An Algebraic Proof Of A Quadratic Relation In MICZ-Kepler Problem

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1 Introduction

1.1 Background Materials

In his articles [3][4][5][6] on the study of the quantum dynamical symmetries of a family of generalized MICZ-Kepler problems, Meng discovered a family of quadratic relations that characterize the representations occurring both in the gauge group representations and the representations of the total Hilbert spaces of these dynamical systems. And these representational results were summarized in the main theorems of Ref. [2].

To describe these quadratic relations, we first review some basic concepts about Clifford algebras and spinor groups.

Let $\mathbb{R}^{p,q}$ be the Euclidean space equipped with the standard (p, q)-form $[\eta_{\mu\nu}]$, which under the standard basis $\{x_\mu = (0, ...0, 1, 0, ...0)|1$ occurs in the μ -th place, $\mu = 1, ..., p + q\}$ of $\mathbb{R}^{p,q}$ takes the form:

$$diag(\underbrace{1, ..., 1}_{p}, \underbrace{-1, ..., -1}_{q}).$$

Let $C_{p,q}$ be the Clifford algebra over $\mathbb{C}$ subject to the relations:

$$x_\mu x_\nu + x_\nu x_\mu = -2\eta_{\mu\nu} \tag{1.1}$$

Denote $M_{\mu\nu} \doteq \frac{\sqrt{-1}}{4}\{x_\mu, x_\nu\}$, and we can check that these operators satisfy the following commutator relations:

$$[M_{\alpha\beta}, M_{\gamma\delta}] = -\sqrt{-1}(\eta_{\beta\gamma}M_{\alpha\delta} - \eta_{\alpha\gamma}M_{\beta\delta} - \eta_{\beta\delta}M_{\alpha\gamma} + \eta_{\alpha\delta}M_{\beta\gamma}).$$

Thus it's readily seen that we have a copy of the Lie algebra $so(p, q)$ embedded in the Clifford algebra, which is the algebra generated by the above $M_{\mu\nu}$'s. Meanwhile there are also copies of the corresponding groups $Spin(p, q)$ inside the Clifford algebras.

In case p or $q = 0$, these spinor groups are known to be compact, and their irreducible unitary representations are well known to be all finite dimensional and have a 1-1 correspondence with the dominant weights.

In case both p and q are nonzero, these groups and their covering groups , denoted by G, are known to be noncompact. And the non-trivial unitary representations of these groups are well-known to be infinite dimensional. In these cases we use K to denote a maximal compact subgroups of these non-compact groups. Harish-Chandra established a 1-1 correspondence between the irreducible unitary representations of G and the irreducible $(\mathfrak{g}, K)$-modules, where $\mathfrak{g}$ stands for the complexified Lie algebra of G. Moreover, recall that a representation of G is called a highest weight module if the underlying $(\mathfrak{g}, K)$-module is a highest weight $\mathfrak{g}$-module. The unitary highest weight $(\mathfrak{g}, K)$-modules are classified by Enright, Howe, Wallach [7] and Jakobsen [8, 9] independently.

In the article [2], Meng proposed and studied the following quadratic relations in the universal enveloping algebras $U(\mathfrak{g})$:

$$\sum_\lambda (M_{\mu\lambda}M^\lambda{}_\nu + M^\lambda{}_\nu M_{\mu\lambda}) = c \cdot \eta_{\mu\nu}.$$

Here the constant c depends only on the representations involved and can be calculated using the quadratic Casimir element. Then Meng proved in the same article that:

Theorem 1.1. Let $n > 0$ be an integer.

(1) An irreducible unitary module of $Spin(2n + 1)$ satisfies the above quadratic relations $\Leftrightarrow$ it's either the trivial representation or the fundamental spinor representation,i.e. the one with highest weight $|\frac{1}{2}, ..., \frac{1}{2}\rangle$;

(2) An irreducible unitary module of $Spin(2n)$ satisfies the above quadratic relations $\Leftrightarrow$ it's a Young power of a fundamental spin representation, namely those with highest weight $||\mu|, ..., |\mu|, \mu\rangle$ where $\mu \in \frac{1}{2}\mathbb{Z}$.

(3) An irreducible unitary $(\mathfrak{so}(2, 2n+1), SO(2)\times SO(2n+1))$-module satisfies the above quadratic relations $\Leftrightarrow$ it's either the trivial one or the one with highest weight $|-(n + \mu - \frac{1}{2}), \mu, ..., \mu\rangle$, where $\mu = 0$ or $1/2$.

(4) An irreducible unitary $(\mathfrak{so}(2, 2n), SO(2) \times SO(2n))$-module satisfies the above quadratic relations $\Leftrightarrow$ it's either the trivial one or the one with highest weight $|-(n + \mu - \frac{1}{2}), \mu, ..., \mu\rangle$, where $\mu \in \frac{1}{2}\mathbb{Z}$

In that paper, Meng gave direct algebraic proofs for part (1) and (2) of the theorem. As for part (3) and (4), he referred to his earlier articles on the MICZ-Kepler problems [5, 6], in which these specific representations are explicitly realized as the Hilbert space of bound states of the generalized hydrogen atoms with magnetic monopoles , whose dynamical symmetry are precisely the corresponding symmetry groups of those algebras in (3) and (4). Moreover, these quadratic relations are explicitly checked using these models by identifying the operators $M_{\mu\nu}$ with a family of differential operators which stand for certain conserved physical quantities.

Representation theoretically, Meng observed [1] that the representations occurred in part 3) are precisely those Wallach representations in Case II ($\mu = 0$) and Case III ($\mu = 1/2$) of the $(\mathfrak{g}, K)$- modules, as on page 128 of Ref. [7]. The representations in part 4) are precisely those Wallach representations in Case II ($\mu = 0$), Case III ($\mu > 0$) and the mirror of Case III of the $(\mathfrak{g}, K)$- modules, as on page 125 of the same article. In the Enright-Howe-Wallach classification diagram for the unitary highest weight modules, there are two reduction points in Case II and one reduction point in Case III. The non trivial representation always sits as the first reduction point, and the trivial representations, which occurs in Case II only, always sits on the second reduction point. In other words, the nontrivial representations in the above theorem are precisely those boundary Wallach points in Case II, III and mirror of Case III.

1.2 Outline of this paper

In the winter of 2007, Meng suggested to the author that he may try to investigate the symplectic case to see whether there is such similar quadratic relations that characterize the harmonic oscillator representations. In trying to investigate this case, the author used direct algebraic computations to reprove the above part (3) and (4) of Theorem 1.1 and also tried for the symplectic cases. Yet there does not seem to be a direct translation of the corresponding quadratic relations that can be used to characterize the harmonic oscillator representations. Instead, we proposed a finer system of quadratic operators which formally satisfy the formal properties of a (super)Riemannian curvature. We found out that the original quadratic relations can be in these languages described as a system of formal "Einstein equations". And so far the only interesting result we obtained in the symplectic case is that the harmonic oscillator representations satisfy the formal "homogeneous manifold"-like equation.

The paper is organized as follows.

In chapter 2, we give a direct algebraic proof of part (3) of Theorem 1.1. The strategy of proof is to reduce the system of quadratic relations to proving only one of them, and then it is checked computationally.

In chapter 3, similar algebraic proofs are carried out for part (4) of Theorem 1.1 and the above mentioned "curvature-type" operators are introduced. We further point out that the original quadratic relations are in this case a system of Einstein equations.

In chapter 4, we investigate the symplectic case and introduce a finer type of quadratic relation that characterizes the harmonic oscillator representations. Furthermore, the "curvature type" operators are introduced and we observe that the new quadratic relations are in this language a "homogeneous manifold"-like condition. Lastly we point out that the Einstein equations in this case are not sufficient to characterize the harmonic oscillator representations.

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[1] c. f. [2] Remark 1.4

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2 The Noncompact Odd Dimensional Case: $\mathfrak{so}(2, 2n+1)$

In this chapter, we will give the proof of the main theorem in the odd dimensional case $\mathfrak{so}(2, 2n+1)$. One side of the proof is done by Meng in his previous paper [2], and is included here for the sake of completeness. The converse is proved as follows: we try to reduce the verification of all $(2n+3) \times (2n+3)$ identities to one particular identity by a symmetry argument, and then we verify that the operator in this equation kills a generating set of vectors in these specific modules.

2.1 Review of Some General Facts

To prove the main theorem in this case, we first review some general facts about the special orthogonal Lie algebra $\mathfrak{so}(2, 2n+1)$. Here we adopt the usual convention of physicists, as can be found in the standard textbook [1].

Recall that the root space of $\mathfrak{so}(2, 2n+1)$ is $\mathbb{R}^{n+1}$, with the standard basis $\{e^0, ..., e^n\}$. The roots of $\mathfrak{so}(2, 2n+1)$ are $\{\pm e^i \pm e^j | 0 \le i < j \le n\}$ and $\{\pm e^k | 0 \le k \le n\}$. As usual, we choose the positive roots to be $\{e^i \pm e^j | 0 \le i < j \le n\}$, together with $\{e^k | 0 \le k \le n\}$. The associated simple roots are then $\{e^0 - e^1, ..., e^{n-1} - e^n, e^n\}$.

A Cartan basis can be chosen as follows (c. f. [2]):

$$\begin{cases} H_0 &= M_{-1,0} \\ H_i &= -M_{2i-1,2i}, \quad 1 \le i \le n \\ E_{\eta e^j + \zeta e^k} &= \frac{1}{2}(M_{2j-1,2k-1} + \sqrt{-1}\eta M_{sj,2k-1} + \sqrt{-1}\zeta M_{2j-1,2k} - \eta\zeta M_{2j,2k}), \\ & \quad 0 \le j < k \le n \\ E_{\eta e^j} &= \frac{1}{\sqrt{2}}(M_{2j-1,2n+1} + \sqrt{-1}\eta M_{2j,2n+1}) \end{cases}$$

The transition from the Clifford algebra interpretation to the root vectors is given by:

$$\begin{cases} M_{2j-1,2k-1} &= \frac{1}{2}(E_{e^j+e^k} + E_{e^j-e^k} + E_{-e^j+e^k} + E_{-e^j-e^k}) \\ M_{2j,2k-1} &= \frac{1}{2\sqrt{-1}}(E_{e^j+e^k} + E_{e^j-e^k} - E_{-e^j+e^k} - E_{-e^j-e^k}) \\ M_{2j-1,2k} &= \frac{1}{2\sqrt{-1}}(E_{e^j+e^k} - E_{e^j-e^k} + E_{-e^j+e^k} - E_{-e^j-e^k}) \\ M_{2j,2k} &= \frac{1}{2}(-E_{e^j+e^k} + E_{e^j-e^k} + E_{-e^j+e^k} - E_{-e^j-e^k}), \end{cases}$$

where $0 \le j < k \le n$; also

$$\begin{cases} M_{2j-1,2n+1} &= \frac{1}{\sqrt{2}}(E_{e^j} + E_{-e^j}) \\ M_{2j,2n+1} &= \frac{1}{\sqrt{-2}}(E_{e^j} - E_{-e^j}), \end{cases}$$

where $0 \le j \le n$; finally, we have:

$$\begin{cases} M_{-1,0} &= H_0 \\ M_{2i-1,2i} &= -H_i, \end{cases}$$

where $1 \le i \le n$.

2.2 Proof of One Side

We start with one side of the proof, which is already done by Meng in his paper [2]. We include it here for the sake of completeness.

Observe that in a unitary highest weight $(\mathfrak{so}(2, 2n+1), SO(2) \times SO(2n+1))$-module, the operators M_{ij}'s act as unitary operators. From definitions, we see that each H_j acts as a hermitian operator, while the root vectors satisfy $E_\alpha^\dagger = E_{-\alpha}$. Thus in a unitary highest weight module with highest weight $|\Omega\rangle = |\lambda_0, \lambda_1, ..., \lambda_n\rangle$,we have the following:

Lemma 2.1. The highest weight satisfies:

$$-\lambda_0 \geq \lambda_1 \geq ... \geq \lambda_n.$$

Moreover, $\{\lambda_i|(i = 1...n)\}$ are all half integers and their differences $\{(\lambda_1 - \lambda_2), ..., (\lambda_{n-1} - \lambda_n)\}$ are all integers.

Proof. For later uses also, we note that the Cartan basis $\{H_i, E_\alpha\}$ satisfies the following commutator relations:

$$\begin{cases} [E_{e^0+e^i}, E_{-e^0-e^i}] = -H_0 - H_i \;\; [E_{e^0-e^i}, E_{-e^0+e^i}] = -H_0 + H_i \\ [E_{e^i+e^j}, E_{-e^i-e^j}] = H_i + H_j \;\; [E_{e^i-e^j}, E_{-e^i+e^j}] = H_i - H_j \\ [E_{\eta e^i}, E_{\zeta e^j}] = -\sqrt{-1}E_{\eta e^i+\zeta e^j} \;\; [E_{e^i}, E_{-e^i}] = H_i \\ [E_{e^0}, E_{-e^0}] = -H_0, \;\; [E_{e^0+e^k}, E_{-e^k}] = iE_{e^0} \\ [E_{e^0+e^k}, E_{-e^k+e^l}] = \sqrt{-1}E_{e^0+e^l} \;\; [E_{e^k+e^l}, E_{-e^l+e^j}] = \sqrt{-1}E_{e^k+e^j} \end{cases}$$

By evaluating the equation, $[E_{e^0+e^i}, E_{-e^0-e^i}] = -H_0 - H_i$, on the highest weight vector $|\Omega\rangle$, we obtain:

$$0 \leq ||E_{-e^0-e^i}|\Omega\rangle||^2 = \langle\Omega|[E_{e^0+e^i}, E_{-e^0-e^i}]|\Omega\rangle = \langle\Omega|H_0 - H_i|\Omega\rangle = -\lambda_0 - \lambda_i$$

Similarly by evaluating the highest weight vector at $[E_{e^i}, E_{-e^i}] = H_i(i \geq 1)$, $[E_{e^0}, E_{-e^0}] = -H_0$ and $[E_{e^i-e^j}, E_{-e^i+e^j}] = H_i - H_j$ we obtain $\lambda_i \geq 0 (i \geq 1)$, $-\lambda_0 \geq 0$ and $\lambda_i - \lambda_j \geq 0$.

Furthermore, the vectors $\{E_{e^i}, E_{-e^i}, H_i\}, i = 1, ..., n$ constituting a copy of $\mathfrak{su}(2)$ shows that the λ_i's are half integers. Similarly $\{E_{\pm e^i \pm e^j}, E_{\mp e^i \mp e^j}, 1/2(\pm H_i \mp H_j)\}$ forming copies of $\mathfrak{su}(2)$ shows that the differences $\lambda_i - \lambda_j$ are integers. □

Now we prove:

Proposition 2.2. An irreducible unitary highest weight $(\mathfrak{so}(2, 2n+1), SO(2)\times SO(2n+1))$-module satisfying the quadratic relations can only be those with highest weights $| - (n + \mu - \frac{1}{2}), \mu, ..., \mu\rangle$, where $\mu = 0$ or $\mu = \frac{1}{2}$, or it is the trivial representation.

Proof. The quadratic relations in the special cases $\mu = \nu = -1, ...2j - 1, ...2n + 1$ read:

$$\begin{cases} \langle\Omega| - \sum_k M_{-1,k}M^k{}_{-1}|\Omega\rangle = const. \\ \langle\Omega| \sum_k M_{2j-1,k}M^k{}_{2j-1}|\Omega\rangle = const. \;\; (j = 1, ..., n) \\ \langle\Omega| \sum_k M_{2n+1,k}M^k{}_{2n+1}|\Omega\rangle = const. \end{cases}$$

Plugging in the above transformation relations, and noticing that since we are evaluating on the highest weight vector, we may well omit those terms occurring in the computation of the form $E_{-\alpha} \cdot * + * \cdot E_\beta$, where α and β are positive roots. Then from the above equations, we obtain:

$$\begin{aligned} c &= \langle\Omega| - \textstyle\sum_k M_{-1,k}M^k_{,-1}|\Omega\rangle \\ &= \langle\Omega|M^2_{-1,0} - \textstyle\sum_{1\leq k\leq n} M^2_{-1,2k-1} - \sum_{1\leq k\leq n} M^2_{-1,2k} - M^2_{-1,2n+1}|\Omega\rangle \\ &= \langle\Omega|H_0^2 - (\tfrac{1}{4}\textstyle\sum_k(E_{e^0+e^k} + E_{e^0-e^k} + E_{-e^0+e^k} + E_{-e^0-e^k})^2 \\ &\quad -\tfrac{1}{4}\textstyle\sum_k(E_{e^0+e^k} - E_{e^0-e^k} + E_{-e^0+e^k} - E_{-e^0-e^k})^2 + \tfrac{1}{2}(E_{e^0} + E_{-e^0})^2)|\Omega\rangle \\ &= \langle\Omega|H_0^2 - \tfrac{1}{2}[E_{e^0}, E_{-e^0}] - \tfrac{1}{2}\textstyle\sum_k(\{E_{e^0+e^k}, E_{e^0-e^k}\} + \{E_{e^0+e^k}, E_{-e^0-e^k}\} \\ &\quad +\{E_{e^0-e^k}, E_{-e^0+e^k}\} + \{E_{e^0-e^k}, E_{-e^0-e^k}\})|\Omega\rangle \\ &= \langle\Omega|H_0^2 + \tfrac{1}{2}H_0 - \tfrac{1}{2}\textstyle\sum_k([E_{e^0+e^k}, E_{-e^0-e^k}] + [E_{e^0-e^k}, E_{-e^0+e^k}])|\Omega\rangle \\ &= \langle\Omega|H_0^2 + \tfrac{1}{2}H_0 - \tfrac{1}{2}\textstyle\sum_k(-H_0 - H_k) - \tfrac{1}{2}\sum_k(-H_0 + H_k)|\Omega\rangle \\ &= \lambda_0 + (n + \tfrac{1}{2})\lambda_0 \end{aligned}$$

Next, for $j = 1, ..., n$, we have:

$$\begin{aligned}
c &= \langle\Omega| - M^2_{2j-1,-1} - M^2_{2j-1,0} + \textstyle\sum_{k\geq 1} M^2_{2j-1,k}|\Omega\rangle \\
&= \langle\Omega| - \tfrac{1}{4}(E_{e^j+e^0} + E_{e^j-e^0} + E_{-e^j+e^0} + E_{-e^j-e^0})^2 + \tfrac{1}{4}(E_{e^j+e^0} - E_{e^j-e^0} + E_{-e^j+e^0} \\
&\quad -E_{-e^j-e^0})^2 + \textstyle\sum_{1\leq k\leq n, k\neq j} \tfrac{1}{4}((E_{e^j+e^k} + E_{e^j-e^k} + E_{-e^j+e^k} + E_{-e^j-e^k})^2 - (E_{e^j+e^k} \\
&\quad -E_{e^j-e^k} + E_{-e^j+e^k} - E_{-e^j-e^k})^2) + H_j^2 + \tfrac{1}{2}(E_{e^j} + E_{-e^j})|\Omega\rangle \\
&= \langle\Omega| - \tfrac{1}{2}(\{E_{e^j+e^0}, E_{e^j-e^0}\} + \{E_{e^j+e^0}, E_{-e^j-e^0}\} + \{E_{e^j-e^0}, E_{-e^j+e^0}\} \\
&\quad +\{E_{-e^j+e^0}, E_{-e^j-e^0}\}) + \tfrac{1}{2}\textstyle\sum_{k\neq j}(\{E_{e^j+e^k}, E_{e^j-e^k}\} + \{E_{e^j+e^k}, E_{-e^j-e^k}\} \\
&\quad \{E_{e^j-e^k}, E_{-e^j+e^k}\} + \{E_{-e^j+e^k}, E_{-e^j-e^k}\}) + H_j^2 + \tfrac{1}{2}(E^2_{e^j} + E^2_{-e^j} + \{E_{e^j}, E_{-e^j}\})|\Omega\rangle \\
&= \langle\Omega|H_j^2 + \tfrac{1}{2}H_j + \textstyle\sum_{k>j}([E_{e^j-e^k}, E_{-e^j+e^k}]) + \sum_{k<j}([E_{-e^j+e^k}, E_{e^j-e^k}]) \\
&\quad -\tfrac{1}{2}([E_{e^j+e^0}, E_{e^j-e^0}] + [E_{e^j+e^0}, E_{-e^j-e^0}] + [E_{-e^j+e^0}, E_{e^j-e^0}] + [E_{-e^j+e^0}, E_{-e^j-e^0}])|\Omega\rangle \\
&= \langle\Omega|H_j^2 + \tfrac{1}{2}H_j + \tfrac{1}{2}\textstyle\sum_{k\neq j}(H_j + H_k) + \tfrac{1}{2}\sum_{k>j}(H_j - H_k) \\
&\quad \tfrac{1}{2}\textstyle\sum_{k<j}(H_k - H_j) - \tfrac{1}{2}(-H_0 - H_j - H_0 + H_j)|\Omega\rangle \\
&= \langle\Omega|H_j^2 + \tfrac{1}{2}H_j + \tfrac{1}{2}\textstyle\sum_{k>j} 2H_j + \tfrac{1}{2}\sum_{k<j} 2H_k + H_0|\Omega\rangle \\
&= \langle\Omega|H_j^2 + (n - j + \tfrac{1}{2})H_j + \textstyle\sum_{k<j} H_k + H_0|\Omega\rangle \\
&= \lambda_j^2 + (n - j + \tfrac{1}{2})\lambda_j + (\lambda_1 + ... + \lambda_{j-1}) + \lambda_0
\end{aligned}$$

Thirdly,

$$\begin{aligned}
c &= \langle\Omega| - M^2_{2n+1,-1} - M^2_{2n+1,0} + \textstyle\sum_{k\geq 1} M^2_{2n+1,k}|\Omega\rangle \\
&= \langle\Omega| - \tfrac{1}{2}(E_{e^0} + E_{-e^0})^2 + \tfrac{1}{2}(E_{e^0} - E_{-e^0})^2 + \tfrac{1}{2}\textstyle\sum_{k\geq 1}((E_{e^j} + E_{-e^j})^2 - (E_{e^j} - E_{-e^j})^2)|\Omega\rangle \\
&= \langle\Omega| - \{E_{e^0}, E_{-e^0}\} + \textstyle\sum_{k\geq 1}\{E_{e^j}, E_{-e^j}\}|\Omega\rangle \\
&= \langle\Omega| - [E_{e^0}, E_{-e^0}] + \textstyle\sum_{k\geq 1}[E_{e^j}, E_{-e^j}]|\Omega\rangle \\
&= \langle\Omega|H_0 + \textstyle\sum_{j\geq 1} H_j|\Omega\rangle \\
&= \lambda_0 + \lambda_1 + ... + \lambda_n
\end{aligned}$$

Now, from the second groups of equations, we deduce by subtracting the neighboring ones:

$$\lambda_{j+1}^2 - \lambda_j^2 + (n - j + \frac{1}{2})(\lambda_{j+1} - \lambda_j) - (\lambda_{j+1} - \lambda_j) = (\lambda_{j+1} + \lambda_j + n - j - \frac{1}{2})(\lambda_{j+1} - \lambda_j) = 0$$

Moreover since $\lambda_j \geq 0$ and the index j runs from 1 to $n-1$, $(\lambda_{j+1} + \lambda_j + n - j - \frac{1}{2}) > 0$, it follows that:

$$\lambda_{j+1} = \lambda_j = \lambda, \quad (j = 1, ..., n-1).$$

Plugging this into any of the second groups of equations and the third equation,we obtain:

$$\lambda^2 + (n - \frac{1}{2})\lambda + \lambda_0 = c = \lambda_0 + n\lambda,$$

which in turn implies:

$$\lambda^2 = \frac{1}{2}\lambda.$$

By equating the first equation with any of the second group of equations, say, the first, we obtain:

$$\lambda_0^2 + (n + \frac{1}{2})\lambda_0 = c = \lambda^2 + (n - \frac{1}{2})\lambda + \lambda_0$$

Hence:

$$(\lambda - \lambda_0)(\lambda + \lambda_0 + n - \frac{1}{2}) = 0.$$

Thus either $\lambda = \lambda_0$ or $\lambda_0 = -(\lambda + n - \frac{1}{2})$. Together with lemma 2.1, we deduce that the only possibilities are as follows:

$$\begin{cases} \lambda_0 = 0 & \lambda = 0 \quad \text{and } |\Omega\rangle = |0, 0, ..., 0\rangle \\ \lambda_0 = -n + \frac{1}{2} & \lambda = 0 \quad \text{and } |\Omega\rangle = |-n + \frac{1}{2}, 0, ..., 0\rangle \\ \lambda_0 = -n & \lambda = \frac{1}{2} \quad \text{and } |\Omega\rangle = |-n, \frac{1}{2}, ..., \frac{1}{2}\rangle. \end{cases}$$

□

2.3 Proof of The Other Side

To prove that the other side holds, we need the following observations to reduce the verification of the $(2n+3)\times(2n+3)$ quadratic relations

$$\sum_\lambda \{M_{\mu\lambda}, M^\lambda_{\ \nu}\} = c\eta_{\mu\nu}$$

to the vanishing of the "off-diagonal" cases in these specific representations. Moreover by another simple symmetry argument we are further reduced to prove for only one special "off-diagonal" case.

Lemma 2.3. The quadratic representations are satisfied in those irreducible unitary highest weight $(\mathfrak{g}, K)$-modules if and only if the following identity holds in these representations:

$$\sum_\lambda \{M_{-1\lambda}, M^\lambda_{\ 0}\} = 0.$$

Proof. One side is clear.

For the converse, we note that the "diagonal operators" $\{M_{\mu\lambda}, M^\lambda_{\ \mu}\} - c\eta_{\mu\mu}$ kill the highest weight vector by similar computations as those in Proposition 2.2. Thus these operators will vanish identically if

$$U \doteq [M_{\alpha\beta}, \sum_\lambda \{M_{\mu\lambda}, M^\lambda_{\ \mu}\} - c\eta_{\mu\mu}] \equiv 0,$$

where $\alpha, \beta \in \{-1, 0, ..., 2n+1\}$ in these representations, since all the other weight vectors are created from repeated actions of $M_{\alpha\beta}$. But we have:

$$\begin{aligned}
U &= [M_{\alpha\beta}, \textstyle\sum_\lambda \{M_{\mu\lambda}, M^\lambda_{\ \mu}\} - c\eta_{\mu\mu}] \\
&= [M_{\alpha\beta}, \textstyle\sum_\lambda (\eta^{\lambda\delta}\{M_{\mu\lambda}, M_{\lambda\mu}\}) - c\eta_{\mu\mu}] \\
&= \textstyle\sum_\lambda (\eta^{\lambda\delta}\{M_{\mu\lambda}, [M_{\alpha\beta}, M_{\delta\mu}]\} + \eta^{\lambda\delta}\{[M_{\alpha\beta}, M_{\mu\lambda}], M_{\delta\mu}\}) \\
&= \textstyle\sum_\lambda (\eta^{\lambda\delta}\{M_{\mu\lambda}, -\sqrt{-1}(\eta_{\beta\delta}M_{\alpha\mu} - \eta_{\alpha\delta}M_{\beta\mu} - \eta_{\beta\mu}M_{\alpha\delta} + \eta_{\alpha\mu}M_{\beta\delta})\} \\
&\quad +\eta^{\lambda\delta}\{-\sqrt{-1}(\eta_{\beta\mu}M_{\alpha\lambda} - \eta_{\alpha\mu}M_{\beta\lambda} - \eta_{\beta\lambda}M_{\alpha\mu} + \eta_{\alpha\lambda}M_{\beta\mu}), M_{\delta\mu}\}) \\
&= \textstyle\sum_\lambda (-\sqrt{-1}\{M_{\mu\lambda}, (\delta_{\beta\lambda}M_{\alpha\mu} - \delta_{\lambda\alpha}M_{\beta\mu} - \eta_{\beta\mu}\eta^{\lambda\delta}M_{\alpha\delta} + \eta_{\alpha\mu}\eta^{\lambda\delta}M_{\beta\delta})\} \\
&\quad -\sqrt{-1}\{(\eta^{\lambda\delta}\eta_{\beta\mu}M_{\alpha\mu} - \eta^{\lambda\delta}\eta_{\alpha\mu}M_{\beta\mu} - \delta_{\beta\delta}M_{\alpha\mu} + \delta_{\alpha\delta}M_{\beta\mu}), M_{\delta\mu}\}) \\
&= -\textstyle\sum_\lambda \sqrt{-1}(\eta_{\beta\mu}\{M_{\mu\lambda}, M^\lambda_{\ \alpha}\} - \eta_{\alpha\mu}\{M_{\mu\lambda}, M^\lambda_{\ \beta}\} + \eta_{\beta\mu}\{M_{\alpha\lambda}, M^\lambda_{\ \mu}\} - \eta_{\alpha\mu}\{M_{\beta\lambda}, M^\lambda_{\ \mu}\}) \\
&= -2\sqrt{-1}\textstyle\sum_\lambda \eta_{\beta\mu}\{M_{\mu\lambda}, M^\lambda_{\ \alpha}\} + 2\sqrt{-1}\sum_\lambda \eta_{\alpha\mu}\{M_{\mu\lambda}, M^\lambda_{\ \beta}\} \\
&= \begin{cases} 2\sqrt{-1}\eta_{\alpha\alpha}\sum_\lambda \{M_{\alpha\lambda}, M^\lambda_{\ \beta}\} & (\alpha \neq \beta \ \ \alpha = \mu) \\ -2\sqrt{-1}\eta_{\beta\beta}\sum_\lambda \{M_{\beta\lambda}, M^\lambda_{\ \alpha}\} & (\alpha \neq \beta \ \ \beta = \mu) \\ 0 & (\alpha, \beta, \mu \text{ all different}). \end{cases}
\end{aligned}$$

Thus we see that if all the "off-diagonal" operators vanish, the "diagonal" ones will be identically 0 in these modules.

Furthermore, consider the three identities below:

$$\begin{cases} A \ : \sum_\lambda \{M_{-1\lambda}, M^\lambda_{\ 0}\} = 0 \\ B : \ \sum_\lambda \{M_{0\lambda}, M^\lambda_{\ 1}\} = 0 \\ C : \ \sum_\lambda \{M_{1\lambda}, M^\lambda_{\ 2}\} = 0. \end{cases}$$

Once the above identities A, B, C hold in the modules, the other "off-diagonal" cases will be true. This is because they just differ by an inner automorphism of the group $Spin(2, 2n+1)$.

Finally, identities B and C can be deduced once identity A holds, this is because:

$$\begin{aligned}
0 &= [M_{1,-1}, \textstyle\sum_\lambda \{M_{-1\lambda}, M^\lambda_{\ 0}\}] \\
&= \textstyle\sum_\lambda (\{[M_{1,-1}, M_{-1\lambda}], M^\lambda_{\ 0}\} + \{M_{-1\lambda}, [M_{1,-1}, M^\lambda_{\ 0}]\}) \\
&= \textstyle\sum_\lambda (\sqrt{-1}\{\eta_{-1,-1}M_{1\lambda} - \eta_{1,-1}M_{-1\lambda} - \eta_{-1,\lambda}M_{1,-1} + \eta_{-1,\lambda}M_{-1,-1}, \eta^{\lambda\delta}M_{\delta 0}\} \\
&\quad -\sqrt{-1}\{\eta^{\lambda\delta}M_{-1,\lambda}, \eta_{-1,\delta}M_{10} - \eta_{1,\delta}M_{-10} - \eta_{-1,0}M_{1,\delta} + \eta_{1,0}M_{-1,\delta}\}) \\
&= \sqrt{-1}(\textstyle\sum_\lambda \{M_{1\lambda}, M^\lambda_{\ 0}\} - \{M_{1,-1}, M_{-1,0}\} - \{M_{-1,1}, M_{-1,0}\}) \\
&= \sqrt{-1}\textstyle\sum_\lambda \{M_{1\lambda}, M^\lambda_{\ 0}\}.
\end{aligned}$$

Similarly, from identity B, we can deduce identity C:

$$\begin{array}{rcl} 0 & = & [M_{0,2},\sum_\lambda\{M_{0\lambda},M^\lambda_{\ 1}\}] \\ & = & \sum_\lambda(\{[M_{0,2},M_{0\lambda}],M^\lambda_{\ 1}\}+\{M_{0\lambda},[M_{0,2},M^\lambda_{\ 1}]\}) \\ & = & \sum_\lambda(\sqrt{-1}\{\eta_{2,0}M_{0\lambda}-\eta_{0,0}M_{2\lambda}-\eta_{2,\lambda}M_{0,01}+\eta_{01,\lambda}M_{2,0},\eta^{\lambda\delta}M_{\delta 1}\} \\ & & -\sqrt{-1}\{\eta^{\lambda\delta}M_{0,\lambda},\eta_{2,\delta}M_{01}-\eta_{0,\delta}M_{21}-\eta_{2,1}M_{0,\delta}+\eta_{0,1}M_{2,\delta}\}) \\ & = & -\sqrt{-1}(\sum_\lambda\{M_{2\lambda},M^\lambda_{\ 1}\}-\{M_{2,0},M_{0,1}\}-\{M_{0,2},M_{0,1}\}) \\ & = & \sqrt{-1}\sum_\lambda\{M_{2\lambda},M^\lambda_{\ 1}\}. \end{array}$$

This finishes the proof of our lemma. □

Now we finish the proof of proposition 2.2 by showing that the above identity holds in our specific modules:

Lemma 2.4. $\sum_\lambda\{M_{-1\lambda},M^\lambda_{\ 0}\}=0$ in these modules.

Proof.

$$L.H.S.=\sum_{k\geq 1}\{M_{-1k},M_{0k}\}=\frac{2}{\sqrt{-1}}\sum_{k\geq 1}(E_{e^0-e^k}E_{e^0+e^k}-E_{-e^0-e^k}E_{-e^0+e^k})+\frac{1}{\sqrt{-1}}(E^2_{e^0}-E^2_{-e^0})$$

Let $O=2\sum_{k\geq 1}E_{-e^0-e^k}E_{-e^0+e^k}+E^2_{-e^0}$, then $L.H.S.=\frac{1}{\sqrt{-1}}(O^\dagger-O)$. Hence it suffices to check that the operator O vanishes identically in these modules. For this purpose it suffices to check that, i), it kills the highest weight vector $|\Omega\rangle$, and ii), it commutes with all (simple) negative roots, since all other weight vectors, which constitute a basis of our $(\mathfrak{g},K)$-module, are obtained by succinct actions of the (simple) negative roots on the highest weight vector $|\Omega\rangle$.

Firstly, we check that it kills the highest weight vector. To do this we will show that $||O|\Omega\rangle||=0$ in these modules:

$$\begin{array}{rcl} \langle\Omega|O^\dagger O|\Omega\rangle & = & \langle\Omega|\underbrace{\sum_{k,l}(4E_{e^0+e^k}E_{e^0-e^k}E_{-e^0+e^l}E_{-e^0-e^l})}_{X}+\underbrace{\sum_k 2E_{e^0+e^k}E_{e^0-e^k}E^2_{-e^0}}_{Y} \\ & & +\underbrace{\sum_l 2E^2_{e^0}E_{-e^0+e^l}E_{-e^0-e^l}}_{Y^\dagger}+\underbrace{E^2_{e^0}E^2_{-e^0}}_{Z}|\Omega\rangle \end{array}$$

Now we compute termwise:

$$\begin{array}{rcl} \langle\Omega|X|\Omega\rangle & = & \langle\Omega|\sum_{k,l}(4E_{e^0+e^k}E_{e^0-e^k}E_{-e^0+e^l}E_{-e^0-e^l})|\Omega\rangle \\ & = & 4\langle\Omega|\underbrace{\sum_{k=1}^n E_{e^0+e^k}E_{e^0-e^k}E_{-e^0+e^k}E_{-e^0-e^k}}_{X_1}|\Omega\rangle \\ & & +4\langle\Omega|\underbrace{\sum_{k\neq l}E_{e^0+e^k}E_{e^0-e^k}E_{-e^0+e^l}E_{-e^0-e^l}}_{X_2}|\Omega\rangle \end{array}$$

For the first term, $\langle\Omega|X_1|\Omega\rangle$ and $\langle\Omega|X_2|\Omega\rangle$, we have respectively:

$$\begin{array}{rcl} \langle\Omega|X_1|\Omega\rangle & = & \langle\Omega|\sum_{k=1}^n(E_{e^0+e^k}E_{e^0-e^k}E_{-e^0+e^k}E_{-e^0-e^k})|\Omega\rangle \\ & = & \langle\Omega|\sum_{k=1}^n(E_{e^0+e^k}E_{-e^0-e^k}E_{e^0-e^k}E_{-e^0-e^k}+E_{e^0+e^k}[E_{e^0-e^k},E_{-e^0+e^k}]E_{-e^0-e^k}))|\Omega\rangle \\ & = & \langle\Omega|\sum_{k=1}^n([E_{e^0+e^k}E_{-e^0-e^k}][E_{e^0-e^k}E_{-e^0-e^k}]+E_{e^0+e^k}(-H_0+H_k)E_{-e^0-e^k}))|\Omega\rangle \\ & = & \langle\Omega|\sum_{k=1}^n(-H_0+H_k)[E_{e^0+e^k},E_{-e^0-e^k}]|\Omega\rangle \\ & = & \langle\Omega|\sum_{k=1}^n(-H_0+H_k)(-H_0-H_k)|\Omega\rangle \\ & = & \sum_{k=1}^n(\lambda_0^2-\lambda_k^2) \end{array}$$

$$\begin{array}{rcl}\langle\Omega|X_2|\Omega\rangle &=& \langle\Omega|\sum_{k\neq l}(E_{e^0+e^k}E_{-e^0+e^l}E_{e^0-e^k}E_{-e^0-e^l}+E_{e^0+e^k}[E_{e^0-e^k},E_{-e^0+e^l}]E_{-e^0+e^l})|\Omega\rangle \\ &=& \langle\Omega|\sum_{k\neq l}([E_{e^0+e^k},E_{-e^0+e^l}][E_{e^0-e^k},E_{-e^0-e^l}]+E_{e^0+e^k}\sqrt{-1}E_{-e^k+e^l}E_{-e^0+e^l})|\Omega\rangle \\ &=& \langle\Omega|\sum_{k\neq l}(\sqrt{-1}E_{e^k+e^l}\sqrt{-1}E_{-e^k-e^l}+\sqrt{-1}[E_{e^0+e^k},E_{-e^k+e^l}]E_{-e^0-e^l} \\ && +\sqrt{-1}E_{-e^k+e^l}E_{e^0+e^k}E_{-e^0-e^l})|\Omega\rangle \\ &=& \langle\Omega|\sum_{k\neq l}(-E_{e^k+e^l}E_{-e^k-e^l}-E_{e^0+e^l}E_{-e^0-e^l}+\sqrt{-1}E_{-e^k+e^l}[E_{e^0+e^k},E_{-e^0-e^l}])|\Omega\rangle \\ &=& \langle\Omega|\sum_{k\neq l}(-[E_{e^k+e^l}E_{-e^k-e^l}]-[E_{e^0+e^l}E_{-e^0-e^l}]-E_{-e^k+e^l}E_{e^k-e^l})|\Omega\rangle \\ &=& \langle\Omega|\sum_{k\neq l}(-(H_k+H_l)-(-H_0-H_l))-\sum_{k>l}[E_{-e^k+e^l},E_{e^k-e^l}]|\Omega\rangle \\ &=& \langle\Omega|\sum_{k\neq l}(H_0-H_k)+\sum_{k>l}(H_k-H_l)|\Omega\rangle \\ &=& \sum_{k\neq l}(\lambda_0-\lambda_k)+\sum_{k>l}(\lambda_k-\lambda_l)\end{array}$$

The remaining terms are similarly computed:

$$\begin{array}{rcl}\langle\Omega|Y|\Omega\rangle &=& \langle\Omega|\sum_k(E_{e^0+e^k}E_{e^0-e^k}E^2_{-e^0})|\Omega\rangle \\ &=& \langle\Omega|\sum_k(E_{e^0+e^k}E_{-e^0}E_{e^0-e^k}E_{-e^0}+E_{e^0+e^k}[E_{e^0-e^k},E_{-e^0}]E_{-e^0})|\Omega\rangle \\ &=& \langle\Omega|\sum_k([E_{e^0+e^k},E_{-e^0}][E_{e^0-e^k},E_{-e^0}]+\sqrt{-1}E_{e^0+e^k}E_{-e^k}E_{-e^0})|\Omega\rangle \\ &=& \langle\Omega|\sum_k(-E_{e^k}E_{-e^k}+\sqrt{-1}[E_{e^0+e^k},E_{-e^k}]E_{-e^0})|\Omega\rangle \\ &=& \langle\Omega|\sum_k(-[E_{e^k},E_{-e^k}]-E_{e^0},E_{-e^0})|\Omega\rangle \\ &=& \langle\Omega|\sum_k(-H_k-[E_{e^0},E_{-e^0}])|\Omega\rangle \\ &=& \langle\Omega|\sum_k(-H_k-H_0)|\Omega\rangle \\ &=& \sum_k(\lambda_0-\lambda_k)\end{array}$$

$$\begin{array}{rcl}\langle\Omega|Z|\Omega\rangle &=& \langle\Omega|E^2_{e^0}E^2_{-e^0}|\Omega\rangle \\ &=& \langle\Omega|(E_{e^0}E_{-e^0}E_{e^0}E_{-e^0}+E_{e^0}[E_{e^0},E_{-e^0}]E_{-e^0})|\Omega\rangle \\ &=& \langle\Omega|([E_{e^0},E_{-e^0}][E_{e^0},E_{-e^0}]+E_{e^0}(-H_0)E_{-e^0})|\Omega\rangle \\ &=& \langle\Omega|(H_0^2+E_{e^0}E_{-e^0}-H_0[E_{e^0}E_{-e^0}])|\Omega\rangle \\ &=& \langle\Omega|(H_0^2+[E_{e^0},E_{-e^0}]+H_0^2)|\Omega\rangle \\ &=& \langle\Omega|(2H_0^2-H_0)|\Omega\rangle \\ &=& \lambda_0^2-\lambda_0\end{array}$$

In sum, we deduce that:

$$\begin{array}{rcl}\langle\Omega|O^\dagger O|\Omega\rangle &=& 4\sum_{k=1}^n(\lambda_0^2-\lambda_k)+4\sum_{k\neq l}(\lambda_k-\lambda_l)+4\sum_{k>l}(\lambda_k-\lambda_l) \\ && +4\sum_k(\lambda_0-\lambda_k)+\lambda_0^2-\lambda_0\end{array}$$

Now, we substitute the λ_i's by those occurring in the highest weight vectors $|\Omega\rangle=|-(n+\mu-\frac{1}{2}),\mu,...\mu\rangle$, where $\mu=0$ or $\frac{1}{2}$, and obtain after a simple calculation that $\langle\Omega|O^\dagger O|\Omega\rangle=(2\mu-1)\mu=0$.
Secondly, we verify that O commutes with all (simple) negative roots:
For $E_{-e^0+e^1}$ we have:

$$\begin{array}{rcl}[O,E_{-e^0+e^1}] &=& [2\sum_{k\geq 1}E_{-e^0-e^k}E_{-e^0+e^k}+E^2_{-e^0},E_{-e^0+e^1}] \\ &=& \sum_{k\geq 1}(\{[E_{-e^0-e^k},E_{-e^0+e^1}],E_{-e^0+e^k}\}+\{E_{-e^0-e^k},[E_{-e^0+e^k},E_{-e^0+e^1}]\}) \\ && +E_{-e^0}[E_{-e^0},E_{-e^0+e^1}]+[E_{-e^0},E_{-e^0+e^1}]E_{-e^0} \\ &=& 0.\end{array}$$

For $E_{-e^k+e^{k+1}}$ where $1\leq k\leq n-1$, we have:

$$\begin{array}{rcl}[O,E_{-e^k+e^{k+1}}] &=& [2\sum_{l\geq 1}E_{-e^0-e^l}E_{-e^0+e^l}+E^2_{-e^0},E_{-e^k+e^{k+1}}] \\ &=& \{[E_{-e^0-e^k},E_{-e^k+e^{k+1}}],E_{-e^0+e^k}\}+\{E_{-e^0-e^k},[E_{-e^0+e^k},E_{-e^k+e^{k+1}}]\} \\ && \{[E_{-e^0-e^{k+1}},E_{-e^k+e^{k+1}}],E_{-e^0+e^{k+1}}\}+\{E_{-e^0-e^{k+1}},[E_{-e^k+e^{k+1}},E_{-e^k+e^{k+1}}]\} \\ &=& 0+\{E_{-e^0-e^k},\sqrt{-1}E_{-e^0+e^{k+1}}\}+\{-\sqrt{-1}E_{-e^0-e^k},E_{-e^0+e^{k+1}}\}+0 \\ &=& 0.\end{array}$$

Finally for E_{-e^n}, we have:

$$\begin{aligned} [O, E_{-e^n}] &= [2\sum_{k\geq 1} E_{-e^0-e^k}E_{-e^0+e^k} + E^2_{-e^0}, E_{-e^n}] \\ &= \{E_{-e^0-e^n}, [E_{-e^0+e^n}, E_{-e^n}]\} + \{E_{-e^0}, [E_{-e^0}, E_{-e^n}]\} \\ &= \{E_{-e^0-e^n}, \sqrt{-1}E_{-e^0}\} + \{E_{-e^0}, -\sqrt{-1}E_{-e^0-e^n}\} \\ &= 0 \end{aligned}$$

This finishes the proof of the lemma. □

In summary of the previous lemmas we conclude with the following proposition, which together with Proposition 2.2 finishes the non-compact odd dimensional part of our main theorem 1.1:

Proposition 2.5. The irreducible unitary highest weight $(\mathfrak{so}(2, 2n+1), SO(2) \times SO(2n+1))$-modules with highest weights $| -(n+\mu-\frac{1}{2}), \mu, ..., \mu\rangle$, for $\mu = 0$ and $\mu = \frac{1}{2}$ and the trivial representation satisfy the quadratic relations:

$$\sum_{\lambda} \{M_{\mu\lambda}, M^{\lambda}{}_{\nu}\} = c\eta_{\mu\nu}.$$

□

3 The Noncompact Even Dimensional Case: $\mathfrak{so}(2, 2n)$; An Einstein Equation

In this chapter, we prove the quadratic relations for the even dimensional cases. Since the methods are completely similar as that of the previous chapter, we just indicate the necessary changes which need to be made of the previous chapter. Next, we make some observations on a formal type of "Einstein equations" that our specific modules satisfy.

Again we start by reviewing some general facts about the special orthogonal lie algebra $\mathfrak{so}(2, 2n)$. Here we also adopt the usual convention of physicists, as can be found in the standard textbook. [1]

Recall that the root space of $\mathfrak{so}(2, 2n)$ is $\mathbb{R}^{n+1}$, with the standard basis $\{e^0, ..., e^n\}$. The roots of $\mathfrak{so}(2, 2n)$ are $\{\pm e^i \pm e^j | 0 \leq i < j \leq n\}$. As usual, we choose the positive roots to be $e^i \pm e^j$. The associated simple roots are $\{e^0 - e^1, ..., e^{n-1} - e^n, e^{n-1} + e^n\}$.

A Cartan basis can be chosen as follows(C. f. [2]):

$$\begin{cases} H_0 = M_{-1,0} \\ H_i = -M_{2i-1,2i}, \quad 1 \leq i \leq n \\ E_{\eta e^j + \zeta e^k} = \frac{1}{2}(M_{2j-1,2k-1} + \sqrt{-1}\eta M_{sj,2k-1} + \sqrt{-1}\zeta M_{2j-1,2k} - \eta\zeta M_{2j,2k}), \\ \qquad 0 \leq j < k \leq n \end{cases}$$

The transition from the Clifford algebra interpretation to the root vectors is given by:

$$\begin{cases} M_{2j-1,2k-1} = \frac{1}{2}(E_{e^j+e^k} + E_{e^j-e^k} + E_{-e^j+e^k} + E_{-e^j-e^k}) \\ M_{2j,2k-1} = \frac{1}{2\sqrt{-1}}(E_{e^j+e^k} + E_{e^j-e^k} - E_{-e^j+e^k} - E_{-e^j-e^k}) \\ M_{2j-1,2k} = \frac{1}{2\sqrt{-1}}(E_{e^j+e^k} - E_{e^j-e^k} + E_{-e^j+e^k} - E_{-e^j-e^k}) \\ M_{2j,2k} = \frac{1}{2}(-E_{e^j+e^k} + E_{e^j-e^k} + E_{-e^j+e^k} - E_{-e^j-e^k}), \end{cases}$$

where $0 \leq j < k \leq n$; and we have:

$$\begin{cases} M_{-1,0} = H_0 \\ M_{2i-1,2i} = -H_i, \end{cases}$$

where $1 \leq i \leq n$.

3.1 Proof of One Side

Again we start with one side of the proof, which is done by Meng in his paper [2].

Note that the Cartan basis $\{H_i, E_\alpha\}$ satisfies the following commutator relations:

$$\begin{cases} [E_{e^0+e^i}, E_{-e^0-e^i}] = -H_0 - H_i \;\; [E_{e^0-e^i}, E_{-e^0+e^i}] = -H_0 + H_i \\ [E_{e^i+e^j}, E_{-e^i-e^j}] = H_i + H_j \;\; [E_{e^i-e^j}, E_{-e^i+e^j}] = H_i - H_j \\ [E_{e^0+e^k}, E_{-e^k+e^l}] = \sqrt{-1}E_{e^0+e^l} \;\; [E_{e^k+e^l}, E_{-e^l+e^j}] = \sqrt{-1}E_{e^k+e^j} \end{cases}$$

By evaluating the equation, $[E_{e^0+e^i}, E_{-e^0-e^i}] = -H_0 - H_i$, on the highest weight vector $|\Omega\rangle$, we obtain

$$0 \leq ||E_{-e^0-e^i}|\Omega\rangle||^2 = \langle\Omega|[E_{e^0+e^i}, E_{-e^0-e^i}]|\Omega\rangle = \langle\Omega|H_0 - H_i|\Omega\rangle = -\lambda_0 - \lambda_i$$

Similarly by evaluating the highest weight vector at the equations $[E_{e^i\mp e^j}, E_{-e^i\pm e^j}] = H_i \mp H_j$ we obtain $\lambda_i \geq 0,\ (n-1 \geq i \geq 1), -\lambda_0 \geq 0$ and $\lambda_i \pm \lambda_j \geq 0$ for $i > j$. Furthermore, the differences and sums are all integers by similar arguments as in Lemma 2.1. We summarize this discussion as:

Lemma 3.1. The highest weight satisfies

$$-\lambda_0 \geq \lambda_1 \geq ... \geq |\lambda_n|.$$

Moreover, $\{\lambda_i|(i = 1, ..., n)\}$ are all half integers and their differences $(\lambda_1 - \lambda_2), ..., (\lambda_{n-1} - \lambda_n)$ are all integers. □

Now we prove:

Proposition 3.2. An irreducible unitary highest weight $(\mathfrak{so}(2, 2n), SO(2)\times SO(2n))$-module satisfying the quadratic relation could only be those with highest weights $|-(n+|\mu|-1), |\mu|, ..., |\mu|, \mu\rangle$, for $\mu \in \frac{1}{2}\mathbb{Z}$ or the trivial representation.

Proof. Again we use the fact that the quadratic relations in the special cases $\mu = \nu = -1, ...2j - 1, ...2n - 1$ gives

$$\begin{cases} \langle\Omega| - M_{-1,k}M^k{}_{-1}|\Omega\rangle = const. \\ \langle\Omega|M_{2j-1,k}M^k{}_{2j-1}|\Omega\rangle = const.\ \ (j = 1, ..., n) \end{cases}$$

Plugging in the above transformation relations, and noticing that since we are evaluating on the highest weight vector, we may well omit those terms occurring in the computation of the form $E_{-\alpha} \cdot * + * \cdot E_\beta$, where α and β are positive roots. Thus for the above equations, we obtain:

$$\begin{aligned} c &= \langle\Omega| - M_{-1,k}M^k{}_{-1}|\Omega\rangle \\ &= \langle\Omega|M^2_{-1,0} - \textstyle\sum_{1\leq k\leq n} M^2_{-1,2k-1} - \sum_{1\leq k\leq n} M^2_{-1,2k}|\Omega\rangle \\ &= \langle\Omega|H_0^2 - (\tfrac{1}{4}\textstyle\sum_k(E_{e^0+e^k} + E_{e^0-e^k} + E_{-e^0+e^k} + E_{-e^0-e^k})^2 \\ &\quad -\tfrac{1}{4}\textstyle\sum_k(E_{e^0+e^k} - E_{e^0-e^k} + E_{-e^0+e^k} - E_{-e^0-e^k})^2)|\Omega\rangle \\ &= \langle\Omega|H_0^2 - \tfrac{1}{2}(\{E_{e^0+e^k}, E_{e^0-e^k}\} + \{E_{e^0+e^k}, E_{-e^0-e^k}\} \\ &\quad +\{E_{e^0-e^k}, E_{-e^0+e^k}\} + \{E_{e^0-e^k}, E_{-e^0-e^k}\})|\Omega\rangle \\ &= \langle\Omega|H_0^2 - \tfrac{1}{2}([E_{e^0+e^k}, E_{-e^0-e^k}] + [E_{e^0-e^k}, E_{-e^0+e^k}])|\Omega\rangle \\ &= \langle\Omega|H_0^2 - \tfrac{1}{2}\textstyle\sum_k(-H_0 - H_k) - \tfrac{1}{2}\sum_k(-H_0 + H_k)|\Omega\rangle \\ &= \lambda_0 + n\lambda_0 \end{aligned}$$

Next, for $j = 1, ..., n$, we have:

$$
\begin{aligned}
c &= \langle\Omega| - M^2_{2j-1,-1} - M^2_{2j-1,0} + \textstyle\sum_{k\geq 1} M^2_{2j-1,k}|\Omega\rangle \\
&= \langle\Omega| - \tfrac{1}{4}(E_{e^j+e^0} + E_{e^j-e^0} + E_{-e^j+e^0} + E_{-e^j-e^0})^2 + \tfrac{1}{4}(E_{e^j+e^0} - E_{e^j-e^0} + E_{-e^j+e^0} \\
&\quad -E_{-e^j-e^0})^2 + \textstyle\sum_{1\leq k\leq n, k\neq j} \tfrac{1}{4}((E_{e^j+e^k} + E_{e^j-e^k} + E_{-e^j+e^k} + E_{-e^j-e^k})^2 - (E_{e^j+e^k} \\
&\quad -E_{e^j-e^k} + E_{-e^j+e^k} - E_{-e^j-e^k})^2) + H_j^2|\Omega\rangle \\
&= \langle\Omega| - \tfrac{1}{2}(\{E_{e^j+e^0}, E_{e^j-e^0}\} + \{E_{e^j+e^0}, E_{-e^j-e^0}\} + \{E_{e^j-e^0}, E_{-e^j+e^0}\} \\
&\quad +\{E_{-e^j+e^0}, E_{-e^j-e^0}\}) + \tfrac{1}{2}\textstyle\sum_{k\neq j}(\{E_{e^j+e^k}, E_{e^j-e^k}\} + \{E_{e^j+e^k}, E_{-e^j-e^k}\} \\
&\quad \{E_{e^j-e^k}, E_{-e^j+e^k}\} + \{E_{-e^j+e^k}, E_{-e^j-e^k}\}) + H_j^2|\Omega\rangle \\
&= \langle\Omega|H_j^2 + \tfrac{1}{2}H_j + \textstyle\sum_{k>j}([E_{e^j-e^k}, E_{-e^j+e^k}]) + \sum_{k<j}([E_{-e^j+e^k}, E_{e^j-e^k}]) \\
&\quad -\tfrac{1}{2}([E_{e^j+e^0}, E_{e^j-e^0}] + [E_{e^j+e^0}, E_{-e^j-e^0}] + [E_{-e^j+e^0}, E_{e^j-e^0}] + [E_{-e^j+e^0}, E_{-e^j-e^0}])|\Omega\rangle \\
&= \langle\Omega|H_j^2 + \tfrac{1}{2}\textstyle\sum_{k\neq j}(H_j + H_k) + \tfrac{1}{2}\sum_{k>j}(H_j - H_k) \\
&\quad \tfrac{1}{2}\textstyle\sum_{k<j}(H_k - H_j) - \tfrac{1}{2}(-H_0 - H_j - H_0 + H_j)|\Omega\rangle \\
&= \langle\Omega|H_j^2 + \tfrac{1}{2}\textstyle\sum_{k>j} 2H_j + \tfrac{1}{2}\sum_{k<j} 2H_k + H_0|\Omega\rangle \\
&= \langle\Omega|H_j^2 + (n-j)H_j + \textstyle\sum_{k<j} H_k + H_0|\Omega\rangle \\
&= \lambda_j^2 + (n-j)\lambda_j + (\lambda_1 + ... + \lambda_{j-1}) + \lambda_0
\end{aligned}
$$

Now, from the second groups of equations, we deduce by subtracting the neighboring ones:

$$\lambda_{j+1}^2 - \lambda_j^2 + (n-j)(\lambda_{j+1} - \lambda_j) - (\lambda_{j+1} - \lambda_j) = (\lambda_{j+1} + \lambda_j + n - j - 1)(\lambda_{j+1} - \lambda_j) = 0$$

Moreover since $\lambda_{j+1} + \lambda_j \geq 0$ and the index j runs from 1 to $n-2$, $(\lambda_{j+1} + \lambda_j + n - j - 1) > 0$, it follows that:

$$\lambda_{j+1} = \lambda_j = \lambda, \quad (j = 1, ..., n-2),$$

and for $j = n-1$:

$$(\lambda_n - \lambda_{n-1})(\lambda_n + \lambda_{n-1}) = 0,$$

or that's to say $|\lambda_n| = \lambda_{n-1}$. It follows that the highest weight must be of the form $|\Omega\rangle = |\lambda_0, \lambda, ..., \lambda, \pm\lambda\rangle$

Plugging this weight into the first and any of the second groups of equations, say, the first one,we obtain:

$$\lambda_0^2 + n\lambda_0 = c = \lambda_0 + (n-1)\lambda + \lambda^2$$

which in turn implies:

$$(\lambda_0 - \lambda)(n - 1 + \lambda_0 + \lambda) = 0.$$

Thus either $\lambda = \lambda_0$ or $\lambda_0 = -(\lambda + n - 1)$. Together with lemma 2.1, we deduce the only possibilities are:

$$
\begin{cases}
\lambda_0 = 0 & \lambda = 0 \quad \text{and } |\Omega\rangle = |0, 0, ..., 0\rangle \\
\lambda_0 = -n - |\mu| + 1 & \lambda = |\mu| \quad \text{and } |\Omega\rangle = |-n - |\mu| + 1, |\mu|, ..., |\mu|, \mu\rangle \\
 & \qquad\qquad \text{where } \mu \text{ is a half integer.}
\end{cases}
$$

□

3.2 Proof of The Other Side

Once again to prove that the other side holds, we turn to Lemma 2.3 for help to reduce the verification of the $(2n+2) \times (2n+2)$ quadratic relations:

$$\sum_\lambda \{M_{\mu\lambda}, M^\lambda{}_\nu\} = c\eta_{\mu\nu}$$

to the verification of only one special off-diagonal case:

$$\sum_\lambda \{M_{-1\lambda}, M^\lambda{}_0\} = 0.$$

Remark 3.3. One more remark is needed here, so that we may reduce by half the number of representations we need to check for the above identity. In fact, we only need to consider the case when the highest weight has $\mu \geq 0$, this is readily seen since the negative cases can be conjugated to this case by an inner automorphism of the group $Pin(2, 2n)$.

And now we prove the corresponding vanishing result in this case for this particular identity,

Lemma 3.4. In these modules, the following equation holds

$$\sum_{\lambda} \{M_{-1\lambda}, M^{\lambda}_{\ 0}\} = 0.$$

Proof.

$$L.H.S. = \sum_{k\geq 1}\{M_{-1k}, M_{0k}\} = \frac{2}{\sqrt{-1}}\sum_{k\geq 1}(E_{e^0-e^k}E_{e^0+e^k} - E_{-e^0-e^k}E_{-e^0+e^k})$$

Let $O = 2\sum_{k\geq 1} E_{-e^0-e^k}E_{-e^0+e^k}$, then $L.H.S. = \frac{1}{\sqrt{-1}}(O^\dagger - O)$ Hence it suffices to check that the operator O vanishes identically in these modules. For this purpose it suffices to check that, i), it kills the highest weight vector $|\Omega\rangle$, and ii), it commutes with all (simple) negative roots, since all other eigenvectors are obtained by succinct actions of the (simple) negative roots on $|\Omega\rangle$.

Firstly, we check that it kills the highest weight vector. To do this we will show that $||O|\Omega\rangle|| = 0$ in these modules:

$$\begin{aligned}
\langle\Omega|O^\dagger O|\Omega\rangle &= \langle\Omega|\textstyle\sum_{k,l}(4E_{e^0+e^k}E_{e^0-e^k}E_{-e^0+e^l}E_{-e^0-e^l})|\Omega\rangle \\
&= \langle\Omega|\textstyle\sum_{k,l}(4E_{e^0+e^k}E_{e^0-e^k}E_{-e^0+e^l}E_{-e^0-e^l})|\Omega\rangle \\
&= 4\langle\Omega|\textstyle\sum_{k=1}^{n} E_{e^0+e^k}E_{e^0-e^k}E_{-e^0+e^k}E_{-e^0-e^k}|\Omega\rangle \\
&\quad +4\langle\Omega|\textstyle\sum_{k\neq l} E_{e^0+e^k}E_{e^0-e^k}E_{-e^0+e^l}E_{-e^0-e^l}|\Omega\rangle \\
&= \textstyle\sum_{k=1}^{n}(\lambda_0^2 - \lambda_k^2) + \sum_{k\neq l}(\lambda_0 - \lambda_k) + \sum_{k>l}(\lambda_k - \lambda_l)
\end{aligned}$$

In case of the highest weight, $|\Omega\rangle = |-n-\mu+1, \mu, ..., , \mu\rangle$, the above number is readily computed to be (notice that we use the above remark 3.3 so that everything in the last term cancels out, of course it's not necessary to have assumed this and direct computations apply.):

$$\begin{aligned}
\langle\Omega|O^\dagger O|\Omega\rangle &= n(n+\mu-1)^2 - n\mu^2 + n(n-1)(-n-2\mu+1) \\
&= n(n-1)(n+2\mu-1-n-2\mu+1) \\
&= 0
\end{aligned}$$

Secondly, we verify that O commutes with all (simple) negative roots:

For $E_{-e^0+e^1}$ we have:

$$\begin{aligned}
[O, E_{-e^0+e^1}] &= [2\textstyle\sum_{k\geq 1} E_{-e^0-e^k}E_{-e^0+e^k}, E_{-e^0+e^1}] \\
&= \textstyle\sum_{k\geq 1}\{[E_{-e^0-e^k}, E_{-e^0+e^1}], E_{-e^0+e^k}\} + \sum_{k\geq 1}\{E_{-e^0-e^k}, [E_{-e^0+e^k}, E_{-e^0+e^1}]\} \\
&= 0.
\end{aligned}$$

For $E_{-e^k+e^{k+1}}$ where $1 \leq k \leq n-1$, we have:

$$\begin{aligned}
[O, E_{-e^k+e^{k+1}}] &= [2\textstyle\sum_{l\geq 1} E_{-e^0-e^l}E_{-e^0+e^l}, E_{-e^k+e^{k+1}}] \\
&= \{[E_{-e^0-e^k}, E_{-e^k+e^{k+1}}], E_{-e^0+e^k}\} + \{E_{-e^0-e^k}, [E_{-e^0+e^k}, E_{-e^k+e^{k+1}}]\} \\
&\quad \{[E_{-e^0-e^{k+1}}, E_{-e^k+e^{k+1}}], E_{-e^0+e^{k+1}}\} + \{E_{-e^0-e^{k+1}}, [E_{-e^k+e^{k+1}}, E_{-e^k+e^{k+1}}]\} \\
&= 0 + \{E_{-e^0-e^k}, \sqrt{-1}E_{-e^0+e^{k+1}}\} + \{-\sqrt{-1}E_{-e^0-e^k}, E_{-e^0+e^{k+1}}\} + 0 \\
&= 0.
\end{aligned}$$

Finally for $E_{-e^{n-1}-e^n}$, we have:

$$\begin{aligned}
[O, E_{-e^{n-1}-e^n}] &= [2\textstyle\sum_{k\geq 1} E_{-e^0-e^k}E_{-e^0+e^k}, E_{-e^{n-1}-e^n}] \\
&= 2\{E_{-e^0-e^{n-1}}, [E_{-e^0+e^{n-1}}, E_{-e^{n-1}-e^n}]\} + 2\{E_{-e^0-e^n}, [E_{-e^0+e^n}, E_{-e^{n-1}-e^n}]\} \\
&= 2\{E_{-e^0-e^{n-1}}, \sqrt{-1}E_{-e^0-e^n}\} + 2\{E_{-e^0-e^n}, -\sqrt{-1}E_{-e^0-e^{n-1}}\} \\
&= 0
\end{aligned}$$

This finishes the proof of the lemma. □

In summary of the previous remarks and lemmas we conclude with the following proposition, which together with Proposition 2.2 finishes the non-compact even dimensional part of our main Theorem 1.1:

Proposition 3.5. The irreducible unitary highest weight $(\mathfrak{so}(2,2n), SO(2)\times SO(2n))$-modules with highest weights $|-(n+|\mu|-1), |\mu|, ..., |\mu|, \mu\rangle$, for $\mu \in \frac{1}{2}\mathbb{Z}$ or the trivial representations satisfy the quadratic relations:

$$\sum_{\lambda}\{M_{\mu\lambda}, M^{\lambda}{}_{\nu}\} = c\eta_{\mu\nu}.$$

3.3 An Einstein Equation

Definition 3.6. Define a family of operators as follows:

$$R_{abcd} \doteq 2\{M_{ab}, M_{cd}\} - \{M_{ac}, M_{db}\} - \{M_{bc}, M_{ad}\}$$

Lemma 3.7. The family of operators R_{abcd} satisfy the following "curvature-type" properties:

(1) $R_{abcd} = -R_{abdc} = -R_{bacd} = R_{cdab}$.

(2) $R_{abcd} + R_{acdb} + R_{adbc} = 0$.

Proof. (1). The first identities follow directly from definitions.
(2). Straightforward calculations, just noticing that $M_{ab} = -M_{ba}$:

$$\begin{aligned} R_{abcd} + R_{acdb} + R_{adbc} &= 2\{M_{ab}, M_{cd}\} - \{M_{ac}, M_{db}\} - \{M_{bc}, M_{ad}\} \\ &\quad +2\{M_{ad}, M_{bc}\} - \{M_{ab}, M_{cd}\} - \{M_{db}, M_{ac}\} \\ &\quad +2\{M_{ac}, M_{db}\} - \{M_{ad}, M_{bc}\} - \{M_{cd}, M_{ab}\} \\ &= 0. \end{aligned}$$

□

Using this family of operators, our main results Propostions 2.2, 2.5, 3.2, 3.5 can be summarized in the following form:

Proposition 3.8. (1) . An irreducible unitary highest weight $(\mathfrak{so}(2,2n+1), SO(2)\times SO(2n+1))$-module satisfies the Einstein equation $\eta^{bc}R_{abcd} = c\eta_{ad}$ if and only if it's either the trivial representation or the one with highest weight

$$|-(n+\mu-\frac{1}{2}), \mu, ..., \mu\rangle,$$

for $\mu = 0$ and $\mu = \frac{1}{2}$.

(2) An irreducible unitary highest weight $(\mathfrak{so}(2,2n), SO(2)\times SO(2n))$-module satisfies the Einstein equation $\eta^{bc}R_{abcd} = c\eta_{ad}$ if and only if it's either the trivial representation or the one with highest weight

$$|-(n+|\mu|-1), |\mu|, ..., |\mu|, \mu\rangle,$$

for $\mu \in \frac{1}{2}\mathbb{Z}$.

Proof. We only need to show that, this "Einstein equation" is in fact equivalent to the original quadratic relations, but:

$$\begin{aligned} \eta^{bc}R_{abcd} &= \eta^{bc}(2\{M_{ab}, M_{cd}\} + \{M_{ac}, M_{bd}\} - \{M_{bc}, M_{ad}\}) \\ &= 2\{M_{ab}, M^{b}{}_{d}\} + \{M_{cd}, \{M_{ac}, M^{c}{}_{d}\}\} \\ &= 3\{M_{ab}, M^{b}{}_{d}\}. \end{aligned}$$

Thus everything follows from what we have done in the above mentioned propositions. □

Remark 3.9. It is interesting to notice that, using this formal language, we have the following results:

1). For compact Lie groups $Spin(2n)$ and $Spin(2n+1)$, a finite dimensional representation is the spinor representation if and only if the module satisfies the formal "homogeneous curvature" equation in the universal enveloping algebra:

$$R_{abcd} = c(\delta_{ad}\delta_{bc} - \delta_{ac}\delta_{bd})$$

2). In the noncompact case, an irreducible unitary $(\mathfrak{so}(2,2n+1), SO(2)\times SO(2n+1))$-module or $(\mathfrak{so}(2,2n), SO(2)\times SO(2n))$-module satisfies the above "homogeneous curvature" type equation if and only if it is the trivial representation.

4 The Symplectic Case

In this chapter we shall focus our attention on the symplectic cases. Similar as in the previous section, we may introduce a formal curvature type operator in the universal enveloping algebra. Only the two fundamental harmonic oscillator representations are proved to satisfy the "homogeneous curvature" equation. However there is a continuous infinite family of irreducible unitary highest weight $(\mathfrak{g}, K)$-modules satisfying the Einstein equation, or equivalently, satisfying the similar quadratic relations in the symplectic case.

4.1 Review of Some General Facts

In this section we briefly review the Heisenberg algebra construction and use it to define and review some basic facts on the symplectic Lie algebra.

Recall that a symplectic vector space is a Euclidean vector space $\mathbb{R}^N$ endowed with a nondegenerate antisymmetric bilinear form. It follows that the dimension N must be even, and the symplectic form maybe written as $\omega = \sum_{ij}\omega_{ij}dx^i dx^j$. The Heisenberg algebra is generated by $x_i \doteq \omega_{ij}x^j$ subject to the commutator relations $[x_i, x_j] = \sqrt{-1}\omega_{ij}$. Define $M_{ij} \doteq \frac{1}{2}\{x_i, x_j\}$, and it's readily checked that these operators generate a copy of symplectic Lie algebra $\mathfrak{sp}(2n)$. Also recall that the symplectic algebra has the root space $\mathbb{R}^n$,with the standard basis $\{e^1, ...e^n\}$. The roots of $\mathfrak{sp}(2n)$ are $\{\pm e^i \pm e^j | 1 \le i \ne j \le n\}$ and $\{\pm 2e^k | 1 \le k \le n\}$. As usual, we choose the positive roots to be $\{e^i \pm e^j | 1 \le i < j \le n\}$, together with $\{e^k | 1 \le k \le n\}$. The associated simple roots are $\{e^1 - e^2, ..., e^{n-1} - e^n, 2e^n\}$.

Now assume without loss of generality that $\omega_{2i-1,2i} = -\omega_{2i,2i-1} = 1$ and all other $\omega_{ij} = 0$. We may choose the following Cartan basis for $\mathfrak{sp}(2n)$:

$$\left\{\begin{array}{rcl} H_i &=& -\frac{1}{2}(M_{2i-1,2i-1} + M_{2i,2i}) \quad 1 \le i \le n \\ E_{\eta e^j + \zeta e^k} &=& \frac{1}{2}(M_{2j-1,2k-1} + \sqrt{-1}\eta M_{sj,2k-1} + \sqrt{-1}\zeta M_{2j-1,2k} - \eta\zeta M_{2j,2k}), \\ && 0 \le j \ne k \le n \\ E_{2\eta e^j} &=& \frac{1}{2\sqrt{2}}(M_{2j-1,2j-1} + \sqrt{-1}2\eta M_{2j-1,2j} - M_{2j,2j}), \;\; 0 \le j \le n \end{array}\right.$$

The transition from the Cartan basis to the Heisenberg algebra interpretation is given by:

$$\left\{\begin{array}{rcl} M_{2j-1,2k-1} &=& \frac{1}{2}(E_{e^j+e^k} + E_{-e^j+e^k} + E_{e^j-e^k} + E_{-e^j-e^k}) \\ M_{2j,2k-1} &=& \frac{1}{2\sqrt{-1}}(E_{e^j+e^k} - E_{-e^j+e^k} - E_{e^j-e^k} - E_{-e^j-e^k}) \\ M_{2j,2k} &=& -\frac{1}{2}(-E_{e^j+e^k} - E_{-e^j+e^k} - E_{e^j-e^k} - E_{-e^j-e^k}), \end{array}\right.$$

where $0 \le j < k \le n$; also

$$\left\{\begin{array}{rcl} M_{2j-1,2j} &=& -\sqrt{\frac{-1}{2}}(E_{2e^j} - E_{-2e^j}) \\ M_{2j-1,2j-1} &=& \frac{1}{\sqrt{2}}(E_{2e^j} + E_{-2e^j}) - H_j \\ M_{2j,2j} &=& -\frac{1}{\sqrt{2}}(E_{2e^j} + E_{-2e^j}) - H_j \end{array}\right.$$

.

Using the commutator relations:

$$\begin{cases}[E_{e^j-e^k}, E_{-e^j+e^k}] = H_j - H_k\\ [E_{e^j+e^k}, E_{-e^j-e^k}] = -H_j - H_k\\ [E_{2e^j}, E_{-2e^j}] = -2H_j,\end{cases}$$

we may conclude as we did for Lemma 2.1 that a unitary highest weight module with highest weight $|\Omega\rangle = |\lambda_1, ..., \lambda_n\rangle$ must satisfy $0 \geq \lambda_1 \geq ... \geq \lambda_n$.

Finally, recall that the fundamental harmonic oscillator representations of the Lie algebra $\mathfrak{sp}(2n, \mathbb{R})$ are the two irreducible components of the representation of the Heisenberg algebra generated by the so called "raising" and "lowering" operators from the vacuum state $|-\frac{1}{2}, -\frac{1}{2}, ..., -\frac{1}{2}\rangle$. Here the raising operator a_k and the lowering operator $a_k^\dagger$ are given by $x_{2j-1} = \frac{1}{\sqrt{2}}(a_j + a_j^\dagger)$ and $x_{2j} = \frac{1}{\sqrt{-2}}(a_j - a_j^\dagger)$. And the Cartan basis may be defined more compactly as:

$$\begin{cases} H_i &= -(a_i^\dagger a_i + \frac{1}{2}) \quad 1 \leq i \leq n\\ E_{-e^j+e^k} &= a_j^\dagger a_k, \quad 0 \leq j \neq k \leq n\\ E_{-e^j-e^k} &= a_j^\dagger a_k^\dagger, \quad 0 \leq j \neq k \leq n\\ E_{-2e^j} &= \frac{1}{\sqrt{2}} a_j^\dagger a_j^\dagger, \ \ 0 \leq j \leq n \end{cases}$$

4.2 A Quadratic Relation

Our main result in this chapter is the following:

Proposition 4.1. An irreducible highest weight module satisfies the quadratic relation

$$\{M_{ab}, M_{cd}\} - \{M_{ac}, M_{bd}\} = \omega_{cb}\omega_{ad} + \frac{1}{2}\omega_{ab}\omega_{cd} - \frac{1}{2}\omega_{ac}\omega_{bd}$$

if and only if it is one of the two fundamental harmonic oscillator representation, i.e. those with highest weight $|-\frac{1}{2}, -\frac{1}{2}, ..., -\frac{1}{2}\rangle$ or $|-\frac{1}{2}, -\frac{1}{2}, ..., -\frac{3}{2}\rangle$.

We break the proof of proposition into the following lemmas.

Lemma 4.2. The fundamental harmonic oscillator representations satisfy the above mentioned quadratic relation.

Proof. The proof is a straightforward computation in the Heisenberg algebra, utilizing only the commutator relations. □

For the converse:

Lemma 4.3. An irreducible highest weight module satisfying the above quadratic relations could only be those with highest weights $|-\frac{1}{2}, -\frac{1}{2}, ..., -\frac{1}{2}\rangle$ or $|-\frac{1}{2}, -\frac{1}{2}, ..., -\frac{3}{2}\rangle$.

Proof. First of all we notice that it suffices to check for $n = 2$ case. Since any other (a, b, c, d) than $(1, 2, 3, 4)$ can be conjugated to it by an inner automorphism of the group $Sp(2n, \mathbb{R})$.

Thus we may well assume that the symplectic Lie algebra is $\mathfrak{sp}(4, \mathbb{R})$, and the quadratic relations for $(a, b, c, d) = (1, 3, 1, 4)$, $(a, b, c, d) = (3, 1, 3, 2)$ read:

$$\begin{cases}1: \ \ \{M_{13}, M_{14}\} - \{M_{11}, M_{34}\} = 0,\\ 2: \ \ \{M_{31}, M_{32}\} - \{M_{33}, M_{12}\} = 0.\end{cases}$$

We will evaluate these identities on the highest weight vector $|\Omega\rangle = |\lambda, \mu\rangle$.

For equation 1, we have:

$$\begin{aligned}
&\{M_{13}, M_{14}\} - \{M_{11}, M_{34}\}|\Omega\rangle \\
= \;& \{\tfrac{1}{2}(E_{e^1+e^2} + E_{-e^1+e^2} + E_{e^1-e^2} + E_{-e^1-e^2}), \tfrac{1}{\sqrt{-2}}(E_{e^1+e^2} + E_{-e^1+e^2} - E_{e^1-e^2} - E_{-e^1-e^2})\} \\
& -\{\tfrac{1}{\sqrt{2}}(E_{2e^1} + E_{-2e^1}) - H_1, -\sqrt{\tfrac{-1}{2}}(E_{2e^2} + E_{-2e^2}\})|\Omega\rangle \\
= \;& \tfrac{\sqrt{-1}}{2}(\underbrace{-E^2_{-e^1+e^2}|\Omega\rangle}_{A} + \underbrace{E^2_{-e^1-e^2}|\Omega\rangle}_{B} + \underbrace{\sqrt{2}(2\lambda+1)E_{-2e^2}|\Omega\rangle}_{C} - \underbrace{2E_{-2e^1}E_{-2e^2}|\Omega\rangle)}_{D}
\end{aligned}$$

And similarly as in chapter 2, we compute readily: $A^\dagger A = 2(\lambda-\mu)^2 - 2(\lambda-\mu)$, $A^\dagger B = A^\dagger C = A^\dagger D = B^\dagger C = C^\dagger D = 0$, $B^\dagger B = 2(\lambda+\mu)^2 - 2(\lambda+\mu)$, $B^\dagger D = -4(2\lambda+1)^2\mu$, and $D^\dagger D = 16\mu\lambda$.

From this we conclude that equation 1 gives rise to the equation

$$4(\lambda^2 + \mu^2 - \lambda + 3\mu - 4\lambda^2\mu) = 0$$

For equation 2, we do the same thing: $\{M_{13}, M_{14}\} - \{M_{11}, M_{34}\}|\Omega\rangle$

$$\begin{aligned}
= \;& \{\tfrac{1}{2}(E_{e^1+e^2} + E_{-e^1+e^2} + E_{e^1-e^2} + E_{-e^1-e^2}), \tfrac{1}{\sqrt{-2}}(E_{e^1+e^2} - E_{-e^1+e^2} + E_{e^1-e^2} - E_{-e^1-e^2})\} \\
& -\{\sqrt{\tfrac{-1}{2}}(E_{2e^1} - E_{-2e^1}), \sqrt{\tfrac{1}{2}}(E_{2e^2} + E_{-2e^2} - H_2)\}|\Omega\rangle \\
= \;& \tfrac{\sqrt{-1}}{2}(\underbrace{E^2_{-e^1+e^2}|\Omega\rangle}_{A} + \underbrace{2E_{-e^1-e^2}E_{-e^1+e^2}|\Omega\rangle}_{B} + \underbrace{\sqrt{2}(2\mu+1)E_{-2e^1}|\Omega\rangle}_{C} \\
& + \underbrace{E^2_{-e^1-e^2}|\Omega\rangle)}_{D} - \underbrace{2E_{-2e^1}E_{-2e^2}|\Omega\rangle)}_{E}
\end{aligned}$$

Also we compute easily that $A^\dagger A = 2(\lambda-\mu)^2 - 2(\lambda-\mu)$, $B^\dagger B = 4\mu^2 - 4\lambda^2$, $B^\dagger C = 4(\lambda-\mu)(1+2\mu)$, $C^\dagger C = 2(1+2\mu)^2(-2\lambda)$, $D^\dagger D = 2(\lambda+\mu)^2 - 2(\lambda+\mu)$, $D^\dagger E = 8\mu$, $E^\dagger E = 16\mu\lambda$, and $A^\dagger B = A^\dagger C = A^\dagger D = A^\dagger E = B^\dagger D = B^\dagger E = C^\dagger D = C^\dagger E$. It follows equation 2 implies the equation $8\mu(1+2\lambda)(1-\mu) = 0$.

Now from these two equations:

$$\begin{cases} 4(\lambda^2 + \mu^2 - \lambda + 3\mu - 4\lambda^2\mu) = 0 \\ 8\mu(1+2\lambda)(1-\mu) = 0, \end{cases}$$

we obtain the following possibilities, with the aide of assumption $0 \geq \lambda \geq \mu$:

$$\begin{cases} \mu = 0, \;\; \lambda = 0 \\ \lambda = -\frac{1}{2} \;\; , \mu = -\frac{1}{2} \\ \lambda = -\frac{1}{2} \;\; , \mu = -\frac{3}{2} \end{cases}$$

□

4.3 Formal Curvature Type Operators

Now similar as what we did for the special Lie orthogonal algebra cases, we introduce the following curvature type quantity and reformulate our main result of this section in this new language.

Definition 4.4. Define a family of operators R_{abcd} as follows:

$$R_{abcd} = 2\{M_{ab}, M_{cd}\} - \{M_{ac}, M_{db}\} - \{M_{bc}, M_{ad}\}.$$

Lemma 4.5. The family of operators satisfy the following (super-)curvature type properties:

(1) $R_{abcd} = R_{bacd} = R_{abdc} = R_{cdab}$,

(2) $R_{abcd} + R_{acdb} + R_{adbc} = 0$.

(3) $\omega^{bc}R_{abcd} = 3\{M_{ab}, M^b{}_d\}$

Proof. (1). These follows from definition.

(2). Similar calculations as we did in, and here we have $M_{ab} = M_{ba}$:

$$\begin{aligned} R_{abcd} + R_{acdb} + R_{adbc} &= 2\{M_{ab}, M_{cd}\} - \{M_{ac}, M_{db}\} - \{M_{bc}, M_{ad}\} \\ &\quad +2\{M_{ad}, M_{bc}\} - \{M_{ab}, M_{cd}\} - \{M_{db}, M_{ac}\} \\ &\quad +2\{M_{ac}, M_{db}\} - \{M_{ad}, M_{bc}\} - \{M_{cd}, M_{ab}\} \\ &= 0. \end{aligned}$$

(3). Notice that the last term is symmetric in bc, and ω^{bc} is antisymmetric in bc:

$$\begin{aligned} \omega^{bc}R_{abcd} &= 2\omega^{bc}\{M_{ab}, M_{cd}\} - \omega^{bc}\{M_{ab}, M_{cd}\} - \omega^{bc}\{M_{ab}, M_{cd}\} \\ &= 3\{M_{ab}, M^b{}_d\} \end{aligned}$$

□

In terms of these formal "curvature-type" quantities, we reformulate Lemma 4.3 as follows:

Proposition 4.6. An irreducible highest weight module satisfies the quadratic relation:

$$R_{abcd} = c(\omega_{ad}\omega_{cb} - \omega_{ac}\omega_{bd})$$

if and only if it's one of the two fundamental harmonic oscillator representation, i.e. those with highest weight $|-\frac{1}{2}, -\frac{1}{2}, ..., -\frac{1}{2}\rangle$ or $|-\frac{1}{2}, -\frac{1}{2}, ..., -\frac{3}{2}\rangle$.

Proof. "⇐": In the harmonic oscillator representations, we have:

$$\{M_{ab}, M_{cd}\} - \{M_{ac}, M_{bd}\} = \omega_{cb}\omega_{ad} + \frac{1}{2}\omega_{ab}\omega_{cd} - \frac{1}{2}\omega_{ac}\omega_{bd}$$

and

$$\{M_{ba}, M_{cd}\} - \{M_{bc}, M_{ad}\} = \omega_{ca}\omega_{bd} + \frac{1}{2}\omega_{ba}\omega_{cd} - \frac{1}{2}\omega_{bc}\omega_{ad}$$

Summing these up, and noticing that $M_{ab} = M_{ba}$, we obtain:

$$R_{abcd} = \frac{3}{2}(\omega_{ad}\omega_{cb} - \omega_{ac}\omega_{bd}).$$

"⇒": In the special cases, $(a, b, c, d) = (1, 1, 3, 4)$ and $(a, b, c, d) = (1, 2, 3, 3)$, the above equations specializes to the two identities we used in the proof of Lemma 4.3, thus the proof goes through as in that case. □

Remark 4.7. By (3) of Lemma 4.5, we may also propose the problem of what the unitary highest weight modules are which satisfy the similar Einstein equations $\{M_{ac}, M^c{}_b\} = c \cdot \omega_{ab}$. Yet after a lengthy calculation similar as those done in Chapter 2, we found out that there is a continuous infinite family of modules satisfying this formal Einstein equation, and in this sense, there does not seem to be a clear duality between the symplectic and orthogonal cases.

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