Categorification at prime roots of unity

Note Title 10/18/2015

§1. Hopfological algebra

In 1994, Crane and Frenkel published their seminal paper "Four-dimensional topological quantum field theory, Hopf categories, and the canonical bases". There they proposed a program called "categorification", hoping to lift the combinatorially defined 3-manifold invariant constructed by Kuperberg to a 4d TQFT

$$\begin{tikzcd}[column sep=large, row sep=large]
\text{Donaldson-Floer} \arrow[r, Leftrightarrow] \arrow[d, Rightarrow] & \boxed{\text{A 4D TQFT ?}} \\
\begin{array}{l}\text{Kuperberg: } u_q^{+}(\mathfrak{sl}_2)\text{-invariant of} \\ \text{3-manifolds}\end{array} \arrow[r, phantom, "\sim"] & \begin{array}{l}\text{WRT-3D TQFT} \\ \text{associated with } u_q(\mathfrak{sl}_2)\end{array} \arrow[u, Rightarrow, "\text{Categorification}"']
\end{tikzcd}$$

- q: a primitive n-th root of unity

Digression: Homological algebra

Assume, for simplicity, that we work over a ground field $\Bbbk$. Homological algebra has the following features.

(0). Chain complexes and their cohomology groups

$$(K^\bullet, d): \quad d: K^\bullet \longrightarrow K^{\bullet+1}, \quad d^2 = 0 \;;$$

(1). Direct sums of chain complexes ;

(2). Tensor products of chain complexes : $K^\bullet \otimes L^\bullet$

(3). Inner homs between chain complexes: $\mathrm{HOM}^\bullet(K^\bullet, L^\bullet)$

$$\mathrm{HOM}^i(K^\bullet, L^\bullet) := \{ f: K^\bullet \longrightarrow L^\bullet \mid f(K^k) \subseteq L^{k+i} \}$$
$$d(f) = d \circ f - (-1)^{|f|} f \circ d.$$

(4). Triangular structures.

Homological shifts / cone constructions / s.e.s. leading to d.t.
TR1 — TR4 etc.

Homological algebra plays a fundamental role in categorification since it gives a systematic lifting of abelian structures

$$\begin{array}{ccc} D^b(\Bbbk) & \xrightarrow{\chi\,(K_0)} & \mathbb{Z} \\ K^\bullet & \mapsto & \sum(-1)^i \dim_i K^i \\ \oplus & \mapsto & \text{addition} \\ \otimes & \mapsto & \text{multiplication} \\ \text{tensor unit} & \mapsto & 1 \\ [1] & \mapsto & \text{multiplication by } (-1). \end{array}$$

Rmk: If we replace vector spaces by graded vector spaces, we get a systematic lifting of "quantum" abelian structures:

$$K_0(\Bbbk\text{-gvect}) \cong \mathbb{Z}[q,q^{-1}]$$

The grading shift $\{1\}$ decategorifies to multiplication by q.

- Observation: Feature (1) – (3) are reminiscent of some basic constructions in representation theory: If G is some group, $H=\Bbbk G$ is a Hopf algebra so that its category of modules H-mod has:

(1)'. $K \oplus L \in H\text{-mod}$

(2)'. $K \otimes L \in H\text{-mod}$ $\quad h(k\otimes \ell) := \sum (h_{(1)}k)\otimes(h_{(2)}\ell)$.

(3)'. $\mathrm{HOM}(K,L) \in H\text{-mod}$ $\quad (h\cdot f)(k) := \sum h_{(2)} f(S^{-1}(h_{(1)}(v)))$.

Thus (1) – (3) above can be viewed as a special case of (1)' – (3)' for the Hopf superalgebra of dual numbers $H=\Bbbk[d]/(d^2)$.

• Question: Are there analogues of the other features of homological algebra for H-mod? For instance, what is "cohomology"?

Any chain complex / $\Bbbk$ decomposes uniquely into direct sums:

$$(\oplus\ 0 \longrightarrow \Bbbk \longrightarrow 0) \oplus (\oplus\ 0 \longrightarrow \Bbbk \longrightarrow \Bbbk \longrightarrow 0)$$

Taking cohomology does nothing but killing the second factor, which is a direct sum of free $\Bbbk[d]/(d^2)$-modules.

Less obvious is the fact that $(0 \longrightarrow \Bbbk \longrightarrow \Bbbk \longrightarrow 0)$ is also injective. In fact, $\Bbbk[d]/(d^2)$ is a Frobenius superalgebra.

Thm. (Sweedler) A Hopf algebra H is Frobenius iff it is finite-dim'l.

Our question reduces to asking how one can systematically kill projective-and-injective modules.

The stable category

Intuitively, the stable category H-$\underline{\text{mod}}$ is the categorical quotient of H-mod by the class of projective / injective objects.

Def. The category H-$\underline{\text{mod}}$ consists of the same objects as H-mod, while the morphism spaces between any objects K, L are given by

$$\mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(K, L) := \mathrm{Hom}_{H\text{-mod}} \Big/ \left(\begin{array}{c}\text{morphisms that factor} \\ \text{through pro/injectives}\end{array}\right)$$

Rmk: The notion of stable categories makes sense for any self-injective algebra, not necessarily those coming from finite-dimensional Hopf algebras.

Thm. (Heller) If H is self-injective, then H-$\underline{\mathrm{mod}}$ is triangulated.

In general, the morphism spaces between objects in some stable category is hard to compute. But for a stable category arising from a finite dimensional Hopf algebra, there is a conceptually easy way to compute them. To do this we need the notion of integrals for Hopf algebras.

Def. Let H be a Hopf algebra$/\Bbbk$. An element $\Lambda \in H$ is called a left integral in H if $\forall h \in H$,

$$h \cdot \Lambda = \epsilon(h)\Lambda.$$

Thm. (Sweedler) Any finite dimensional H has a non-zero integral, unique up to a non-zero constant.

Examples (1). $H = \Bbbk G$ (G: finite group). $\Lambda = \sum_{g\in G} g$.

(2). $H = \Bbbk[d]/(d^2)$, $\Lambda = d$.

(3). $H = \Bbbk[\partial]/(\partial^p)$, $(\mathrm{char}\,\Bbbk = p > 0)$, $\Lambda = \partial^{p-1}$.

Prop. Let H be a finite-dim'l Hopf algebra, and K, L be H-modules. Then

$$\begin{aligned}\mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(K,L) &\cong \mathrm{Hom}_H(K,L)/\Lambda\cdot \mathrm{HOM}(K,L)\\ &\cong \mathrm{HOM}(K,L)^H/\Lambda\cdot\mathrm{HOM}(K,L)\end{aligned}$$

We will prove the prop shortly. Before that, we look at a couple of examples.

Examples. (1) $H = \Bbbk G$, a finite group, $\Lambda = \sum_{g\in G} g$. Recall that H is semisimple iff $\Bbbk$ is a projective module. This is equivalent to requiring that $\mathrm{Hom}_{H\text{-}\underline{mod}}(\Bbbk, \Bbbk) = 0$. But $\Lambda \cdot \mathrm{HOM}(\Bbbk, \Bbbk) = \epsilon(\Lambda)\Bbbk = |G| \cdot \Bbbk$. Thus $\Bbbk G$ is semisimple iff $|G| \in \Bbbk^*$.

(2) $H = \Bbbk[d]/(d^2)$: $\Lambda \cdot f = d \cdot f = d \circ f - (-1)^{|f|} f \circ d$

(3) $H = \Bbbk[\partial]/(\partial^{p-1})$ $(\mathrm{char}\,\Bbbk = p > 0)$

$$\Lambda \cdot f = \partial^{p-1}(f) = \sum_{i=0}^{p-1} \partial^i \circ f \circ \partial^{p-1-i}.$$

Lemma. An H-module map $K \longrightarrow L$ factors through an injective H-module iff there exists a factorization

$$\begin{tikzcd} K \arrow[rr] \arrow[dr, "\mathrm{Id}\otimes\Lambda"'] & & L \\ & K\otimes H \arrow[ur] & \end{tikzcd}$$

Proof. It suffices to show this when L is injective. Consider the following commutative diagram

$$\begin{tikzcd} K \arrow[r, "\varphi"] \arrow[d, "\mathrm{Id}_K\otimes\Lambda"'] & L \arrow[d, "\mathrm{Id}_L\otimes\Lambda"'] \\ K\otimes H \arrow[r, "\varphi\otimes\mathrm{Id}"] & L\otimes H \arrow[u, bend right, "g"'] \end{tikzcd},$$

since L is injective, the injection $\mathrm{Id}_L \otimes \Lambda : L \longrightarrow L \otimes H$ must split. Choose a splitting g. Then φ factors as $g \circ (\varphi \otimes \mathrm{Id}) \circ (\mathrm{Id}_K \otimes \Lambda)$. □

Lemma. An H-module map $\varphi : K \longrightarrow L$ factors through $\mathrm{Id}_K \otimes \Lambda : K \longrightarrow K \otimes H$ iff there is a $\Bbbk$-linear map ψ s.t. $\varphi = \Lambda \cdot \psi$.

Proof. If $\varphi = \Lambda \cdot \psi$ for some $\psi \in \mathrm{Hom}_{\Bbbk}(K, L)$, we will extend ψ to $\tilde{\psi} : K \otimes H \longrightarrow L$ by

$$\tilde{\psi}(k \otimes h) := (h \cdot \psi)(k) = h_{(2)} \psi(S^{-1}(h_{(1)}) k)$$

Then $\tilde{\psi}$ is H-linear: $\forall x, h \in H$, $k \in K$, we have

$$\tilde{\psi}(x \cdot (k \otimes h)) = \tilde{\psi}(x_{(1)} k \otimes x_{(2)} h)$$

$$
\begin{aligned}
&= (x_{(2)}h)_{(2)}\tilde{\Psi}(S^{-1}((x_{(2)}h)_{(1)})\, x_{(1)}k) \\
&= x_{(3)}h_{(2)}\tilde{\Psi}(S^{-1}(h_{(1)})\, S^{-1}(x_{(2)})x_{(1)}k) \\
&= x_{(2)}h_{(2)}\tilde{\Psi}(S^{-1}(h_{(1)})\, \epsilon(x_{(1)})k) \\
&= x\, h_{(2)}\tilde{\Psi}(S^{-1}(h_{(1)})k) \\
&= x\,(h\cdot\tilde{\Psi})(k) \\
&= x\,\tilde{\Psi}(k\otimes h)
\end{aligned}
$$

Then, φ factors through $\varphi: K \xrightarrow{\mathrm{Id}\otimes\Lambda} K\otimes H \xrightarrow{\tilde{\Psi}} L$.

Conversely, given such a factorization of H-module maps

$$\varphi: K \xrightarrow{\mathrm{Id}\otimes\Lambda} K\otimes H \xrightarrow{\tilde{\Psi}} L.$$

Let ψ be the $\Bbbk$-linear composition map $K \xrightarrow{\cong} K\otimes 1 \hookrightarrow K\otimes H \xrightarrow{\tilde{\Psi}} L$. Then $\varphi = \Lambda\cdot\psi$. Indeed, $\forall k\in K$,

$$
\begin{aligned}
(\Lambda\cdot\psi)(k) &= \Lambda_{(2)}\psi(S^{-1}(\Lambda_{(1)})k) \\
&= \Lambda_{(2)}\tilde{\Psi}(S^{-1}(\Lambda_{(1)})k\otimes 1) \\
&= \tilde{\Psi}(\Lambda_{(2)}\cdot(S^{-1}(\Lambda_{(1)})k\otimes 1)) \\
&= \tilde{\Psi}(\Lambda_{(2)}S^{-1}(\Lambda_{(1)})k\otimes\Lambda_{(3)}) \\
&= \tilde{\Psi}(\epsilon(\Lambda_{(1)})k\otimes\Lambda_{(2)}) \\
&= \tilde{\Psi}(k\otimes\Lambda) = \varphi(k).
\end{aligned}
$$

□

Relation to categorification

Def. Let H be the graded Hopf algebra $\Bbbk[\partial]/(\partial^p)$, $\deg\partial := 1$. We call H-gmod the category of p-complexes, while H-$\underline{\text{gmod}}$ the homotopy category of p-complexes.

Historically, the first consideration of p-complexes and their homotopy category is due to Mayer (1942). In the definition of simplicial homology theory, the differential $d = \sum_i (-1)^i d_i$ satisfies $d^2 \equiv 0$. Mayer noticed that, if we work over a field of char $p>0$, and set $\partial := \sum_i d_i$. Then $\partial^p \equiv 0$, and there are the corresponding notions of (Mayer) homology. However, Spanier soon found out that Mayer's homology can be recovered from the usual homology groups ($d^2=0$), and thus are less interesting.

Then, why do we care about p-complexes? This is due to the following simple observation.

Lemma (Bernstein-Khovanov). If $H = \Bbbk[\partial]/(\partial^p)$, $\deg(\partial)=1$, then

$$K_0(H\text{-gmod}) \cong \mathbb{Z}[q,q^{-1}]$$
$$K_0(H\text{-}\underline{\text{gmod}}) \cong \mathbb{Z}[q,q^{-1}]/(1+q+\cdots+q^{p-1}) := \mathcal{O}_p.$$

Indeed, K_0 of the homotopy category is generated by the symbol $[\Bbbk]$, subject to the only relation

$$0 = [H] = [\Bbbk] + [\Bbbk\{1\}] + \cdots + [\Bbbk\{p-1\}] = (1+q+\cdots+q^{p-1})[\Bbbk].$$

In other words, H-$\underline{\text{gmod}}$ is a categorical interpretation of the ring of the pth cyclotomic integers.

$$\begin{array}{ccc} H\text{-}\underline{\text{gmod}} & \xrightarrow{K_0} & \mathbb{Z}[q,q^{-1}]/(1+\cdots+q^{p-1}) \\ \oplus \quad \otimes & \longmapsto & +,\ - \\ [1] & \longmapsto & -1 \\ \{1\} & \longmapsto & q \end{array}$$

Here, the homological shift is defined as follows: $M \in H$-mod, then we have the canonical

$$\varphi_M : M \xrightarrow{\mathrm{Id}\otimes\Lambda} M\otimes H\{-\deg\Lambda\}$$

Then

$$M[1] := \mathrm{coker}(\varphi_M).$$

To utilize this categorical $\mathcal{O}_p$, we need to find interesting "algebras" in H-gmod. Then the Grothendieck groups of these "algebras" will be $\mathcal{O}_p$-modules. As a motivation, note that many interesting algebra objects in the usual homotopy category of chain complexes ($H = \Bbbk[d]/(d^2)$) arise as differential graded algebras (DG algebras).

Def. A p-DG algebra A over a field of $\mathrm{char}\,\Bbbk = p > 0$ is a graded algebra together with a differential ∂ s.t. $\forall\, a, b \in A$,

$$\partial(ab) = \partial(a)b + a\partial(b),$$
$$\partial^p(a) = 0.$$

More generally, one has the notion of an H-module algebra, which in turn gives rise to an algebra object in H-mod. We refer to the study of homological properties of algebra objects in H-mod as "hopfological algebra."

In analogy with the usual DG-algebras, we have

$$(A,\partial)\text{-mod} \xrightarrow[\text{nullhomotopy}]{\text{mod out}} \mathcal{C}(A,\partial) \xrightarrow[\text{qis}]{\text{inverting}} \mathcal{D}(A,\partial)$$

Much of my thesis is about developing some necessary tools in establishing the following result.

Thm. (Khovanov, Qi) The homotopy and derived categories of a p-DG algebra are module-categories over H-gmod. Under taking Grothendieck groups (in some appropriate sense), $K_0(\mathcal{D}(A,\partial))$ has the structure of an $\mathcal{O}_p$-module.

In other words, we have the following diagram:

$$\begin{tikzcd} H\text{-}\underline{\mathrm{gmod}} \times \mathcal{D}(A,\partial) \arrow[r, "\otimes"] \arrow[d, Rightarrow, "K_0"] & \mathcal{D}(A,\partial) \arrow[d, Rightarrow, "K_0"] \\ \mathcal{O}_p \times K_0(A,\partial) \arrow[r, "\text{mult}"] & K_0(A,\partial) \end{tikzcd}$$

Question: Are there other symmetric monoidal categories whose Grothendieck rings are isomorphic to rings of integers in number fields? Or $\mathbb{Q}$ / $\mathbb{R}$ / $\mathbb{C}$ etc.?

§2. A new year's resolution

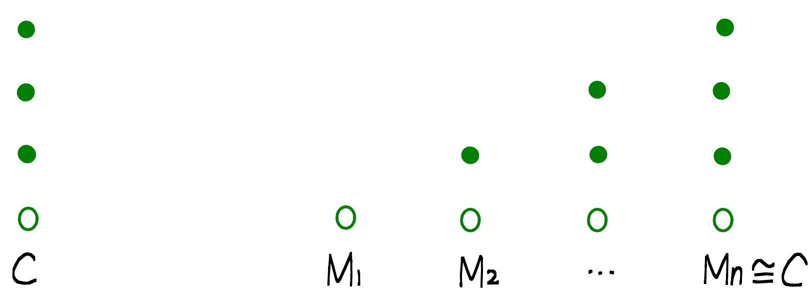
Previously, we have introduced a categorification of the cyclotomic ring $\mathcal{O}_p$, over which quantum topological invariants live. The goal of this talk is to discuss a relatively simple example.

The zig-zag algebra

We start with a very simple algebra $C := \Bbbk[x]/(x^{n+1}) \cong H^*(\mathbb{P}^n, \Bbbk)$. Equip C with the derivation: $\partial(x) = x^2$, and extended to all of C by the Leibnitz rule.

Lemma. If $\mathrm{char}(\Bbbk) = p > 0$, then (C, ∂) is a p-DG algebra.

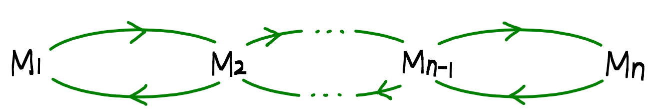
From now on, we fix $\Bbbk$ to be of positive characteristic p. The p-DG algebra C has some natural p-DG modules. Since (x^k) is a ∂-stable ideal, $C/(x^k)$ is naturally a p-DG quotient module of C.



Form the graded endomorphism algebra (allowing maps of all degrees)

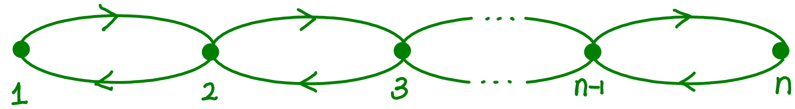
$$A_n^! := \mathrm{End}_C(\oplus_{i=1}^n M_i)$$

We see that $A_n^!$ has the following quiver algebra description:

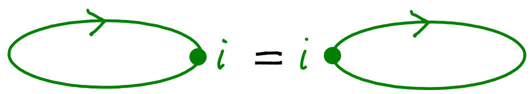


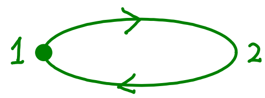
where the arrows pointing towards the right are given by multiplication by x, while the leftwards pointing arrows are the natural projection maps.

Lemma. (1). The algebra $A_n^!$ has the following path algebra presentation:

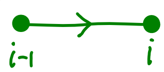
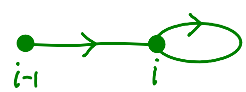
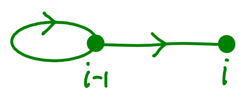


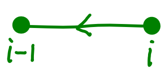
subject to relations

 $\quad (i = 2, \cdots, n-1)$

 $= 0$

(2). The differential acts on $A_n^!$ as follows.

$\partial\Big($  $\Big) :=$  $=$ 

$\partial\Big($  $\Big) := 0$

Proof. (1) is easy. To see (2), we recall that, if A is an H-module algebra, and M, N are A-modules with a compatible H-action, then H acts on $\mathrm{Hom}_A(M, N)$ by

$$(h \cdot \varphi)(m) := h_{(2)}(\varphi(S^{-1}(h_{(1)})m)).$$

For $H = \Bbbk[\partial]/(\partial^p)$, we then have $(\partial \cdot \varphi)(m) = \partial(\varphi(m)) - \varphi(\partial(m))$. If φ is multiplication by x, then

$$(\partial \cdot x)(m) = \partial(xm) - x\,\partial(m) = \partial(x)m = x^2 m,$$

while if $\varphi = \text{projection}: M_k \longrightarrow M_{k-1}$, $\partial \cdot \varphi = 0$ since φ commutes with differentials on M_k, M_{k-1}. $\square$

We will consider the p-DG algebra $(A_n^!, \partial)$ and its derived category $D(A_n^!, \partial)$ below.

Hopfological properties

As in usual homological algebra, some p-DG modules have relatively nicer hopfological properties.

Def. Let A be a p-DG algebra, and P be a p-DG module over A.

(1). P is a cofibrant module if given any $M \longrightarrow N$ a surjective qis of p-DG modules, the induced map of p-complexes

$$\mathrm{Hom}_A(P, M) \longrightarrow \mathrm{Hom}_A(P, N)$$

is a homotopy equivalence.

(2). P is a finite cell module if P has a finite step filtration whose subquotients are isomorphic, up to grading shifts, to p-DG direct summands of A.

(3). P is called a compact object in the derived category $D(A, \partial)$ if $\mathrm{Hom}_{D(A,\partial)}(P, -)$ commutes with taking arbitrary direct sums.

An easy induction shows that any finite cell module is cofibrant, and is also compact in the derived category.

Rmk: The reason why one wants to consider compact modules is that, to define Grothendieck groups, one has to limit the size of modules allowed to avoid trivial cancellations.

Examples: We consider some cofibrant and finite cell modules over $(A_n^!, \partial)$.

(1). $A_n^!$ itself.

(2) Cofibrant modules $P_i :=$ span $\langle$ all paths that end at $i \rangle$. It is a p-DG summand of $A_n^!$

(3) Consider $M := P_{i+2} \oplus P_i$ with the upper triangular differential

$$\partial_M := \begin{pmatrix} \partial_{P_{i+2}}, & (i|i+1|i+2) \\ 0 & \partial_{P_i} \end{pmatrix}$$

Then $\partial_M^p \equiv 0$, and (M, ∂_M) is a two-step finite cell module.

Thm (Q-Sussan) The derived category $\mathcal{D}(A_n^!, \partial)$ affords categorical Temperley-Lieb algebra TL_n and braid group Br_n actions. These actions categorify the Burau representation at a pth root of unity.

Sketch of proof: A New Year's Resolution

Without ∂, the above categorical actions are given by derived tensor product with some bimodules

Example. The Temperley-Lieb algebra generators are given by

$$\left(L_i \otimes (L_i)^*\right) \otimes_{A_n^!} (\ ?\ )$$

To show this in the usual derived category of modules over $A_n^!$, we need a nice projective resolution of L_i's.

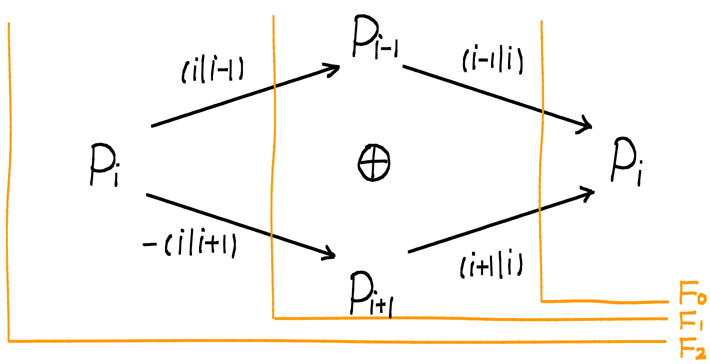
$$\begin{tikzcd}
 & & P_{i-1} \arrow[dr, "(i-1|i)"] & & & \\
0 \arrow[r] & P_i \arrow[ur, "(i|i-1)"] \arrow[dr, "-(i|i+1)"'] & \oplus & P_i \arrow[r] & L_i \arrow[r] & 0 \\
 & & P_{i+1} \arrow[ur, "(i+1|i)"'] & & &
\end{tikzcd}$$

In other words, in $D(A_n^!)$, we may replace L_i by its projective resolution:

$$\begin{tikzcd}
 & P_{i-1} \arrow[dr, "(i-1|i)"] & \\
P_i \arrow[ur, "(i|i-1)"] \arrow[dr, "-(i|i+1)"'] & \oplus & P_i \arrow[dd, "qis"] \\
 & P_{i+1} \arrow[ur, "(i+1|i)"'] & \\
 & & L_i
\end{tikzcd}$$

But, in the presence of ∂, the maps $-(i-1|i)$, $(i|i+1)$ are no longer maps of $(A_n^!, \partial)$-module maps.

Looking back at the above resolution, it can be understood as a filtered dg module over $(A_n^!, d \equiv 0)$, whose subquotients are projective.



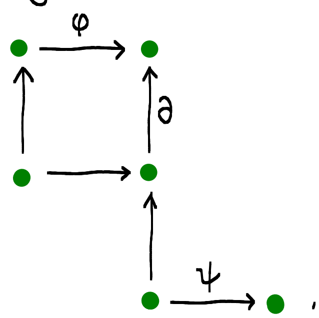
This motivates us to look for p-DG modules $\mathbb{p}(L_i)$ s.t. it is cofibrant, or even better, finite-cell, and it is quasi-isomorphic to L_i. Such a cofibrant replacement plays the role of a "projective resolution." It always exists for any p-DG module, but not necessarily small enough to be finite cell.

Lemma. If $0 \longrightarrow A \xrightarrow{\varphi} B \xrightarrow{\psi} C \longrightarrow 0$ is a s.e.s. of p-DG modules. then, in char p, the filtered module

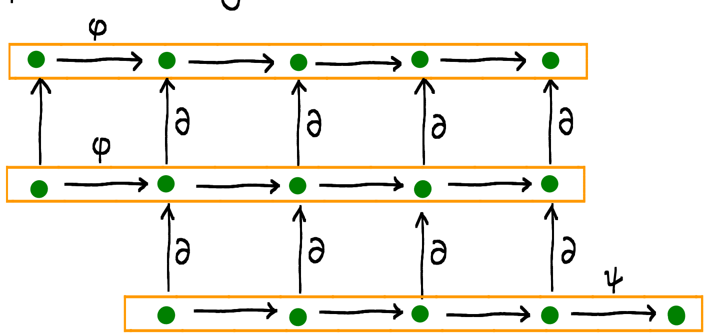
$$0 \longrightarrow A \xrightarrow{\varphi} \underbrace{B = \cdots = B}_{(p-1)\text{ terms}} \xrightarrow{\psi} C \longrightarrow 0$$

is acyclic.

Proof by example. If A. B, C are as below,

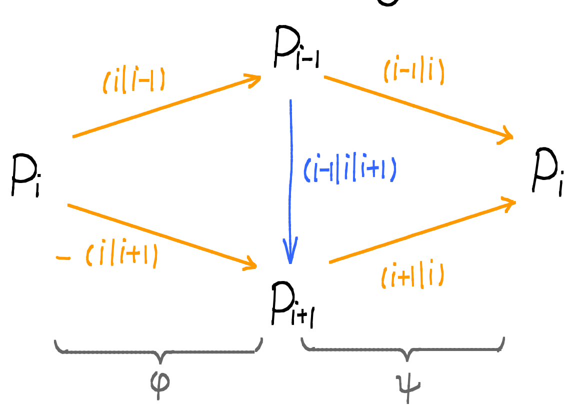


then the middle p-1 extension gives us



The boxed terms are all acyclic, and thus so is the total p-complex. $\square$

Now we tweak the usual resolution by introducing an internal differential



The two P_i's on both ends are finite-cell. The middle term is of type (3) in the Example above, and thus is also finite-cell.

The maps φ and ψ are now maps that commute with ∂. For instance

$$
\begin{tikzcd}
 & \partial(a)(i) \arrow[r, "\varphi"] & \partial(a)(i|i-1) - \partial(a)(i|i+1) \arrow[d, equal] \\
a(i) \arrow[ur, "\partial"] \arrow[dr, "\varphi"'] & & \begin{array}{l} \partial(a)(i|i-1) + a(i|i-1|i|i+1) \\ - \partial(a)(i|i+1) - a(i|i+1|i|i+1) \end{array} \\
 & a(i|i-1) - a(i|i+1) \arrow[ur, "\partial"] &
\end{tikzcd}
$$

Applying the lemma, we get a resolution of L_i by a finite-cell module

$$
\mathbb{P}(L_i): \quad
\begin{tikzcd}
 & P_{i-1} \arrow[r, equal] \arrow[d, "(i-1|i|i+1)"] & \cdots \arrow[r, equal] & P_{i-1} \arrow[dr, "(i-1|i)"] \arrow[d, "(i-1|i|i+1)"] & \\
P_i \arrow[ur, "(i|i-1)"] \arrow[dr, "-(i|i+1)"'] & & & & P_i \arrow[d, "qis"] \\
 & P_{i+1} \arrow[r, equal] & \cdots \arrow[r, equal] & P_{i+1} \arrow[ur, "(i+1|i)"'] & L_i
\end{tikzcd}
$$

This is our "New Year's Resolution" $\mathbb{P}(L_i)$ for the simple p-DG module L_i. Using these resolutions, and with the appropriate grading shifts, one can show that the functors

$$\mathbb{U}_i := (L_i \otimes (L_i)^*) \otimes_{A_n^!} (?)$$

satisfy the Temperley-Lieb relations:

$$\mathbb{U}_i^2 \cong \mathbb{U}_i[1]\{1\} \oplus \mathbb{U}_i[-1]\{-1\}$$
$$\mathbb{U}_i\mathbb{U}_j \cong \mathbb{U}_j\mathbb{U}_i \ (\cong 0) \quad \text{if } |i-j|>1$$
$$\mathbb{U}_i\mathbb{U}_{i\pm1}\mathbb{U}_i \cong \mathbb{U}_i .$$

§3 Categorified Quantum $\mathfrak{sl}(2)$ at Prime Roots of Unity 🚩

- **Why do we want to categorify $U_q(\mathfrak{sl}_2)$?**

- Reshetikhin - Turaev - Witten :

  $U_q(\mathfrak{sl}_2)$ is the quantized gauge group of 3d Chern-Simons theory.

- Crane - Frenkel :

  Categorify 3d Chern-Simons to a 4d-TQFT.

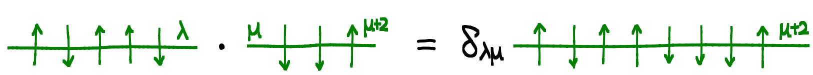
  $\mathcal{U}_q(\mathfrak{sl}_2)$: quantized 2-gauge group ?

- **Quantum $\mathfrak{sl}(2)$ at roots of unity.**

We are interested in the idempotented version of $U_q(\mathfrak{sl}_2)$. It is generated over $\mathbb{Z}[q,q^{-1}]$ by pictures of the form

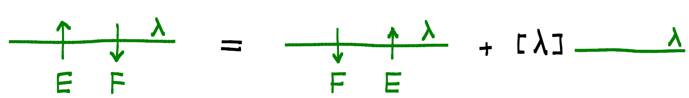
$(\lambda \in \mathbb{Z})$

with the algebra structure

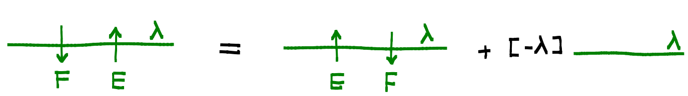


(etc)

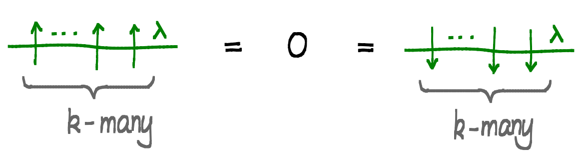
Modulo relations (at a $2k$-th root of unity, k odd)



$(\lambda \geq 0)$



$(\lambda \leq 0)$



(Nilpotency relation)

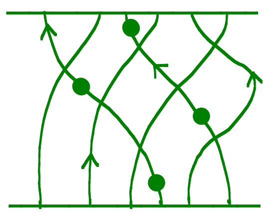
• Categorification of $U_q(\mathfrak{sl}_2)$

Below we present Lauda's diagrammatic calculus for $U_q(\mathfrak{sl}_2)$ at a generic q-value.
The rough idea is that:

- Pictures = Isomorphism class / symbol of some modules
- Sum of pictures = symbol of direct sum of modules
- Equalities of pictures = isomorphisms of modules.

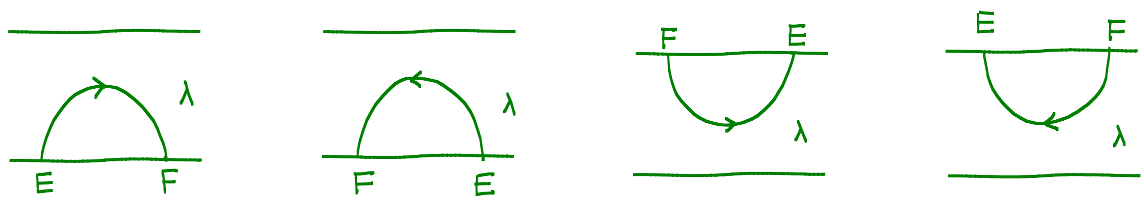
In general, isomorphisms are rare between modules. Instead, study homomorphisms between them. Intuitively, homomorphisms = evolution of pictures, which is not necessarily reversible.

- Maps just among E's (or F's) (Khovanov - Lauda - Rouquier)



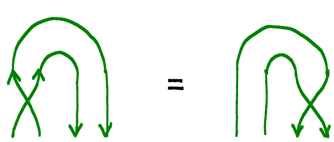
(Nil-Hecke algebra)

- To categorically "Drinfeld-double" E's, Lauda introduces cups and caps



Together with the nilHecke algebra generators, cups and caps satisfy certain relations

(i) Biadjointness   E.g.

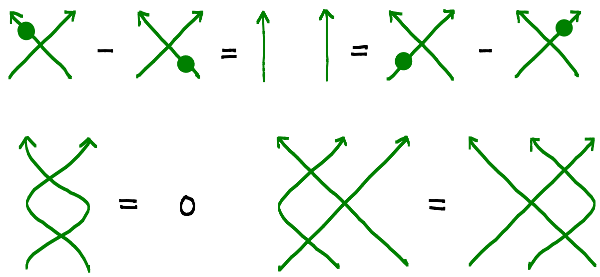


(ii) Bubble positivity (degrees of

must be ≥ 0)

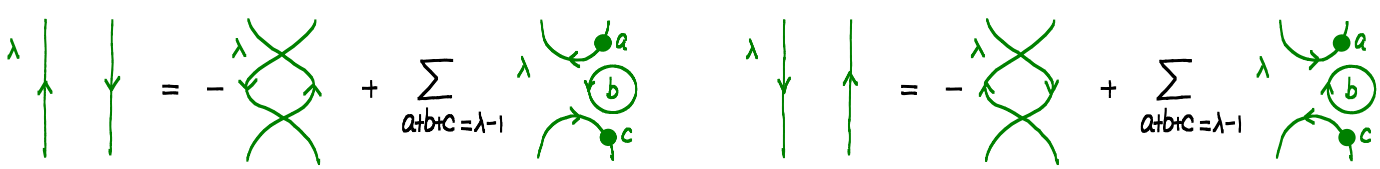
$k = \frac{1}{2}(m+1-\lambda) \geq 0$ $\qquad$ $\ell = \frac{1}{2}(m+1+\lambda) \geq 0$

(iii). NilHecke relations



(iv) Reduction to bubbles

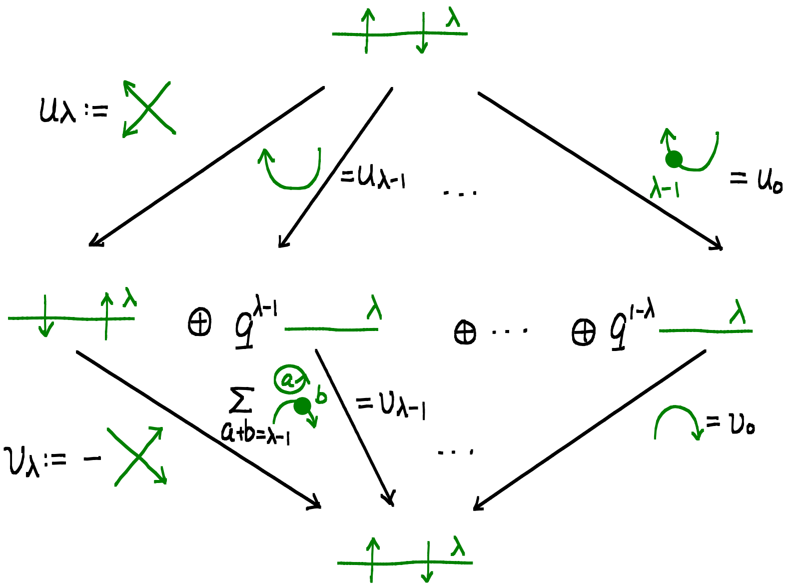
(v). Identity decomposition



Thm. (Lauda) This graphical calculus is non-degenerate and categorifies $\dot{U}_q(\mathfrak{sl}_2)$ at a generic q-value.

Rmk: Lauda's calculus is a 2-dim'l idempotented algebra, i.e. it has two compatible multiplication structures (vertical and horizontal). Such idempotented algebras are also known as a 2-category)

To see the plausibility of this categorification, we consider how $EF\mathbb{1}_\lambda$ can "evolve" into $FE\mathbb{1}_\lambda \oplus \mathbb{1}_\lambda^{\oplus[\lambda]}$



These elements $\{u_\lambda\}$, $\{v_\lambda\}$ satisfy

$$\begin{cases} u_i v_i u_i = u_i \\ v_i u_i v_i = v_i \\ v_i u_j = 0 \quad (i \neq j) \end{cases},$$

which follows from the identity decomposation relation. Consequently $\{u_i v_i \mid i = 0, \cdots, \lambda\}$ form an orthogonal set of idempotents in $\mathrm{End}_{\dot{\mathcal{U}}}(\mathcal{E}\mathcal{F}\mathbb{1}_\lambda)$

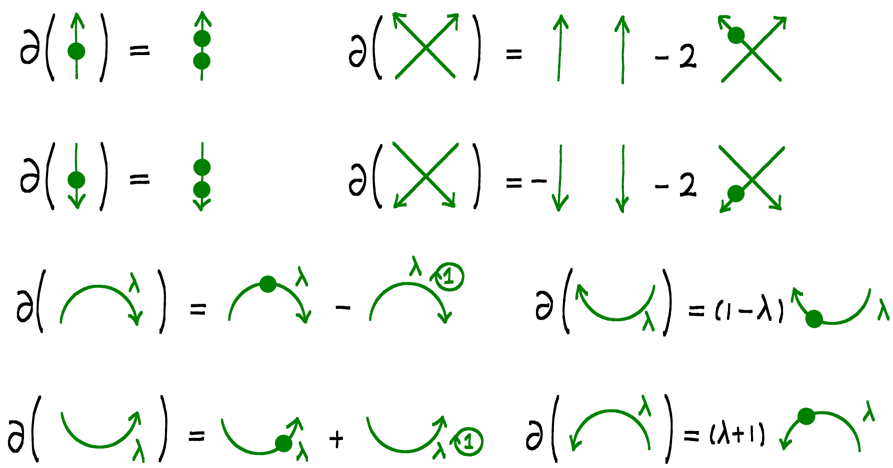
(<u>Factorization of idempotents</u>)

• Enhancing $\dot{\mathcal{U}}$ with a p-differential

As we have learnt from §1, if A is a p-DG algebra, then the derived category of p-DG modules over A is a module-category over the homotopy category of p-complexes.

$$\begin{array}{ccccc} \Bbbk[\partial]/(\partial^p)\text{-}\underline{gmod} & \times & \mathcal{D}(A,\partial) & \xrightarrow{\otimes} & \mathcal{D}(A,\partial) \\ \Big\Downarrow K_0 & & \Big\Downarrow K_0 & & \Big\Downarrow K_0 \\ \mathbb{O}_p & \times & K_0(A,\partial) & \xrightarrow{\times} & K_0(A,\partial) \end{array}$$

Def. Let $(\mathcal{U}, \partial)$ be Lauda's 2-dimensional algebra equipped with the differential ∂-action on generators given by



Lemma. The above ∂ preserves all relations of $\mathcal{U}$, and it is p-nilpotent over a field of characteristic $p>0$.

Thm. (Elias-Q.) The derived module category $\mathcal{D}^b(\mathcal{U}, \partial)$ is Karoubian, and it categorifies $\dot{U}_q(\mathfrak{sl}_2)$ at a p-th primitive root of unity.

$$K_0(\mathcal{U}, \partial) \cong \dot{U}_q(\mathfrak{sl}_2)$$

- Decomposition v.s. filtration.

In Lauda's abelian categorification, the relations in $\dot{U}_q(\mathfrak{sl}_2)$ are usually realized as different ways of decomposing projective $\mathcal{U}$-modules.

In the realm of triangulated categories, direct sum decompositions are very rare. Instead, a short exact sequence of p-DG $\mathcal{U}$-modules gives rise to a distinguished triangle in $\mathcal{D}(\mathcal{U}, \partial)$.

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \qquad \text{in} \quad (\mathcal{U}, \partial)\text{-mod}$$

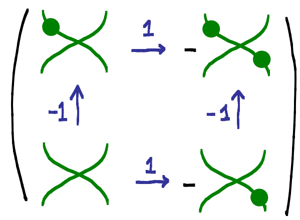
$$\Downarrow$$

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] \quad \text{in} \quad \mathcal{D}(\mathcal{U}, \partial) \Longrightarrow [B] = [A] + [C] \in K_0(\mathcal{U}, \partial)$$

More generally, a filtered p-DG module $(M, F^\bullet)$ presents M as a convolution (Postnikov tower) of $\mathrm{gr} F^\bullet$.

Example In the nilHecke algebra NH_2:

$$NH_2 \cong Sym_2 \cdot$$



$\Rightarrow$ $0 \longrightarrow P_2\{1\} \longrightarrow NH_2 \longrightarrow P_2\{1\} \longrightarrow 0$ is a s.e.s. of $(\mathcal{U}, \partial)$-modules.

$\Rightarrow$ In $K_0(\mathcal{U}, \partial)$, $E^2 = [(NH_2, \partial)] = q[P_2] + q^{-1}[P_2] = (q+q^{-1})E^{(2)}$

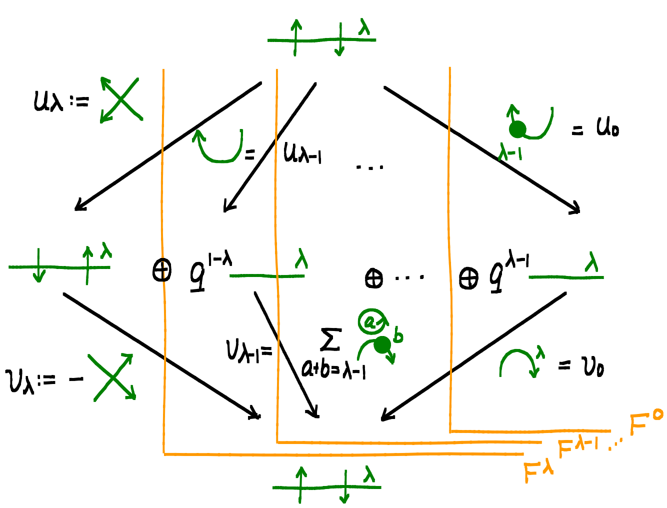
Prop. Let $\{(u_i, v_i) \mid i \in I\}$ be factorization of idempotents in a p-DG algebra R. If there is a total ordering on I such that

$$\begin{cases} v_i \partial(u_i) = 0 \\ u_i \partial(v_i) \equiv 0 \quad \text{(modulo lower order terms)} \end{cases}$$

Then if $\varepsilon = \sum_{i \in I} u_i v_i$, then the p-DG module $R\varepsilon$ admits a filtration $F^\bullet$ whose subquotients are isomorphic to Rv_iu_i's

Cor. (Fantastic!) In the situation of the Prop. $[R\varepsilon] = \sum_{i \in I} [Rv_iu_i]$.

Cor. Under the differential defined earlier on $\mathcal{U}$, there is a filtration on $\mathcal{E}\mathcal{F}1_\lambda$



• Uniqueness: a small surprise!

Lauda's factorization of idempotents, in general, is not unique.

However, in the presence of a diagrammatically local differential (not necessarily the differential we defined here, but any ∂ compatible with the local relations of $\mathcal{U}$), we have, up to conjugation by diagrammatic automorphisms

- The differential we defined here is the unique differential such that the modules $\mathcal{EF}\mathbb{1}_\lambda$ $(\lambda \geq 0)$ admit filtrations whose subquotients are isomorphic to $\mathcal{FE}\mathbb{1}_\lambda$, $\mathbb{1}_\lambda\{1-\lambda\}, \cdots \mathbb{1}_\lambda\{\lambda-1\}$.

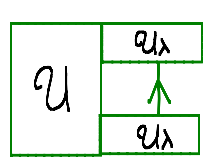
- Lauda's factorization of idempotents is the unique choice that is compatible with the differential. (Fantastic Filtration)

• Application: Categorification of simple modules

As an application of $(\mathcal{U}, \partial)$, we can also categorify simple $\dot{U}(\mathfrak{sl}_2)$-modules following ideas of Khovanov-Lauda-Rouquier.

Recall that $\mathcal{U}$ is a "2-dim'l algebra", and to construct "2-dim'l modules" over such an algebra, it's natural to consider "2-dim'l quotients" of the free module $\mathcal{U}$ by ideals.

Def. For each $\lambda \in \mathbb{N}$, let $\mathcal{U}^\lambda$ be the left $\mathcal{U}$ projective module $\mathcal{U} \cdot \mathbb{1}_\lambda$, and let I^λ be the submodule generated by $\mathcal{E}\mathbb{1}_\lambda$. Diagrammatically, I^λ is spanned by pictures of the form



The cyclotomic quotient category $\mathcal{V}^\lambda$ of Khovanov-Lauda and Rouquier is the $\mathcal{U}$-module category $\mathcal{U}^\lambda / I^\lambda$.

Thm (Khovanov-Lauda, Rouquier) $K_0(\mathcal{V}^\lambda) \cong V^\lambda$, the $U_q(\mathfrak{sl}_2)$-module of highest weight λ.

Since the ideal I^λ is obviously ∂-stable, $\mathcal{V}^\lambda$ inherits a differential ∂ from $(\mathcal{U}, \partial)$, and becomes a module-category over $(\mathcal{U}, \partial)$.

Cor (Elias-Q.) If $\lambda \in \{0, 1, \cdots, p-1\}$, $(\mathcal{V}^\lambda, \partial)$ categorifies the simple $u_{\mathbb{O}}(\mathfrak{sl}_2)$-module of highest weight λ.