Report on Fourier-Mukai

Note Title 4/8/2010

Ref:

1. S. Mukai. Duality Between $D(X)$ And $D(\hat{X})$ With It's Application To Picard Sheaves
2. D. Huybrechts. Fourier-Mukai Transforms in Algebraic Geometry.
3. D. Mumford. Abelian Varieties

§1. Generalities

Let X, Y be smooth projective varieties and denote:

$$\begin{tikzcd} & X\times Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X & & Y \end{tikzcd}$$

Def. Let $\mathcal{P} \in D^b(X\times Y)$. The induced Fourier-Mukai transform is the functor

$$\begin{aligned} \mathcal{J}_{X\to Y,\mathcal{P}}: D^b(X) &\longrightarrow D^b(Y) \\ \mathcal{E} &\longmapsto R\pi_{Y*}(\mathcal{P} \overset{\mathbb{L}}{\otimes} \pi_X^*\mathcal{E}) \end{aligned}$$

Examples:

(1). If $f: X \longrightarrow Y$ is a morphism, and consider $\mathcal{P} = \mathcal{O}_{\Gamma_f} \in D^b(X\times Y)$. Then

$$\begin{aligned} \mathcal{J}_{X\to Y,\mathcal{P}}(\mathcal{E}) &= R\pi_{Y*}(\mathcal{O}_{\Gamma_f} \overset{\mathbb{L}}{\otimes} \pi_X^*\mathcal{E}) \\ &= R\pi_{Y*}(Ri_*\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \pi_X^*\mathcal{E}) \qquad i: X \xrightarrow{\cong} \Gamma_f \subseteq X\times Y \\ &= R\pi_{Y*}(Ri_*(\mathcal{O}_X \overset{\mathbb{L}}{\otimes} Li^*\circ\pi_X^*\mathcal{E})) \qquad \text{(projection formula)} \\ &= Rf_*(\mathcal{E}) \qquad (\pi_Y\circ i = f, \quad \pi_X\circ i = id_X) \end{aligned}$$

$$\begin{aligned} \mathcal{J}_{Y\to X,\mathcal{P}}(\mathcal{F}) &= R\pi_{X*}(\mathcal{O}_{\Gamma_f} \overset{\mathbb{L}}{\otimes} \pi_Y^*\mathcal{F}) \\ &= R\pi_{X*}(Ri_*\mathcal{O}_X \overset{\mathbb{L}}{\otimes} \pi_Y^*\mathcal{F}) \\ &= R\pi_{X*}Ri_*(\mathcal{O}_X \overset{\mathbb{L}}{\otimes} Li^*(\pi_Y^*\mathcal{F})) \\ &= Lf^*\mathcal{F}. \end{aligned}$$

(2). Any line bundle $\mathcal{L}$ on X defines $\mathcal{E} \longrightarrow \mathcal{E}\otimes\mathcal{L}$ an automorphism of $D^b(X)$. The corresponds to $\mathcal{J}_{X\to X,\mathcal{P}}$, where $\mathcal{P} = \Delta_*\mathcal{L} \in Coh(X\times X)$.

(3). The shift functor $T: D^b(X) \longrightarrow D^b(X)$ is given by the FM kernel $\mathcal{O}_\Delta[1]$.

(4). If $\mathcal{P} \in Coh(X\times Y)$ is flat over X, $x\in X$ a closed point:

$$\mathcal{J}_{X\to Y,\mathcal{P}}(k(x)) \cong \mathcal{P}|_{\{x\}\times Y} \in Coh(Y).$$

Composition of FMT's.

Let X, Y, Z be smooth projective varieties. $\mathcal{P} \in D^b(X\times Y)$, $\mathcal{Q} \in D^b(Y\times Z)$. Then $\mathcal{R} \in D^b(X\times Z)$ is defined to be

$$\mathcal{R} = \pi_{XZ*}\left(\pi_{XY}^*\mathcal{P} \overset{L}{\otimes} \pi_{YZ}^*\mathcal{Q}\right)$$

Prop 1. (Mukai). The composition

$$D^b(X) \xrightarrow{\mathcal{J}_P} D^b(Y) \xrightarrow{\mathcal{J}_Q} D^b(Z)$$

is isomorphic to the FMT $\mathcal{J}_R$.

Pf:

$$\begin{tikzcd}
 & X\times Y\times Z \arrow[dl, "\pi_{XY}"'] \arrow[d, "\pi_{XZ}"] \arrow[dr, "\pi_{YZ}"] & \\
(\mathcal{P}-)\ X\times Y \arrow[d, "r_X"'] \arrow[dr, "r_Y"] & X\times Z \arrow[dl, "p_X"'] \arrow[dr, "p_Z"] & Y\times Z\ (-\mathcal{Q}) \arrow[dl, "q_Y"'] \arrow[d, "q_Z"] \\
X & Y & Z
\end{tikzcd}$$

We have:

$$\begin{aligned}
\mathcal{J}_{X\to Z,\mathcal{R}}(\mathcal{E}) &= p_{Z*}(\mathcal{R}\otimes p_X^*\mathcal{E}) \\
&= p_{Z*}(\pi_{XZ*}(\pi_{XY}^*\mathcal{P}\otimes \pi_{YZ}^*\mathcal{Q})\otimes p_X^*\mathcal{E}) \\
&= p_{Z*}(\pi_{XZ*}(\pi_{XY}^*\mathcal{P}\otimes \pi_{YZ}^*\mathcal{Q}\otimes \pi_X^*\mathcal{E})) \quad \text{(projection formula)} \\
&= \pi_{Z*}(\pi_{XY}^*\mathcal{P}\otimes \pi_{XY}^*\circ r_X^*\mathcal{E})\otimes \pi_{YZ}^*\mathcal{Q}) \\
&= \pi_{Z*}(\pi_{XY}^*(\mathcal{P}\otimes r_X^*\mathcal{E})\otimes \pi_{YZ}^*\mathcal{Q}) \\
&= q_{Z*}\,\pi_{YZ*}((\pi_{XY}^*(\mathcal{P}\otimes r_X^*\mathcal{E})\otimes \pi_{YZ}^*\mathcal{Q}) \\
&= q_{Z*}((\pi_{YZ*}(\pi_{XY}^*(\mathcal{P}\otimes r_X^*\mathcal{E}))\otimes \mathcal{Q}) \quad \text{(projection formula)} \\
&= q_{Z*}(q_Y^*(r_{Y*}(\mathcal{P}\otimes r_X^*\mathcal{E}))\otimes \mathcal{Q}) \quad \text{(flat base change)} \\
&= \mathcal{J}_{Y\to Z,\mathcal{Q}}\circ \mathcal{J}_{X\to Y,\mathcal{P}}(\mathcal{E}).
\end{aligned}$$

$\square$

§2. Application I : Abelian Varieties

Let X be an abelian variety of dim g, $\hat{X}$ its dual.
$\mathcal{P}$: the normalized Poincaré bundle on $X\times\hat{X}$, normalized meaning that $\mathcal{P}|_{X\times\hat{0}}$ & $\mathcal{P}|_{0\times\hat{X}}$ are trivial.

$\mathcal{J} \triangleq \mathcal{J}_{\hat{X}\to X,\mathcal{P}}$. $\hat{\mathcal{J}} = \mathcal{J}_{X\to\hat{X},\mathcal{P}}$.

Thm 1. (Mukai) There are isomorphism of functors:

$$R\mathcal{J}\circ R\hat{\mathcal{J}} \cong (-\mathbb{1}_X)^*[-g]$$
$$R\hat{\mathcal{J}}\circ R\mathcal{J} \cong (-\mathbb{1}_{\hat{X}})^*[-g]$$

In other words, $R\mathcal{J}$ gives an equivalence of $D^b(X)$ and $D^b(\hat{X})$, whose quasi-inverse is given by $(-\mathbb{1}_{\hat{X}})^*\circ R\hat{\mathcal{J}}[g]$.

Pf: It suffices to show the first isomorphism, since $\hat{\hat{X}} \cong X$. By Prop 1,

$$R\mathcal{J}\circ R\hat{\mathcal{J}} \cong R\mathcal{J}_{X\to X,H},$$

where $H = R\pi_{12*}(R\pi_{13}^*\mathcal{P}\otimes R\pi_{23}^*\mathcal{P}) \in D^b(X\times X)$, and $\pi_{12}: X\times X\times\hat{X} \longrightarrow X\times X$.
We are reduced to calculating H.

Thm. of cube $\Longrightarrow$ $P_{13}^*\mathcal{P}\otimes P_{23}^*\mathcal{P} \cong (m\times 1)^*\mathcal{P}$. Thus

$$\begin{aligned} H &= R\pi_{12*}((m\times 1)^*\mathcal{P}) \\ &= m^*(R\pi_{X*}\mathcal{P}) \quad \text{(flat base change)} \end{aligned}$$

$$\begin{tikzcd} X\times X\times\hat{X} \arrow[r, "\pi_{12}"] \arrow[d, "m\times 1"'] & X\times X \arrow[d, "m"] \\ X\times\hat{X} \arrow[r, "\pi_X"] & X \end{tikzcd}$$

To this point, we quote the following theorem, whose proof is given below:

Thm. 2. (Mumford). $R\pi_{X*}\mathcal{P} \cong k(0)[-g]$.

It follows from the diagram:

$$\begin{tikzcd} \Gamma_\iota \arrow[r, hook, "i"] \arrow[d, "p"] & X\times X \arrow[d, "m"] \\ 0 \arrow[r, hook, "j"] & X \end{tikzcd}$$

$H = m^*(R\pi_{X*}\mathcal{P}) = m^*(k(0)[-g]) = \mathcal{O}_{\Gamma_f}[-g]$. The thm. follows from our example (i). □

Proof of Mumford's theorem.

Lemma 1. If $\mathcal{L} \in \mathrm{Pic}^0(X)$ and $\mathcal{L}$ is nontrivial, then

$$H^k(X, \mathcal{L}) = 0, \quad \forall k \in \mathbb{Z}.$$

Pf: Let $\mathcal{L} \in \mathrm{Pic}^0(X)$, and $\mathcal{L} \ncong \mathcal{O}_X$, $s \in H^0(X, \mathcal{L}) \Rightarrow$

$$\mathcal{O}_X \xrightarrow{\cdot s} \mathcal{L}$$

$c_1(\mathcal{L}) = 0 \Rightarrow \mathrm{div}(s) = \phi \Rightarrow \mathcal{O}_X \xrightarrow{\cong} \mathcal{L}$. contradiction.

Inductively, suppose we have shown that $H^k(X, \mathcal{L}) = 0$, $\forall k < n$. Then

$$\begin{tikzcd} X \arrow[r, "{(1,0)}"] \arrow[rr, bend right, "id"'] & X \times X \arrow[r, "m"] & X \end{tikzcd}$$

$$\Rightarrow \quad \begin{tikzcd} H^n(X,\mathcal{L}) & H^n(X\times X, m^*\mathcal{L}) \arrow[l, "{(1,0)^*}"'] \arrow[d, phantom, "\cong\ (\text{See-Saw})"] & H^n(X,\mathcal{L}) \arrow[l, "m^*"'] \arrow[ddl] \\ & H^n(X \times X, \pi_1^*\mathcal{L} \otimes \pi_2^*\mathcal{L}) \arrow[d, phantom, "\cong\ (\text{Kunneth})"] & \\ & \bigoplus_{k=0}^{n} H^k(X,\mathcal{L}) \otimes H^{n-k}(X,\mathcal{L}) \arrow[uul] & \end{tikzcd}$$

$$\bigoplus_{k=0}^{n} H^k(X,\mathcal{L}) \otimes H^{n-k}(X,\mathcal{L}) \; \| \; 0 \quad (H^0(X,\mathcal{L}) = 0 \text{ and induction hypothesis})$$

$$\Rightarrow \quad H^n(X, \mathcal{L}) = 0. \qquad \square$$

Lemma 2. $R\pi_{X*}(\mathcal{P}) \in D^{[0,g]}(X)$ has cohomology supported at 0.

Pf: That $R\pi_{X*}(\mathcal{P}) \in D^{[0,g]}(X)$ follows from

$$X \times \hat{X} \xrightarrow{\pi_X} X$$

being smooth of relative dimension g.

Next, $\forall x \in X$ a closed point,

$$\begin{tikzcd} x \times \hat{X} \arrow[r, "j"] \arrow[d] & X \times \hat{X} \arrow[d, "\pi_X"] \\ x \arrow[r, hook, "i"] & X \end{tikzcd}$$

The def. of Poincaré line bundle says that $\mathcal{P}|_{x \times \hat{X}} \cong P_x \in \mathrm{Pic}^0(\hat{X})$ is non-trivial iff $x = 0$. $\Rightarrow$ if $x \neq 0$

$$Li^* \circ R\pi_{X*}(\mathcal{P}) = R\Gamma(Lj^*\mathcal{P}) = R\Gamma(P_x) = 0$$

By semi-continuity, $R\pi_{X*}(\mathcal{P}) \in D^b_{coh}(X)$ has cohomology only supported at o. The lemma follows. $\square$

Lemma 3.

$$\begin{tikzcd} o \times \hat{X} \arrow[r, "j"] \arrow[d] & X \times \hat{X} \arrow[d, "\pi_X"] \\ o \arrow[r, hook, "i"] & X \end{tikzcd}$$

Then

$$\begin{aligned} Li_o^* R\pi_{X*}(\mathcal{P}) &= R\Gamma(Lj^*\mathcal{P}) \\ &= R\Gamma(\mathcal{O}_{\hat{X}}) \\ &\cong \bigoplus H^i(\hat{X}, \mathcal{O}_{\hat{X}})[-i] \end{aligned}$$

Moreover, $H^n(\hat{X}, \mathcal{O}_{\hat{X}}) \cong H^0(\hat{X}, \omega_{\hat{X}})^* \cong k$. $\square$

Since $R\pi_{X*}(\mathcal{P})$ is supported only at o, to see what it is, it suffices to make the flat base change:

$$\begin{tikzcd} \operatorname{Spec} \mathcal{O}_{X,o} \times \hat{X} \arrow[r, "j"] \arrow[d, "p"] & X \times \hat{X} \arrow[d, "\pi_X"] \\ \operatorname{Spec} \mathcal{O}_{X,o} \arrow[r, "i"] & X \end{tikzcd}$$

$$\begin{aligned} \Rightarrow i^* R\pi_{X*}\mathcal{P} &= Rp_*(j^*\mathcal{P}) \\ &\cong K^\bullet : (K^0 \to K^1 \to \cdots \to K^g) \end{aligned}$$

where $K^\bullet$ is a complex of free $\mathcal{O}_{\hat{X},o}$-modules. By lemma 1, $K^\bullet$ has cohomology Artinian $\mathcal{O}_{X,o}$-modules supported at o.

Lemma 4. (Mumford) Let $\mathcal{O}$ be a regular local ring of dimension g. Let

$$K^\bullet : 0 \to K^0 \to K^1 \to \cdots \to K^g \to 0$$

a complex of free modules over $\mathcal{O}$. If $H^i(K^\bullet)$ are Artinian modules, we have $H^i(K^\bullet) = 0$, $0 \le i < g$.

Pf: [Mumford, §13, P118]

Nothing to prove for $g=0$

Choose $x \in m$ belonging to a system of parameters so that $\mathcal{O}/x\mathcal{O}$ is regular local of dim $g-1$.

$$\Rightarrow \quad 0 \to K^\bullet \xrightarrow{x} K^\bullet \to \bar{K}^\bullet \to 0$$

$$\Rightarrow \quad \cdots \to H^{i-1}(\bar{K}^\bullet) \to H^i(K^\bullet) \xrightarrow{x} H^i(K^\bullet) \to H^i(\bar{K}^\bullet) \to \cdots$$

$\Rightarrow \quad H^i(K^\bullet) \xrightarrow{x} H^i(K^\bullet)$ is injective for $0 \le i-1 < g-1$, by induction

$\Rightarrow \quad H^i(K^\bullet) = 0 \quad 0 \le i < g.$ $\square$

Pf of Thm 2.

By lemma 4, we know that

$$K^\bullet \cong H^g(K^\bullet)[-g] \in D^b(\mathcal{O})$$

To calculate $H^g(K^\bullet)$, we use

$$\begin{aligned} H^g(K^\bullet) \otimes_{\mathcal{O}} k(0) &\cong H^g(K^\bullet \otimes k(0)) \quad (\text{since } R^{g+1}\pi_* \mathcal{P} = 0) \\ &\cong H^g(\mathcal{P}|_{0 \times \hat{X}}) \\ &\cong H^g(\mathcal{O}_{\hat{X}}) \\ &\cong k \end{aligned}$$

$\Rightarrow \quad 0 \to K^0 \to \cdots \to K^g \to k \to 0$ has cohomology Artinian and only supported at deg g. Thus $K^g/\mathrm{Im}K^{g-1} \twoheadrightarrow k$ and when reduced mod m_0, it's an isomorphism. By Nakayama's lemma,

$$K^g/\mathrm{Im}K^{g-1} \cong k$$

and thus $K^\bullet \to k[-g]$ is an isomorphism in $D^b(\mathcal{O})$. The thm. follows $\square$

Cor. 1. $$H^i(X \times \hat{X}, \mathcal{P}) = \begin{cases} k, & i = g \\ 0, & \text{otherwise.} \end{cases}$$ $\square$

Cor. 2. $$H^i(X, \mathcal{O}_X) \cong \Lambda^i k$$

Pf: Since any free resolution in $\mathrm{Mod}(\mathcal{O})$ of k are homotopic,

$$K^\bullet \cong \mathrm{Koszul}^\bullet \in D^b(\mathcal{O})$$

where $\mathrm{Koszul}^\bullet = \bigotimes_{i=1}^{g} (\mathcal{O} \xrightarrow{x_i} \mathcal{O})$ ($x_i \in m$, $i=1,\dots,g$ form a system of local coordinates). Thus by lemma 3:

$$R\Gamma(\mathcal{O}_{\hat{X}}) = (\mathrm{Koszul}^\bullet) \otimes k$$
$$\cong \bigotimes_{i=1}^{g} (k \xrightarrow{0} k)$$

The result follows. $\square$

Easy consequences:

Def. We say that weak index thm. (WIT) holds for $F \in \mathrm{Coh}(X)$ if $R^i\hat{\mathcal{J}}(F)=0$ for all but one i. This i, denoted $i(F)$ is called the index of F and the coherent sheaf $R^{i(F)}\hat{\mathcal{J}}(F)$ on $\hat{X}$ is denoted $\hat{F}$ & called the Fourier transform of F.

We say that index thm (IT) holds for F if $H^i(X, F\otimes\mathcal{L})=0$ for all $\mathcal{L}\in \mathrm{Pic}^0(X)$ and all but one i.

Rmk: Base change thm $\Rightarrow$ (IT $\Rightarrow$ WIT). The pf of thm says that $\mathcal{O}_X$ satisfies WIT but not IT.

Cor. If WIT holds for F, then so does for $\hat{F}$ and $i(\hat{F}) = g - i(F)$ $\square$

Cor. Assume that WIT holds for F & G. Then

$$\mathrm{Ext}^i_{\mathcal{O}_X}(F,G) \cong \mathrm{Ext}^{i+\mu}_{\mathcal{O}_{\hat{X}}}(\hat{F},\hat{G})$$

$\forall i$, where $\mu = i(F) - i(G)$. In particular

$$\mathrm{Ext}^i_{\mathcal{O}_X}(F,F) \cong \mathrm{Ext}^i_{\mathcal{O}_{\hat{X}}}(\hat{F},\hat{F}).$$

Pf:
$$\begin{aligned}\mathrm{Ext}^i_{\mathcal{O}_X}(F,G) &= \mathrm{Hom}_{D^b(X)}(F, G[i]) \\ &= \mathrm{Hom}_{D^b(\hat{X})}(R\mathcal{J}_{X\to\hat{X}}(F), R\mathcal{J}_{X\to\hat{X}}(G)[i]) \\ &= \mathrm{Hom}_{D^b(\hat{X})}(\hat{F}[-i(F)], \hat{G}[-i(G)+i]) \\ &= \mathrm{Ext}^{i(F)-i(G)+i}_{\mathcal{O}_{\hat{X}}}(\hat{F},\hat{G}).\end{aligned}$$
$\square$

Example: Let $k(\hat{x})$ denote the skyscraper sheaf supported by $\hat{x} \in \hat{X}$. Since $H^i(X, k(\hat{x}) \otimes \mathcal{L}) = 0$, $\forall\, i>0$, $\mathcal{L} \in Pic^0(\hat{X})$. IT holds for $k(\hat{x})$, $i(k(\hat{x})) = 0$ & $\widehat{k(\hat{x})} \cong P_{\hat{x}} \Rightarrow$ WIT holds for $P_{\hat{x}}$, $i(P_{\hat{x}}) = g$ & $\hat{P}_{\hat{x}} = k(-\hat{x})$. But IT doesn't hold for $P_{\hat{x}}$.

Cor. Assume WIT holds for a coherent sheaf F on X. Then we have:

$$H^i(X, F \otimes P_{\hat{x}}) \cong Ext^{g-i(F)+i}_{\mathcal{O}_{\hat{X}}}(k(\hat{x}), \hat{F})$$
$$Ext^i_{\mathcal{O}_X}(k(x), F) \cong H^{i-i(F)}(\hat{X}, \hat{F} \otimes P_{-x})$$

Pf: $P_{\hat{x}}$ locally free $\Rightarrow$

$$\begin{aligned} H^i(X, F \otimes P_{\hat{x}}) &\cong Ext^i(P_{-\hat{x}}, F) \\ &\cong Ext^{i+i(P_{-\hat{x}})-i(F)}(\widehat{P_{-\hat{x}}}, \hat{F}) \\ &\cong Ext^{i+g-i(F)}(k(\hat{x}), \hat{F}) \end{aligned}$$ $\square$

Example: A vector bundle $\mathcal{U}$ on X is called unipotent if it has a filtration:

$$0 = \mathcal{U}_0 \subseteq \mathcal{U}_1 \subseteq \cdots \subseteq \mathcal{U}_{n-1} \subseteq \mathcal{U}_n = \mathcal{U}$$

s.t. $\mathcal{U}_i / \mathcal{U}_{i-1} \cong \mathcal{O}_X$, $i = 1, \cdots, n$. Since $R^i\mathcal{J}_{X\to\hat{X}}$ is exact in the middle, WIT holds for $\mathcal{U}$, $i(\mathcal{U}) = g$ and the sheaf $\hat{\mathcal{U}}$ is supported at $\hat{o} \in \hat{X}$. Hence:

$R^g\mathcal{J}_{X\to\hat{X}}$: ((Unipotent vector bundles)) $\xrightarrow{\sim}$ ((Skyscraper sheaves at $\hat{o}$))

$SL(2,\mathbb{Z})$-action.

The following beautiful result is due to Mukai:

Thm 3 (Mukai) Let (X, L) be a principally polarized abelian variety, with the isomorphism:

$$\varphi_L : X \xrightarrow{\cong} \hat{X}$$
$$x \longmapsto T_x^*L \otimes L^{-1}.$$

Let $\mathcal{J}: D^b(X) \longrightarrow D^b(\hat{X}) \xrightarrow{\varphi_L^*} D^b(X)$ be the composition. Then:

(i). $\mathcal{J}^4 \cong [-2g]$

(ii). $(L \otimes (\mathcal{J}(-)))^3 = [-g]$

i.e. modulo dimension shifting, this defines an $SL(2,\mathbb{Z})$-action on $D^b(X)$, by assigning:

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \longmapsto \mathcal{J}$$

$$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \longmapsto \otimes L$$