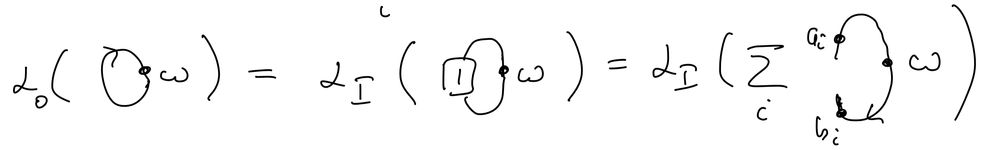
meaning

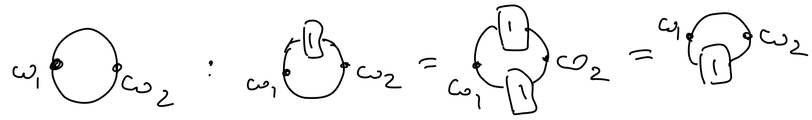


(ignoring L_o)

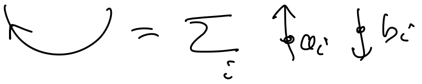
$$\alpha(xy) = \sum_i \alpha(xa_i)\,\alpha(yb_i)$$

Define  and "decomposition of identify"





$L_I \rightsquigarrow L_o$

Then  holds in C_α (only for this α_o).

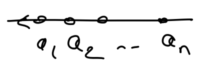
$$A(\varepsilon\varepsilon') \cong A(\varepsilon) \otimes A(\varepsilon')$$

(isom, not just inclusion)

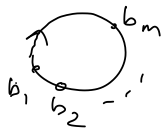
<u>Prop</u> Let L_I be a regular language and assume $A(-)$ is a distributive lattice. Then, for L_o as above, α gives rise to a $\mathbb{B}$-valued TQFT, with tensor product decomposition for state spaces.

Summary

1) Category C_Σ of Σ-decorated oriented 1-cobordisms. Choose multiplicative evaluation α of "closed" cobordisms



$$\xrightarrow[\alpha_I]{\alpha} \mathbb{B}$$



$$\xrightarrow[\alpha_0]{\alpha} \mathbb{B}$$

Equivalently, choose a language L_I and a circular language L_0

2) Form category C_α of $\mathbb{B}$-lin. combinations of cobordisms modulo relations from the universal construction for α

Thm Hom spaces in C_α are finite iff L_I, L_0 are regular languages

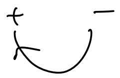
Equivalently, state spaces $A(\varepsilon) = \mathrm{Hom}_{C_\alpha}(\emptyset, \varepsilon)$ are finite.

depends on L_I only

$A(-)$ spanned by $\overset{-}{\downarrow} w$, $w \in \Sigma^*$

$$\begin{tikzcd} \text{min NFA} & \Sigma \circlearrowright \mathbb{B}^J \arrow[d] & \text{free } \mathbb{B}\text{-semimodule} \\ & \Sigma \circlearrowright A(-) \arrow[d, phantom, "\cup"] & \text{state space, } \mathbb{B}\text{-semimodule} \\ \text{min DFA} & \Sigma \circlearrowright Q & \text{"}\Sigma^*\text{-orbit" of } \langle \emptyset | \end{tikzcd}$$

Thm $A(-)$-projective $\mathbb{B}$-semimodule $\Leftrightarrow \exists!$ circular language L_0 s.t. (L_I, L_0) is a $\mathbb{B}$-valued TQFT.



$$= \sum_i \overset{+}{\uparrow} a_i \; \overset{-}{\downarrow} b_i \qquad \begin{array}{l} A(+-) \simeq A(+) \otimes A(-) \\ A(\varepsilon\varepsilon') \simeq A(\varepsilon) \otimes A(\varepsilon') \end{array}$$

From (L_I, L_0) get an entire rigid symmetric monoidal category C_α

Min DFA, NFA for L_I are captured by $A(-)$. Syntactic monoid $\simeq A(+-)$.

Relation to lattices/semilattices. TQFT if have decomposition of identity.

THANK YOU!

LET US THANK THE ORGANIZERS OF THE SERIES!