Quiver Representations and Spectral Sequences

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A selected topic for a course on
representation theory
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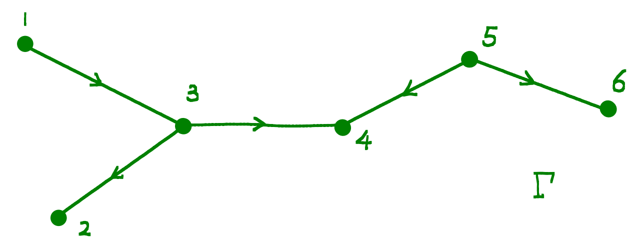
§1. Quivers, Dynkin Diagrams and Positive Roots

Let Γ be a finite oriented graph, $v(\Gamma)$ the set of vertices in Γ, and $e(\Gamma)$ the set of edges in Γ. Let $\Bbbk$ be a fixed ground field.



Def. The path algebra $\Bbbk[\Gamma]$ is the $\Bbbk$-vector space with a basis spanned by all the oriented paths in Γ (including vertices as length 0 paths), with the product structure given by concantenation of paths.

In the above example, Γ has as a basis the paths of:

length 0: $(1), (2), (3), (4), (5), (6)$

length 1: $(13), (34), (54), (56), (32)$

length 2: $(134), (132)$

with products:

$(1)\cdot(13) = (13)$, $(13)\cdot(1) = 0$, $(134)(1) = 0$, $(13)(34) = (134)$, $(34)(13) = 0$, etc.

It follows from def. that $\Bbbk[\Gamma]$ is associative. It's finite dimensional iff Γ doesn't contain any oriented 1-cycles.

The next two properties are clear:

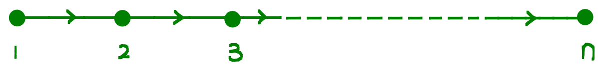
(1). $(i)(j) = \delta_{ij}(i)$, so that vertices (i), $i \in v(\Gamma)$, are idempotents.

(2). $1 = \sum_{i\in v(\Gamma)} (i)$ is the unit of the algebra: it being the left/right unit is equivalent to saying that any path in Γ starts/ends at some vertex.

Thus $\Bbbk[\Gamma]$ is always a unital associative algebra.

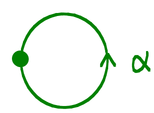
Examples

(1). Let Γ be:



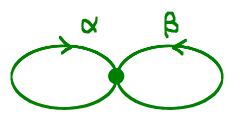
It's easy to check that $\Bbbk[\Gamma] \cong \{n\times n \text{ upper triangular matrices}\}$, identifying the path $(i, i+1, \cdots, j)$ with the matrix E_{ij} $(i \leq j)$.

(2). The Jordan quiver:



$\Bbbk[\Gamma] \cong \Bbbk[\alpha]$, the polynomial ring on α.

(3).



$\Bbbk[\Gamma] \cong \Bbbk\langle\alpha, \beta\rangle$, the free $\Bbbk$-algebra generated by 2 words.

Recall that if G is a finite group, the group algebra $\mathbb{C}[G]$ is semisimple so that any $\mathbb{C}[G]$-module is projective. Such rings, or equivalently their category of modules, are said to be of homological dimension 0.

The next case to look at would then be those rings of homological dimension 1, i.e. those rings whose modules always admit a 2-term resolution by projective modules. Equivalently, this is equivalent to saying that all submodules of projective modules are themselves projective. This last property is also known as being hereditary. Important examples arise in number theory / commutative algebra, namely the ring of integers $\mathcal{O}_F$ of some number field F / smooth affine curves over $\Bbbk$.

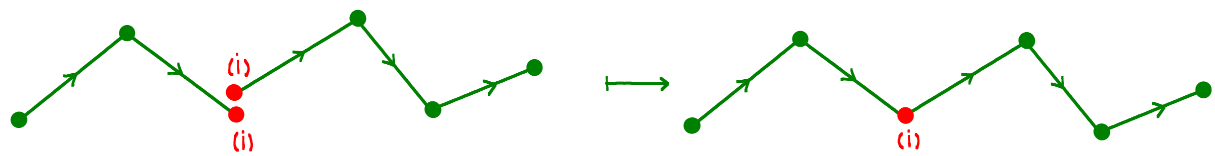
Thm 1. $\Bbbk[\Gamma]$ has homological dimension 1.

Pf: We shall prove that, any $\Bbbk[\Gamma]$-module admits a 2-term projective resolution. We first show that the ring $\Bbbk[\Gamma] \triangleq A$ itself admits a 2-term projective $A \otimes_{\Bbbk} A$ - bimodule resolution.

Since $1 = \sum_{i \in v(\Gamma)} (i)$ and $(i)(j) = \delta_{ij}(i)$, $A(i) / (i)A$ are left/right projective A-modules, $\forall\, i \in v(\Gamma)$. Thus we have a projective (A,A)-bimodule $A(i) \otimes_{\Bbbk (i)} A$ for each vertex $i \in v(\Gamma)$. Consider

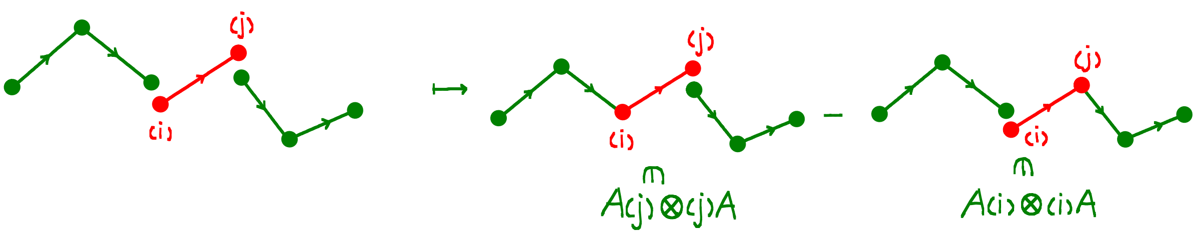
$$\bigoplus_{i \in v(\Gamma)} A(i) \otimes_{\Bbbk (i)} A \longrightarrow A$$
$$\textstyle\sum_i x_i (i) \otimes (i) y_i \longmapsto \sum_i x_i y_i$$

where x_i / y_i stands for a path that starts/ends at the vertex i:



The map is clearly surjective, and we claim that the kernel is given by the projective bimodule:

$$\bigoplus_{\alpha : i \to j \in e(\Gamma)} A(i) \otimes_{\Bbbk (j)} A \longrightarrow \bigoplus_{i \in v(\Gamma)} A(i) \otimes_{\Bbbk (i)} A$$
$$\textstyle\sum_\alpha x_i \otimes y_j \longmapsto \sum x_i \alpha \otimes y_j - x_i \otimes \alpha y_j$$



and in fact, we obtain the desired bimodule resolution:

$$0 \longrightarrow \bigoplus_{\alpha : i \to j \in e(\Gamma)} A(i) \otimes_{\Bbbk (j)} A \xrightarrow{d_1} \bigoplus_{i \in v(\Gamma)} A(i) \otimes_{\Bbbk (i)} A \xrightarrow{d_0} A \longrightarrow 0$$

The injectivity on the l.h.s. and $d_0 \circ d_1 = 0$ is clear. It suffices to check that $\ker d_0 \subseteq \operatorname{im} d_1$.

Let $z \in \ker d_0$. Note that it suffices to prove for z consisting of paths whose image under d_0 lie on a fixed path, say $(i, i+1, \cdots, j)$, i.e.

$$d_0 : z = \textstyle\sum_{k=i}^{j} a_k (i, \cdots, k) \otimes (k, \cdots, j) \longmapsto \sum a_k (i, \cdots, k, \cdots, j) = 0$$

We prove by induction on the "length" $|j-i|$. The length 0 case is trivial.

Consider

$$z + d_1 (a_i (i) \otimes (i+1, \cdots, j)) = \textstyle\sum_{k=i}^{j} a_k (i, \cdots, k) \otimes (k, \cdots, j) + a_i (i, i+1) \otimes (i+1, \cdots, j) - a_i (i) \otimes (i, i+1, \cdots, j)$$

$$= \sum_{k=i+1}^{j} a_k'(i,\cdots,k)\otimes(k,\cdots,j)$$
$$= (i,i+1)\left(\sum_{k=i+1}^{j} a_k'(i+1,\cdots,k)\otimes(k,\cdots,j)\right)$$
$$\xmapsto{d_0} (i,i+1)\, d_0\left(\sum_{k=i+1}^{j} a_k'(i+1,\cdots,k)\otimes(k,\cdots,j)\right)$$
$$= 0$$
$$\Longrightarrow \quad d_0\left(\sum_{k=i+1}^{j} a_k'(i+1,\cdots,k)\otimes(k,\cdots,j)\right) = 0,$$

since $(i,i+1)$ is a non-zero divisor on $\Bbbk\cdot(i+1,\cdots,j)$. By induction

$$\sum_{k=i+1}^{j} a_k'(i+1,\cdots,k)\otimes(k,\cdots,j) = d_1 z'$$
$$\Longrightarrow \quad z = d_1\, a_i(i)\otimes(i,\cdots,j) + (i,i+1)d_1 z'$$
$$= d_1\left(a_i(i)\otimes(i,\cdots,j) + (i,i+1)\cdot z'\right),$$

and this finishes the induction step.

Once this resolution is established, we obtain resolution of A-modules for free. We simply tensor it up with M:

$$0 \longrightarrow \bigoplus_{\alpha:i\to j\in e(\Gamma)} A(i)\otimes_{\Bbbk}(j)M \xrightarrow{d_1} \bigoplus_{i\in v(\Gamma)} A(i)\otimes_{\Bbbk}(i)M \xrightarrow{d_0} M \longrightarrow 0$$

the sequence remains exact since it (without the M term) computes:

$$\mathrm{Tor}_A^i(A,M) \cong \begin{cases} M & i=0 \\ 0 & i\neq 0. \end{cases}$$

Moreover, the sequence gives rise for any left A-module M a 2-step projective resolution. The theorem follows. $\square$

Quiver Representations

Oriented graphs are also called quivers, and by a quiver representation we mean a module over the path algebra $\Bbbk[\Gamma]$. In what follows we shall study right $\Bbbk[\Gamma]$-modules, so that the pictures agree with the orientation of Γ. Switching to left modules just means reversing all the arrows in the pictures.

Let M be a (right) $\Bbbk[\Gamma]$-module. Since $1 = \sum_{i\in v(\Gamma)}(i)$, we have

$$M \cong \bigoplus_{i\in v(\Gamma)} M\cdot(i) \qquad (\text{as } \Bbbk\text{-vector spaces}).$$

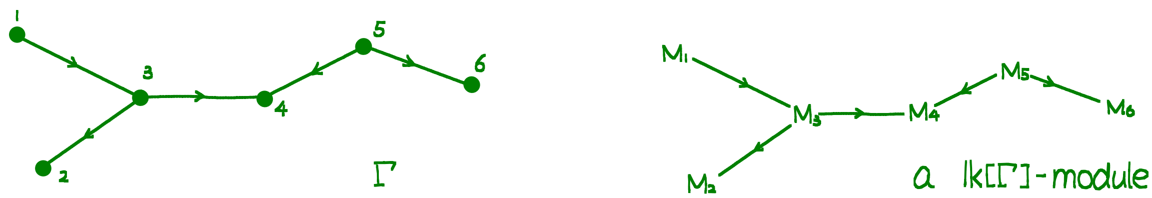
Then $\alpha: i \longrightarrow j \in e(\Gamma)$ gives rise to linear maps:

$$M\cdot(i) \xrightarrow{\alpha} M\cdot(j)$$

$$m\cdot(i) \longmapsto m(i)\cdot\alpha = m(i,j)$$

In this way, we set up a 1-1 correspondence:

$$\{\text{Quiver Representations}\} \longleftrightarrow \left\{\begin{array}{l}\text{Collections of datum: } \Bbbk\text{-vector} \\ \text{spaces } M_i,\ i \in v(\Gamma) \text{ and linear} \\ \text{maps } M_i \xrightarrow{(ij)} M_j,\ (ij) \in e(\Gamma).\end{array}\right\}$$



Before going on, we make some general remarks about the Krull-Schmidt property of quiver representations.

Let A be a ring and M an A-module. M is said to satisfy the Krull-Schmidt property if

(1). M can be decomposed into a direct sum of indecomposables:

$$M \cong \oplus_i M_i^{n_i}$$

(2). Up to permutation, the decomposition is unique.

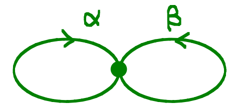
It's a very general fact that any finite length A-module ($\Leftrightarrow$ modules satisfying both the ascending chain condition and the desending chain condition) satisfies the Krull-Schmidt property. Thus for any $\Bbbk$-algebra A, the abelian category of finite dimensional A-modules is Krull-Schmidt.

Historically, Kummer falsely assumed the Krull-Schmidt property for $\mathcal{O}_F$-modules, where F is a number field, and "proved" Fermat's last theorem.

Thus to understand the category of quiver representation, we need to first understand the collection of indecomposables.

Examples:

(i). Consider:



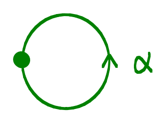
A representation of this quiver is equivalent to an n-dimensional vector space together with 2 endomorphisms on it. Thus the isomorphism classes of such representations are parametrized by:

$$(\mathrm{Mat}(n,\Bbbk)\times \mathrm{Mat}(n,\Bbbk))/GL(n,\Bbbk),$$

which roughly has dimension n^2. Inside this set a generic isomorphism class is indecomposable, and the classification of indecomposables is hard.

Such phenomenon occurs for most quivers, and they are said to be of wild representation type.

(ii). The Jordan quiver



Assume that $\Bbbk=\overline{\Bbbk}$. It's a classical theorem of linear algebra that in this case for each $n\geq 0$, the set of indecomposables of dimension n is parametrized by the Jordan canonical form of α:

$$\left\{ \begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{pmatrix} \,\middle|\, \lambda\in\Bbbk \right\}$$

Thus in this case, for each fixed dimension we have a 1-parameter family of indecomposables, and is more tangible than the previous case. Such quivers are said to be of tame representation type.

(iii). Type A quiver:

In this case, one can check that the indecomposables are all of the form:

$$\begin{tikzcd} 0 \arrow[r] & 0 \arrow[r] & \Bbbk \arrow[r, "1"] & \Bbbk \arrow[r, dashed, "1"] & \Bbbk \arrow[r] & 0 \\ \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r] & \bullet \arrow[r, dashed] & \bullet \arrow[r] & \bullet \end{tikzcd}$$
$$1 \quad \cdots \quad i \quad i+1 \quad \cdots \quad j \quad \cdots \quad n$$

Observe that in this case the indecomposables are in bijection with the set of positive roots of the underlying Dynkin diagram, i.e. $\{\varepsilon_i - \varepsilon_j = \alpha_i + \cdots + \alpha_{j-1},\ \alpha_i = \varepsilon_i - \varepsilon_{i+1}\}$. In this case, we say that Γ is of finite representation type, and for this type of quivers, we have the following:

Thm. 2. $\Bbbk[\Gamma]$ has finite representation type iff the underlying graph of Γ is finite Dynkin. Moreover, in this situation, the indecomposables are in bijection with the positive roots of the associated root system.

We shall give a sketch of the proof. Before that we need to introduce some basic notions.

Def. Let M be a finite dimensional $\Bbbk[\Gamma]$-module. The dimension vector of M is defined to be:

$$\underline{\dim M} \triangleq (\dim M(i))_{i \in V(\Gamma)} \in \mathbb{Z}_{\geq 0}^{\oplus |V(\Gamma)|}$$

The dimension vector is clearly additive on $\mathrm{Rep}(\Gamma)$, and serves the usual purpose of passing

$$\underline{\dim}: \mathrm{Rep}(\Gamma) \longrightarrow K_0(\mathrm{Rep}(\Gamma))$$

(or rather, $D^b(\mathrm{Rep}(\Gamma)) \longrightarrow K_0(\mathrm{Rep}(\Gamma)) \cong \bigoplus_{S_i:\text{ simple rep's}} \mathbb{Z}[S_i]$).

Def. For each vertex $i \in V(\Gamma)$, we define the skyscraper module S_i to be the collection of datum:

$$\begin{cases} M_i \cong \Bbbk, \quad M_j = 0,\ j \neq i \\ \text{all maps between } M_i, M_j \text{ are } 0. \end{cases}$$

In case Γ has no oriented cycles, one easily checks that the simples in $\mathrm{Rep}(\Gamma)$ are exactly the skyscraper modules supported at each vertex $i \in v(\Gamma)$. In particular, $K_0(\mathrm{Rep}(\Gamma)) \cong \bigoplus_{i\in v(\Gamma)} \mathbb{Z}[S_i]$ forms a lattice that is independent of the orientations on Γ, but only depends on the underlying graph of Γ. We put on the lattice the usual metric that is associated with any graph occuring in Lie theory:

$$<[S_i],[S_j]> \triangleq \begin{cases} 2 & i=j \\ -\#\{\text{lines connecting } i \text{ and } j\} & i\neq j \end{cases}$$

assuming that Γ has no oriented cycles. (If Γ does have oriented cycles, we should look at the category of nilpotent representations of $\Bbbk[\Gamma]$ instead.)

Def. (Source and sink). A vertex $i \in v(\Gamma)$ is called a source if all the arrows connecting it point off of it; it's called a sink if all the arrows connecting it point inwards instead:

Sketch of proof of thm. 2.

From now on we assume that all quivers involved are simply-laced.

The main idea of the proof is to "lift" the Weyl group action on the lattice $K_0(\mathrm{Rep}(\Gamma))$ to functors acting on $\mathrm{Rep}(\Gamma)$. More precisely, we shall construct, for a source or sink i of Γ, another quiver

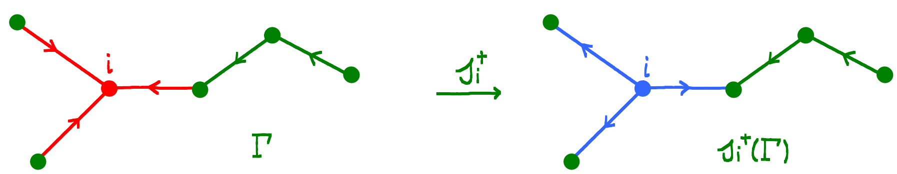
Γ' with the same underlying graph, and functors $\mathcal{J}_i^+$, $\mathcal{J}_i^-$, so that they lift the reflection $S_i : K_0(\mathrm{Rep}(\Gamma)) \xrightarrow{\cong} K_0(\mathrm{Rep}(\Gamma'))$:

$$\begin{tikzcd}
\mathrm{Rep}(\Gamma) \arrow[r, shift left, "\mathcal{J}_i^+"] \arrow[d, "\dim"] & \mathrm{Rep}(\Gamma') \arrow[l, shift left, "\mathcal{J}_i^-"] \arrow[d, "\dim"] \\
K_0(\mathrm{Rep}(\Gamma)) \arrow[r, shift left, "S_i"] & K_0(\mathrm{Rep}(\Gamma')) \arrow[l, shift left, "S_i"]
\end{tikzcd}$$

Then the theorem will follow from a clever use of some elementary property of Coxeter groups.

Def. (Gabriel / Bernstein - Gelfand - Ponomarev reflection functors)

(1). Let i be a sink in a quiver Γ, M a $\Bbbk[\Gamma]$-module. We define $\mathcal{J}_i^+(\Gamma)$ to be the quiver obtained from Γ be reversing all the arrows connected to i :



For any $\Bbbk[\Gamma]$-module, we define a new $\Bbbk[\mathcal{J}_i^+(\Gamma)]$-module $\mathcal{J}_i^+(M)$ as follows :

Let ν be the map $(\bigoplus_{j\to i} M(j)) \xrightarrow{\Sigma(j i)} M(i)$. Define

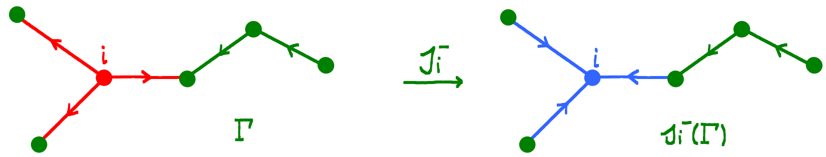
$$(\mathcal{J}_i^+(M))(j) \triangleq \begin{cases} \ker\nu, & j=i \\ M(i), & j\neq i \end{cases}$$

and the maps to be the ones in M if the edges are disjoint from i, and to be the composite:

$$\ker\nu \hookrightarrow (\bigoplus_{j\to i} M(j)) \xrightarrow{\pi_j} M(j),$$

if the edge connects i.

(2). Let i be a source in a quiver Γ, M a $\Bbbk[\Gamma]$-module. We define $\mathcal{J}_i^-(\Gamma)$ to be the quiver obtained from Γ by reversing all the arrows connected to i:



For any $\Bbbk[\Gamma]$-module, we define a new $\Bbbk[\mathcal{J}_i^-(\Gamma)]$-module $\mathcal{J}_i^-(M)$, as follows:

Let η be the map $M(i) \xrightarrow{\oplus(ij)} (\bigoplus_{i\to j} M(j))$. Define

$$(\mathcal{J}_i^-(M))(j) \triangleq \begin{cases} \operatorname{coker}\eta, & j=i \\ M(i), & j\neq i \end{cases}$$

and the maps to be the ones in M if the edges are disjoint from i, and to be the composite:

$$M(j) \hookrightarrow (\bigoplus_{j\to i} M(j)) \longrightarrow \operatorname{coker}\eta$$

if the edge connects i.

The following lemma is an easy exercise.

Lemma. 3. (1). Let i be a sink in Γ, M an indecomposable module which is not a skyscraper module supported at i. Then the map ν above is surjective.

(2). Let i be a source in Γ, M as in (1). Then the map η above is injective.

(3). Let M be an indecomposable module as in (1). Then the canonical maps:

$$\mathcal{J}_i^+\mathcal{J}_i^- M \longleftarrow M \longrightarrow \mathcal{J}_i^-\mathcal{J}_i^+ M$$

are isomorphisms. $\square$

This lemma shows that the functors $\mathcal{J}_i^+$, $\mathcal{J}_i^-$ lift the Weyl group actions of s_i on dimension vectors, at least for M indecomposable and not a skyscraper module. Indeed, if i is a sink and M as in the lemma, we have:

$$0 \longrightarrow \ker\nu \longrightarrow \oplus_{j\to i} M(j) \longrightarrow M(i) \longrightarrow 0$$

$$\Longrightarrow \underline{\dim \mathcal{J}_i^+ M} = (\dim M(1), \cdots, \dim\ker\nu, \cdots, \dim M(n))$$
$$= (\dim M(1), \cdots, \textstyle\sum_{j\to i} \dim M(j) - \dim M(i), \cdots, \dim M(n))$$
$$= s_i(\underline{\dim M}).$$

Here recall that:

$$s_i(\Sigma a_j\alpha_j) = \Sigma\, a_j s_i(\alpha_j)$$
$$= \Sigma\, a_j(\alpha_j - \langle\alpha_j, \alpha_i^\vee\rangle\alpha_i)$$
$$= \textstyle\sum_{j\neq i} a_j\alpha_j + \sum_{j-i} -a_j\langle\alpha_j,\alpha_i^\vee\rangle\alpha_i - a_i\alpha_i$$
$$= \textstyle\sum_{j\neq i} a_j\alpha_j + (\sum_{j-i} a_j - a_i)\alpha_i \quad \text{(simply-laced)}$$

Similarly, the result holds for $\mathcal{J}_i^-$ as well.

In what follows we shall assume Γ has its underlying graph finite Dynkin.

Now from basic Lie theory, we know that for any root $\alpha > 0$ of a semisimple Lie algebra, there exists a sequence of simple reflections

$$s_{i_{k-1}} \circ \cdots \circ s_{i_1}(\alpha) > 0$$
$$s_{i_k} \circ s_{i_{k-1}} \circ \cdots \circ s_{i_1}(\alpha) < 0$$

and moreover, $s_{i_{k-1}} \circ \cdots \circ s_{i_1}(\alpha) = \alpha_{i_k} \in \Delta$. If we could functorially lift these reflections $\mathcal{J}_i : \mathrm{Rep}(\Gamma) \longrightarrow \mathrm{Rep}(\Gamma)$, this would give us:

$$\mathcal{J}_{i_{k-1}} \circ \cdots \circ \mathcal{J}_{i_1}(M) = S_{i_k}$$

the skyscraper module supported at i_k. Unfortunately, the reflection functors $\mathcal{J}_i^+$, $\mathcal{J}_i^-$ do not preserve Γ. Instead, we shall use a clever trick from Coxeter groups.

Let Γ be a finite Dynkin graph and label its vertices by $\{1, 2, \cdots, n\}$ arbitrarily. Let $\mathbb{R}^\Gamma$ be the associated inner product space ($\cong K_0(\mathrm{Rep}(\Gamma)) \otimes_{\mathbb{Z}} \mathbb{R}$), and $W(\Gamma)$ be the Weyl group.

Def. A Coxeter element $c \in W(\Gamma)$ is defined to be
$$c = \prod_{i \in v(\Gamma)} s_i$$
in any order.

Any two Coxeter elements are conjugate by some element in $O(\mathbb{R}^\Gamma)$.

Prop. 4. c has no fixed points in $\mathbb{R}^\Gamma$ other than 0.

Pf: Let v be a fixed point of c. Since
$$s_n(v) = v - \langle v, \alpha_n^\vee \rangle \alpha_n,$$
$$\begin{aligned} s_{n-1}s_n(v) &= s_{n-1}(v - \langle v, \alpha_n^\vee \rangle \alpha_n) \\ &= v - \langle v, \alpha_n^\vee \rangle \alpha_n - \langle v, \alpha_{n-1}^\vee \rangle \alpha_{n-1} + \langle v, \alpha_n^\vee \rangle \langle \alpha_n, \alpha_{n-1}^\vee \rangle \alpha_{n-1}. \end{aligned}$$
and further applying $s_1, \cdots, s_{n-2}$ only modifies $s_{n-1}s_n(v)$ by multiples of $\alpha_1, \cdots, \alpha_{n-2}$. Thus $c(v) = v \Longrightarrow \langle v, \alpha_n^\vee \rangle = 0$.

Inductively, we have $\langle v, \alpha_i^\vee \rangle = 0$, $\forall i$. Since $\{\alpha_i\}$ forms a basis of $\mathbb{R}^\Gamma$, this proves that $v = 0$. □

Prop 5. If $0 \neq v = \sum a_i \alpha_i \in \mathbb{R}^\Gamma$ has $a_i \geq 0$ for all i, then for some $m \in \mathbb{N}$, $c^m v$ is no longer positive.

Pf: Otherwise, $c^m v$ were positive for all $m \in \mathbb{N}$. Since $W(\Gamma)$ is a finite group, $c^h = 1$ for some $h \in \mathbb{N}$ (the minimal such h is called the Coxeter number of $W(\Gamma)$). Then we would have:
$$\begin{aligned} & (1 + c + \cdots + c^{h-1})(v) > 0 \\ \Longrightarrow\ & 0 \neq (1-c)(1+c+\cdots+c^{h-1})(v) \quad \text{(by prop. 4)} \\ & = (1-c^h)v \\ & = 0. \end{aligned}$$
Contradiction. □

Now let Γ be a quiver whose underlying graph is finite Dynkin. It turns out that a Coxeter element $c \in W(\Gamma)$ can be lifted to a functor
$$\mathcal{C} : \mathrm{Rep}(\Gamma) \longrightarrow \mathrm{Rep}(\Gamma)$$
In fact, since Γ is a tree, Γ will always have a sink, say i_n.

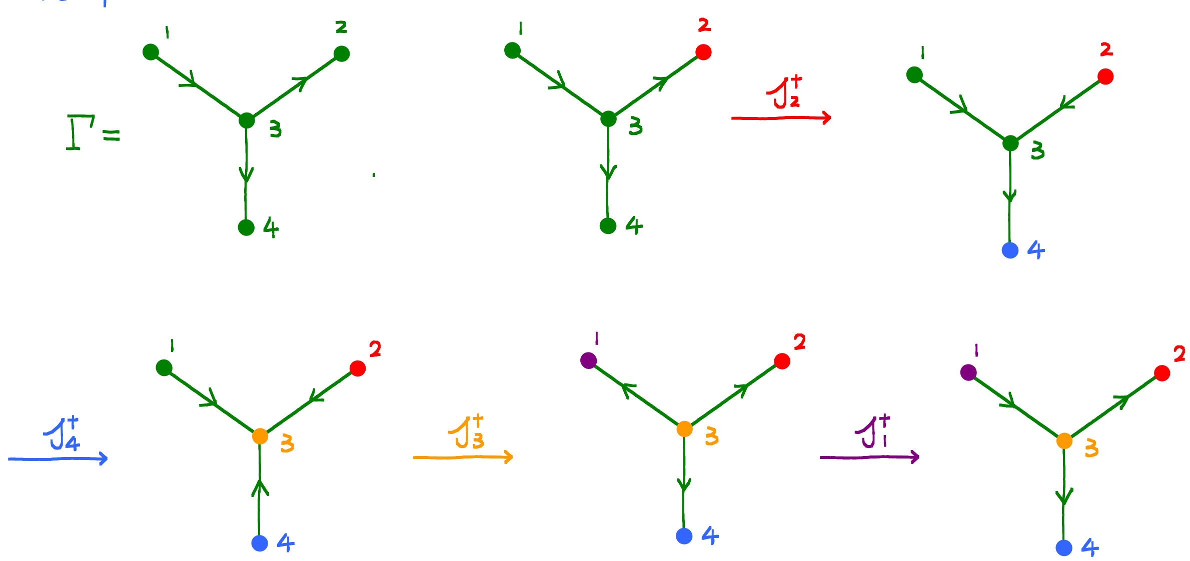
Delete i_n and all edges connected to i_n. The remaining graph is still a tree (may be disconnected) and contains another sink i_{n-1}. Repeat the process for i_{n-1} and keep going. Inductively, we will obtain a sequence of sinks i_{n-k-1} for $\mathcal{J}^+_{i_{n-k}} \circ \cdots \circ \mathcal{J}^+_{i_n}(\Gamma)$. And finally i_n $\mathcal{J}^+_{i_1} \circ \cdots \circ \mathcal{J}^+_{i_n}(\Gamma)$. Since every edge is reversed twice, we get back $\mathcal{J}^+_{i_1} \circ \cdots \circ \mathcal{J}^+_{i_n}(\Gamma) = \Gamma$.

Def. The functor obtained by composition:

$$\mathcal{C} \triangleq \mathcal{J}^+_{i_1} \circ \cdots \circ \mathcal{J}^+_{i_n} : \mathrm{Rep}(\Gamma) \longrightarrow \mathrm{Rep}(\Gamma)$$

is called the Coxeter functor.

Example:



In this example

$$\mathcal{C} = \mathcal{J}^+_1 \circ \mathcal{J}^+_3 \circ \mathcal{J}^+_4 \circ \mathcal{J}^+_2 : \mathrm{Rep}(\Gamma) \longrightarrow \mathrm{Rep}(\Gamma).$$

Now the proof of the theorem is clear. Start with an indecomposable M, and consider it's dimension vector $\underline{\dim} M \in \mathbb{Z}^{\Gamma}_{\geq 0}$. By prop 5, $\exists\, m \in \mathbb{N}$ s.t. $c^m(\underline{\dim} M) \not\geq 0$ but $c^{m-1}(\underline{\dim} M) \geq 0$. Then $\exists\ 1 \leq k \leq n$ s.t.

$$\underline{\dim}\,(\mathcal{J}^+_{i_k} \circ \cdots \circ \mathcal{J}^+_{i_n} \circ \mathcal{C}^{m-1} M) = s_{i_k} \cdots s_{i_n} c^{m-1}(\underline{\dim} M) \geq 0$$

but

$$\underline{\dim}(\mathcal{J}^+_{i_{k-1}}\circ\cdots\circ\mathcal{J}^+_{i_n}\circ\mathcal{C}^{m-1}M) = s_{i_{k-1}}\cdots s_{i_n}c^{m-1}(\underline{\dim M}) \not> 0$$

Since $\mathcal{J}^+_{i_k}\circ\cdots\circ\mathcal{J}^+_{i_n}\circ\mathcal{C}^{m-1}M$ is indecomposable (lemma 3.(3)), this says that:

$$\mathcal{J}^+_{i_k}\circ\cdots\circ\mathcal{J}^+_{i_n}\circ\mathcal{C}^{m-1}M \cong S_{i_{k-1}} \in \mathrm{Rep}(\mathcal{J}^+_{i_k}\circ\cdots\circ\mathcal{J}^+_{i_n}(\Gamma)),$$

is the skyscraper module supported at i_{k-1}. Apply the inverses:

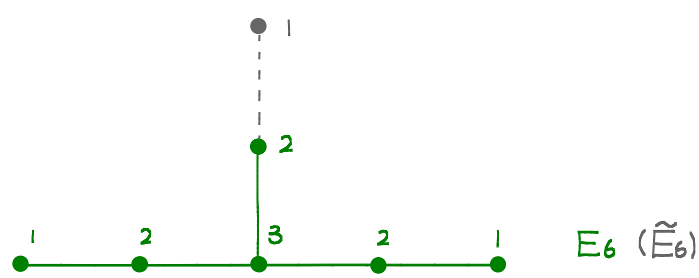
$$(\mathcal{C}^-)^{m-1}\circ\mathcal{J}^-_{i_n}\circ\cdots\circ\mathcal{J}^-_{i_k}(S_{i_{k-1}}) \cong M$$

where $\mathcal{C}^- = \mathcal{J}^-_{i_n}\circ\cdots\circ\mathcal{J}^-_{i_1}$. In particular,

$$\underline{\dim M} = c^{m-1}\circ s_{i_n}\cdots s_{i_k}(\alpha_{i_{k-1}}) \in W(\Gamma)\cdot\Delta$$

is a positive root. This sets up the desired 1-1 correspondence between positive roots and indecomposable $\mathbb{k}[\Gamma]$-modules.

Example: For any finite Dynkin diagram, we have a maximal root α, which corresponds, under the correspondence of thm.2, to a largest indecomposable module M_α. Let's find this module for E_6.



Recall that the maximal $\alpha = \sum d_i\alpha_i$ of a Dynkin graph can be constructed as follows:

(1). Adjoin 1 extra root to make the graph affine.

(2). Label the vertices on the affine graph by $d_i \in \mathbb{N}$ subject to the normalization conditions: $2d_i = \sum_{j-i} d_j$, and the added in root is labeled 1.

(3). Remove the extra root and $\alpha = \sum d_i\alpha_i$ is the desired maximal root.

The d_i's for E_6 is depicted as above. Thus M_α looks like (in some orientation of E_6):

$$\begin{tikzcd} & & \mathbb{k}^2 & & \\ \mathbb{k} & \mathbb{k}^2 \arrow[l] & \mathbb{k}^3 \arrow[l] \arrow[u] \arrow[r] & \mathbb{k}^2 \arrow[r] & \mathbb{k} \end{tikzcd}$$

The maps involved are all surjections (otherwise one can split M_α so that it won't be indecomposable), and the kernels of the maps should be in "generic" position. By the $GL(3,\mathbb{k})$ action, we may assume that the kernel of the 3 projections are the coordinate axis x, y, z resp.. Furthermore, by the 3 copies of $GL(2,\mathbb{k})$ actions, we may reduce the projections into the canonical forms:

$$\begin{tikzcd} & \mathbb{k}^2 & \\ \mathbb{k}^2 & \mathbb{k}^3 \arrow[l, "\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}"'] \arrow[u, "\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}"'] \arrow[r, "\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}"'] & \mathbb{k}^2 \end{tikzcd}$$

Upto this point, the above diagram still carries $\mathbb{k}^* \times \mathbb{k}^* \times \mathbb{k}^*$ automorphisms coming from rescaling the kernels. If we further require the two maps $\mathbb{k}^2 \to \mathbb{k}$ to be of canonical form $(1,0)$, we cut down the automorphisms to only $\mathbb{k}^*$. This shows that the module of "canonical form" below is the desired indecomposable M_α:

$$\begin{tikzcd} & & \mathbb{k}^2 & & \\ \mathbb{k} & \mathbb{k}^2 \arrow[l, "(1,0)"'] & \mathbb{k}^3 \arrow[l, "\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}"'] \arrow[u, "\begin{pmatrix}0&1&0\\0&0&1\end{pmatrix}"'] \arrow[r, "\begin{pmatrix}1&0&0\\0&0&1\end{pmatrix}"'] & \mathbb{k}^2 \arrow[r, "(1,0)"'] & \mathbb{k} \end{tikzcd}$$

M_α for E_6

Exercise: Find M_α for D_n $(n \geq 4)$, E_7, E_8.

Further remarks

(1). In classical Lie theory, a simple Lie algebra $\mathfrak{g}$ with Dynkin diagram

Γ has a decomposition:

$$\mathfrak{g} \cong \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+$$

and positive roots occur in the decomposition

$$\mathfrak{n}^+ \cong \oplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$$

This story has much longer history than that of quiver representation (Lusztig, Ringel etc. 1970's – 1980's)

Furthermore, classically, PBW theorem says that.

$$\mathcal{U}(\mathfrak{n}^+) \triangleq \mathcal{U}^+ \cong \oplus_{\nu \geq 0} \mathcal{U}^+(\nu)$$

where $\nu = \sum a_i \alpha_i \geq 0$. $\mathcal{U}^+(\nu)$ has as basis $x_{\underline{\alpha}} \triangleq x_{\alpha_1}^{i_1} \cdots x_{\alpha_n}^{i_n}$ with $\sum i_k \alpha_k = \nu$. And for $\nu, \nu' \geq 0$, we have:

$$\mathcal{U}^+(\nu) \cdot \mathcal{U}^+(\nu') \subseteq \mathcal{U}^+(\nu+\nu')$$

On the quiver representation side, for each fixed dimension vector $\nu = \sum i_k \alpha_k$, we can consider the "moduli space" of $\Bbbk[\Gamma]$-modules with a fixed dimension vector ν:

$$\{M \mid \underline{\dim} M = \nu\} / \text{iso.}$$

Upon choosing a basis for each $M(i)$, the isomorphism classes are parametrized by the quotient space:

$$\left(\prod_{\alpha: i \to j \in e(\Gamma)} \mathrm{Hom}(\Bbbk^{a_i}, \Bbbk^{a_j})\right) / \prod_{i \in v(\Gamma)} GL(a_i, \Bbbk).$$

By the Krull-Schmidt property, any finite dimensional module is a direct sum of indecomposables, thus the orbits are in bijection with

$$\{\oplus_i M_{\alpha_i}^{b_i} \mid M_\alpha : \text{indecomposable with } \underline{\dim M_\alpha} = \alpha, \sum b_i \alpha_i = \nu\}.$$

Hence:

$$\#\,\text{orbits} = \dim \mathcal{U}^+(\nu).$$

Lusztig pushed this further by studying the topology of the "moduli spaces" (actually they are quotient stacks) by looking at ℓ-adic sheaves on them. By doing so he was able to construct a canonical basis of $\mathcal{U}^+$ (or rather, its quantum deformations $\mathcal{U}_q^+$) satisfying amazing integral and positivity properties (Lusztig-Kashiwara basis). Recently, Khovanou-Lauda found a combinatorial way of describing this basis.

Example: Let's look at one example of the above correspondence:

$$\underset{\alpha_1}{\bullet} \longrightarrow \underset{\alpha_2}{\bullet}$$

For a fixed dimension vector $m\alpha_1 + n\alpha_2$, $\mathrm{Hom}(\mathbb{k}^m, \mathbb{k}^n)/GL(m,\mathbb{k}) \times GL(n,\mathbb{k})$ are parametrized by the set (W.L.O.G. assume $m \geq n$):

$$\left\{ P_r = \begin{pmatrix} I_r & O_{m-r} \\ O_{n-r} & O \end{pmatrix}_{m\times n} \;\middle|\; 0 \leq r \leq n = \min\{m,n\} \right\}$$

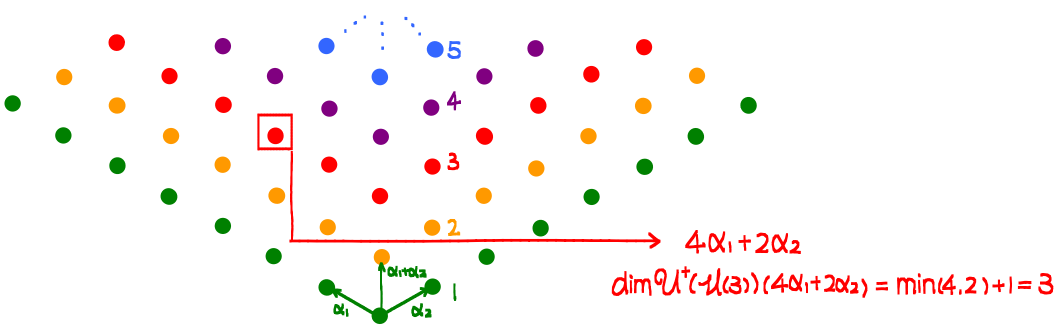
For each fixed r, the corresponding isomorphism class of $\mathbb{k}[\Gamma]$-modules is the direct sum of indecomposables:

$$(\mathbb{k}^m \xrightarrow{P_r} \mathbb{k}^n) \cong (\mathbb{k} \xrightarrow{1} \mathbb{k})^{\oplus r} \oplus (\mathbb{k} \longrightarrow 0)^{\oplus (m-r)} \oplus (0 \longrightarrow \mathbb{k})^{\oplus (n-r)}$$

and thus

$$\# \mathrm{Hom}(\mathbb{k}^m, \mathbb{k}^n)/GL(m,\mathbb{k}) \times GL(n,\mathbb{k}) = 1 + \min\{m,n\}$$

which is the same as $\dim \mathcal{U}^+(\mathfrak{sl}(3))(m\alpha_1 + n\alpha_2)$:



(2). The following result is worth mentioning:

Thm. If a finite dimensional algebra over an algebraically closed field $\mathbb{k}$ has finite representation type and homological dimension 1, then it's Morita equivalent to $\prod_{i=1}^N \mathbb{k}[\Gamma_i]$, where Γ_i is an oriented Dynkin diagram.

In this case, being Morita equivalent to $\prod_{i=1}^N \mathbb{k}[\Gamma_i]$ just means that the algebra itself is isomorphic to $\prod_{i=1}^N \mathrm{Mat}(n_i, \mathbb{k}[\Gamma_i])$, $(n_i \in \mathbb{N})$, so that the representation category of the algebra is isomorphic to that of $\prod_{i=1}^{N} \Bbbk[\Gamma_i]$.

The representation theory of finite dimensional algebras $/\mathbb{C}$ can be viewed as starting from $\mathrm{Rep}(\mathbb{C}[G])$, where G is a finite group. The category is semisimple, and thus:

(i) It's of homological dimension 0, i.e. all modules are projective.
(ii) It's of finite representation type.

If we start to loosen any of the requirements, we obtain many more objects:

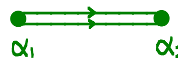
(i') If we allow homological dimension 1, without finite representation type requirements, we obtain rings like $\Bbbk[\Gamma]$ for any oriented graph Γ.
(ii') If we only keep the finite representation type requirement but drop the homological dimension condition, we have rings like $\mathbb{C}[x]/(x^n)$.

Thm 1, Thm 2 and the thm above says that the rings that satisfy both (i') and (ii'), we essentially can only get path algebras of A, D, E type!

Problem: Find this analogue in number theory, i.e. find similar conditions as (i') (ii') above for $\mathcal{O}_F$, and classify these number fields F's.

(iii). Before ending the discussion, we mention some examples of affine and wild type quivers.

Example: For affine graphs, we have the associated Kac-Moody algebras and now the root system consists of real roots and imaginary roots. For positive real roots (real meaning $(\alpha,\alpha)=2$ in the associated Cartan form), the story is similar as for finite Dynkin case, and we have a unique indecomposable $\Bbbk[\Gamma]$-module. However, for each imaginary root, we have a 1-parameter family of indecomposables. We illustrate this phenomenon with the example of Kronecker quiver, whose associated Kac-Moody algebra is $\widehat{sl(2)}$:



The Cartan form is given by $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$. The positive real roots are

$$\{n\alpha_1 + (n+1)\alpha_2 \mid n \geq 0\} \cup \{(n+1)\alpha_1 + n\alpha_2 \mid n \geq 0\}$$

and the associated indecomposables are:

$$\mathbb{C}^n \underset{P_2}{\overset{P_1}{\rightrightarrows}} \mathbb{C}^{n+1} \qquad\qquad \mathbb{C}^{n+1} \underset{Z_2}{\overset{Z_1}{\rightrightarrows}} \mathbb{C}^n$$

where

$$P_1 = \begin{pmatrix} I_{n\times n} \\ 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 \\ I_{n\times n} \end{pmatrix} \qquad\qquad Z_1 = (I_n, 0), \quad Z_2 = (0, I_n)$$

respectively.

The positive imaginary roots are:

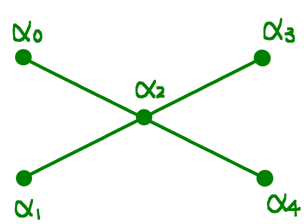
$$\mathbb{N}\cdot\delta = \{n\alpha_1 + n\alpha_2 \mid n \geq 1\}$$

(δ is the null root $\alpha_1+\alpha_2$). For each $n\alpha$, we have a family, parametrized by $\lambda \in \mathbb{C}$, of indecomposables:

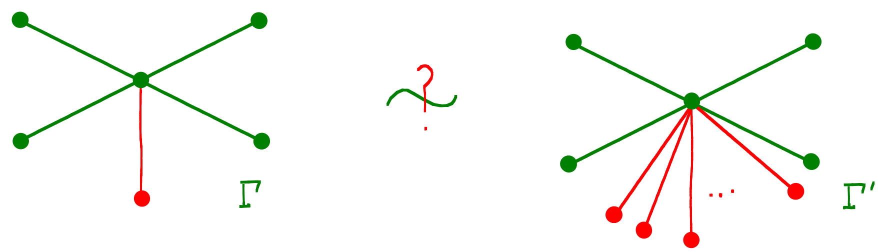
$$\mathbb{C}^n \underset{J_{n,\lambda}}{\overset{Id}{\rightrightarrows}} \mathbb{C}^n$$

where $J_{n,\lambda}$ is the Jordan matrix $\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & & \lambda \end{pmatrix}$, $\lambda \in \mathbb{C}$.

Exercise: For $\widetilde{D}_4$, find a family of indecomoposables for $n\delta$, $n \geq 1$, where $\delta = \alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4$



Beyond the affine case, the problem becomes really difficult. For instance, Gelfand showed that the problem of classifying modules over $\Gamma = \widetilde{D}_4$ with one extra vertex adjoined to the central vertex is in some sense equally as difficult as that for $\widetilde{D}_4$ with any number of extra vertices adjoined!



Gelfand's result says that for any fixed dimension vector ν of Γ', the "moduli space" of $\mathbb{k}[\Gamma']$-modules can be embedded in that of Γ of some large enough dimension vector μ. And vice versa!

§2. Applications on Spectral Sequences

The goal of this section is to understand, from a representation theoretic point of view, why the differentials d_r appear naturally in the spectral sequences of double complexes over a field $\Bbbk$.

(Co)Homology of complexes

From representation theoretic point of view, a complex $(V^\bullet, d)$ over $\Bbbk$:

$$\cdots \xrightarrow{d} V^{i-1} \xrightarrow{d} V^{i} \xrightarrow{d} V^{i+1} \xrightarrow{d} \cdots$$

is nothing but a grade module over the graded ring $\Bbbk[d]/(d^2)$, where $\deg d = 1$. Note that $\Bbbk[d]/(d^2) \cong H^*(S^1, \Bbbk)$.

Graded indecomposable modules over $\Bbbk[d]/(d^2)$ are easy to classify. They are:

$$(1).\ S_i^\bullet : \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \Bbbk \longrightarrow 0 \longrightarrow 0 \longrightarrow \cdots$$

where the only non-trivial term sits in homological degree i, $i \in \mathbb{Z}$. These are exactly all the simples.

$$(2).\ P_i^\bullet : \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \Bbbk \xrightarrow{\cdot 1} \Bbbk \longrightarrow 0 \longrightarrow \cdots$$

where the first non-trivial term $\Bbbk$ sits in degree i, $i \in \mathbb{Z}$. They are all free modules and thus projective. Actually they are injectives as well. (c.f. the proof of the classification result below).

It's readily seen that any graded module $V^\bullet$ is just a direct sum of these indecomposables (Krull-Schmidt):

$$V^\bullet \cong \bigoplus_{i \in \mathbb{Z}} S_i^{n_i} \oplus P_i^{m_i}$$

and taking (co)homology just picks out the simples:

$$H^*(V^\bullet) \cong \bigoplus_{i \in \mathbb{Z}} S_i^{n_i}$$

Bicomplexes and spectral sequences

A bicomplex $V^{\bullet,\bullet}/\Bbbk$ consists of a lattice of vector spaces $V^{i,j}$, $i, j \in \mathbb{Z}$ equipped with differentials d_1 (horizontal), d_2 (vertical) satisfying:

$$d_1^2 = 0 = d_2^2, \quad d_1 d_2 + d_2 d_1 = 0$$

$$\begin{tikzcd}
& \vdots & \vdots & \vdots & \\
\cdots \arrow[r, "d_1"] & V^{i-1,j+1} \arrow[r, "d_1"] \arrow[u, "d_2"] & V^{i,j+1} \arrow[r, "d_1"] \arrow[u, "d_2"] & V^{i+1,j+1} \arrow[r, "d_1"] \arrow[u, "d_2"] & \cdots \\
\cdots \arrow[r, "d_1"] & V^{i-1,j} \arrow[r, "d_1"] \arrow[u, "d_2"] & V^{i,j} \arrow[r, "d_1"] \arrow[u, "d_2"] & V^{i+1,j} \arrow[r, "d_1"] \arrow[u, "d_2"] & \cdots \\
\cdots \arrow[r, "d_1"] & V^{i-1,j-1} \arrow[r, "d_1"] \arrow[u, "d_2"] & V^{i,j-1} \arrow[r, "d_1"] \arrow[u, "d_2"] & V^{i+1,j-1} \arrow[r, "d_1"] \arrow[u, "d_2"] & \cdots \\
& \vdots \arrow[u, "d_2"] & \vdots \arrow[u, "d_2"] & \vdots \arrow[u, "d_2"] &
\end{tikzcd}$$

There are several cohomologies we can take:

(1). Horizontal cohomology: $H^*(V^{\bullet,\bullet}, d_1)$

(2). Vertical cohomology: $H^*(V^{\bullet,\bullet}, d_2)$

(3). Total cohomology: This is where we collapse the bigrading into a single one and take cohomology $H^*(\mathrm{Tot}^\bullet(V^{\bullet,\bullet}), D)$, where

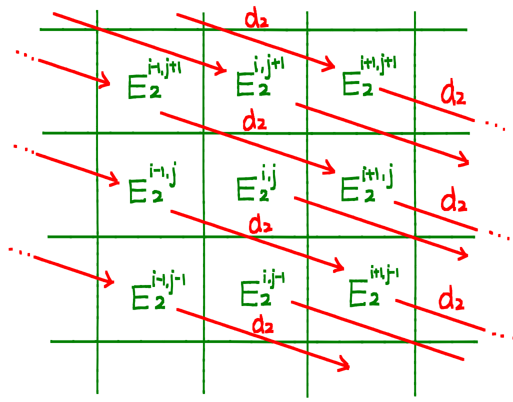
$$\begin{cases} \mathrm{Tot}^k(V^{\bullet,\bullet}) \triangleq \bigoplus_{i+j=k} V^{i,j} \\ D = d_1 + d_2 : \mathrm{Tot}^k(V^{\bullet,\bullet}) \longrightarrow \mathrm{Tot}^{k+1}(V^{\bullet,\bullet}) \end{cases}$$

A spectral sequence of the double complex $V^{\bullet,\bullet}$ says that we can calculate the total cohomology (at least as vector spaces) via the following procedure:

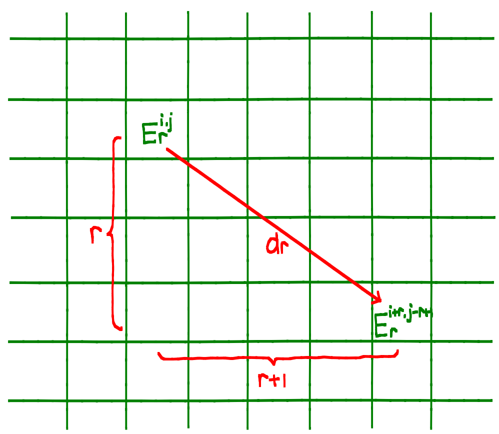
(1). Take the vertical cohomology $H^{\bullet,\bullet}(V^{\bullet,\bullet}, d_2) \triangleq E_1^{\bullet,\bullet}$. Note that d_1 still acts as a differential on it, horizontally:

$$\begin{tikzcd}
\cdots \arrow[r, "d_1"] & E_1^{i-1,j+1} \arrow[r, "d_1"] & E_1^{i,j+1} \arrow[r, "d_1"] & E_1^{i+1,j+1} \arrow[r, "d_1"] & \cdots \\
\cdots \arrow[r, "d_1"] & E_1^{i-1,j} \arrow[r, "d_1"] & E_1^{i,j} \arrow[r, "d_1"] & E_1^{i+1,j} \arrow[r, "d_1"] & \cdots \\
\cdots \arrow[r, "d_1"] & E_1^{i-1,j-1} \arrow[r, "d_1"] & E_1^{i,j-1} \arrow[r, "d_1"] & E_1^{i+1,j-1} \arrow[r, "d_1"] & \cdots
\end{tikzcd}$$

(2). Take the cohomology of $H^{\bullet,\bullet}(E_1^{\bullet,\bullet}, d_1) \triangleq E_2^{\bullet,\bullet}$, and a new differential d_2:



(3). Inductively, form the cohomology complex $H^{\bullet,\bullet}(E_{r-1}^{\bullet,\bullet}, d_{r-1}) \triangleq E_r^{\bullet,\bullet}$, and equip it with a differential $d_r : E_r^{i,j} \longrightarrow E_r^{i+r,\, j-r+1}$:



(4). Passing to $E_\infty^{\bullet,\bullet}$, $\bigoplus_{i+j=k} E_\infty^{i,j}$ will be isomorphic to $H^k(\mathrm{Tot}^\bullet(V^{\bullet,\bullet}), D)$ as $\mathbb{k}$-vector spaces. (More precisely, there is a filtration on $H^k(\mathrm{Tot}^\bullet(V^{\bullet,\bullet}), D)$ whose associated graded module is isomorphic to $\bigoplus_{i+j=k} E_\infty^{i,j}$).

Remark that we may equally start with taking horizontal cohomology as E_1 page. We just reflect all pages $E_2, E_3, \cdots$ and E_∞ about $i=j$ axis.

The main goal here is to understand why taking cohomologies of all d_r's is necessary.

As with complexes, we start by reinterpreting any double complex as a bigraded module over the bigraded ring $\Bbbk[d_1, d_2]/(d_1^2, d_2^2, d_1d_2 + d_2d_1) \triangleq \Lambda_2$, where d_1 has degree $(1,0)$, and d_2 has degree $(0,1)$. Note that $\Lambda_2 \cong H^*(S^1 \times S^1, \Bbbk)$.

Thm 1. (*Classification of indecomposable modules over Λ_2*)

Let $V^{\bullet,\bullet}$ be a bicomplex, bounded in some finite region of $\mathbb{Z}^2$, but might be infinite dimensional in each fixed degree $V^{i,j}$. Then it decomposes into a direct sum of indecomposables, which are classified as follows:

Simples: $S^{i,j}$:
$(i,j \in \mathbb{Z})$

$$\begin{tikzcd} & 0 & \\ 0 \arrow[r] & \Bbbk^{\{i,j\}} \arrow[r] \arrow[u] & 0 \\ & 0 \arrow[u] & \end{tikzcd}$$

Frees: $P^{i,j}$
$(i,j \in \mathbb{Z})$

$$\begin{tikzcd} \Bbbk \arrow[r, "\cdot 1"] & \Bbbk \\ \{i,j\}\ \Bbbk \arrow[r, "\cdot 1"] \arrow[u, "\cdot 1"'] & \Bbbk \arrow[u, "\cdot(-1)"'] \end{tikzcd}$$

And the following zig-zag types, $i,j \in \mathbb{Z}$, $\ell \in \mathbb{N}$ and ℓ denotes the number of arrows in the diagram:

$$\begin{tikzcd} \{i,j\}\ \Bbbk & & \\ \Bbbk \arrow[u, "\cdot 1"'] \arrow[r, "\cdot 1"] & \Bbbk & \\ & \Bbbk \arrow[u, "\cdot 1"] \arrow[r, "\cdot 1"] & \Bbbk \\ Z^{i,j}_{\uparrow,\ell} & & \Bbbk \arrow[u, "\cdot 1"] \end{tikzcd}
\qquad
\begin{tikzcd} \{i,j\}\ \Bbbk \arrow[r, "\cdot 1"] & \Bbbk & & \\ & \Bbbk \arrow[u, "\cdot 1"] \arrow[r, "\cdot 1"] & \Bbbk & \\ & & \Bbbk \arrow[u, "\cdot 1"] \arrow[r, "\cdot 1"] & \Bbbk \\ Z^{i,j}_{\rightarrow,\ell} & & & \Bbbk \arrow[u, "\cdot 1"] \end{tikzcd}$$

The proof of the thm. will be defered. But now let's look at its implications. We compare, for each type of indecomposable above, its contribution to the cohomology groups:

(a). $H^{\bullet,\bullet}(-, d_2)$

(b). $H^{\bullet}(-, D)$

Recall that (a) amouts to forget about the horizontal arrows in these modules and compute its vertical cohomology, while (b) collapses the bigrading into a single one and take cohomology. For instance, for $P^{i,j}$:

$$\begin{array}{ccc} \Bbbk & \xrightarrow{\cdot 1} & \Bbbk \\ \uparrow{\scriptstyle \cdot 1} & & \uparrow{\scriptstyle \cdot(-1)} \\ (i,j)\ \Bbbk & \xrightarrow{\cdot 1} & \Bbbk \end{array}$$

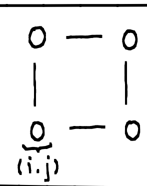
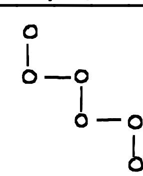
(a):

$$\begin{array}{ccc} \Bbbk & \oplus & \Bbbk \\ \uparrow{\scriptstyle \cdot 1} & & \uparrow{\scriptstyle \cdot(-1)} \\ (i,j)\ \Bbbk & \oplus & \Bbbk \end{array} \xrightarrow[\text{cohomology}]{\text{Take } d_2} \begin{array}{ccc} 0 & \oplus & 0 \\ | & & | \\ (i,j)\ 0 & \oplus & 0 \end{array}$$

(b):

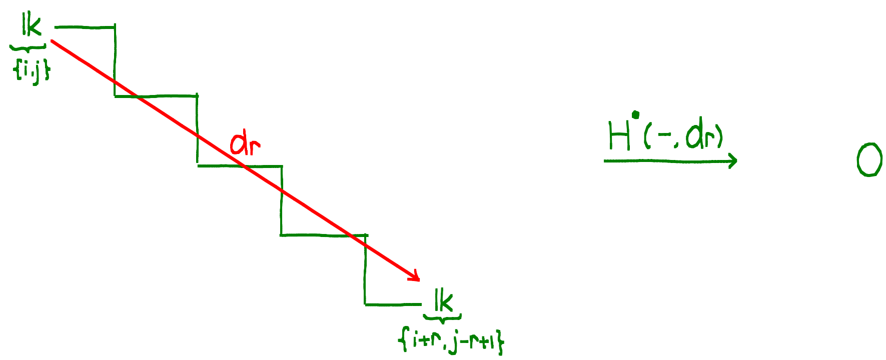
$$\cdots\ 0 \xrightarrow{D} \underbrace{\Bbbk}_{i+j} \xrightarrow{D} \underbrace{\Bbbk^2}_{i+j+1} \xrightarrow{D} \underbrace{\Bbbk}_{i+j+2} \xrightarrow{D} 0\ \cdots \xrightarrow[\text{cohomology}]{\text{Take } D} \cdots\ 0 \longrightarrow \underbrace{0}_{i+j} \longrightarrow \underbrace{0}_{i+j+1} \longrightarrow \underbrace{0}_{i+j+2} \longrightarrow 0\ \cdots$$

We tabulate these results as follows:

	$S^{i,j}$	$P^{i,j}$	$Z^{i,j}_{\uparrow,\ell}$, ℓ: odd
$H^{\bullet,\bullet}(-, d_2)$	$\Bbbk\{i,j\}$		
$H^{\bullet}(-, D)$	$\Bbbk\{i+j\}$	0	0

	$Z^{i,j}_{\uparrow,\ell}$, ℓ : even	$Z^{i,j}_{\to,\ell}$ ℓ: even	$Z^{i,j}_{\to,\ell}$ ℓ: odd
$H^{\bullet,\bullet}(-, d_2)$	$\circ - \Bbbk\{i+\frac{\ell}{2}, j-\frac{\ell}{2}\}$	$\{i,j\}\Bbbk - \circ$	$\underbrace{\Bbbk}_{\{i,j\}} - \circ$; $\circ - \underbrace{\Bbbk}_{\{i+\frac{\ell+1}{2}, j-\frac{\ell+1}{2}+1\}}$
$H^{\bullet}(-, D)$	$\Bbbk\{i+j\}$	$\Bbbk\{i+j\}$	0

From this comparison we conclude that the only discrepancy comes about when taking cohomologies of $Z^{i,j}_{\to,\ell}$, ℓ: odd. Then these differences are killed off by d_r's in E_r:



Hence step by step, a spectral sequence removes all the discrepancies caused from $Z^{i,j}_{\to,\ell}$, ℓ: odd, and returns with an accurate account of the size of $H^{\bullet}(\mathrm{Tot}(V^{\bullet,\bullet}), D)$.

Example: Hodge to de Rham spectral sequence.

Let X be a closed almost complex manifold and J the associated almost complex structure $J^2 = -1$ on $T_{\mathbb{R}}X$. Upon choosing a compatible metric, we may equip the cotangent bundle $T^*_{\mathbb{R}}X$ with the same complex structure acting as an isometric endomorphism of $T^*_{\mathbb{R}}X$. Then:

$$T^*_{\mathbb{C}}X = T^*_{\mathbb{R}}X \otimes_{\mathbb{R}} \mathbb{C} \cong T^{1,0}(X) \oplus T^{0,1}(X)$$

decomposes into $\pm i$-eigen spaces of J, and so does the associated de Rham complex:

$$(\Omega^*(X;\mathbb{C}), d) \cong (\oplus_{p,q}\Omega^{p,q}(X), d)$$

where $\Omega^k(X) = \Gamma(X, \Lambda^k T^*_{\mathbb{C}}X)$ and $\Omega^{p,q}(X) = \Gamma(X, \Lambda^p T^{1,0}(X) \otimes \Lambda^q T^{0,1}(X))$ are the spaces of smooth sections. The famous thm. of Newlander and Nirenberger states that J is a complex structure iff

$$d: \Omega^{p,q}(X) \longrightarrow \Omega^{p+1,q}(X) \oplus \Omega^{p,q+1}(X).$$

(C.f. Huybrechts, Complex geometry, an introduction, §2.6). If this happens, $d = \partial + \bar{\partial}$, where:

$$\begin{cases} \Omega^{p,q}(X) \xrightarrow{\partial} \Omega^{p+1,q}(X) \\ \Omega^{p,q}(X) \xrightarrow{\bar{\partial}} \Omega^{p,q+1}(X), \end{cases}$$

and the condition $d^2 = 0 \iff$

$$\partial^2 = 0, \quad \bar{\partial}^2 = 0, \quad \partial\bar{\partial} + \bar{\partial}\partial = 0.$$

Thus to any complex manifold, there is the associated Hodge to de Rham spectral sequence:

$$E_1^{p,q} = H^q(\Omega^{p,*}(X), \bar{\partial}) \Longrightarrow H^{p+q}(\Omega^*(X,\mathbb{C}), d).$$

Now if we assume furthermore that X is Kähler, we have the:

Lemma (The $\partial\bar{\partial}$-lemma) Let X be a compact Kähler manifold. Then for a d-closed form α of type (p,q), the following are equivalent:

i). α is d-exact

ii). α is ∂-exact

iii). α is $\bar{\partial}$-exact

iv). α is $\partial\bar{\partial}$-exact, i.e $\alpha = \partial\bar{\partial}\beta$ for some β of type $(p-1, q-1)$.

(C.f. Huybrechts, Complex geometry, an introduction, Cor. 3.2.10).

In our context, if α belongs to some indecomposable summand of the $\mathbb{C}[\partial, \bar{\partial}]$-module $\oplus_{p,q}\Omega^{p,q}(X)$, then α arises as

$$\begin{tikzcd} & 0 & \\ & \alpha \arrow[u, dashed] \arrow[r, dashed] & 0 \\ \beta \arrow[r, "\bar\partial"] & {} \arrow[u, "\partial"'] & \end{tikzcd}$$

Checking our list of indecomposables, this could only happen for modules of type $S^{i,j}$ and $P^{i,j}$. Thus we conclude that

$$\bigoplus_{p,q} \Omega^{p,q}(X) = \bigoplus_{i,j} \left((S^{i,j})^{\oplus m_{ij}} \oplus (P^{i,j})^{\oplus n_{ij}} \right)$$

and the spectral sequence degenerates at E_1. This establishes the well-known Hodge decomposition theorem for Kähler manifolds:

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^q(\Omega^{p,*}, \bar\partial).$$

Classification of indecomposables over Λ_2.

In this part we shall prove thm. 1.

Let $V^{\bullet,\bullet}$ be a bigraded module over Λ_2. We first show that, if $v \in V^{i,j}$ is a homogeneous vector, and $d_1 d_2 v \neq 0$ (so that $d_2 d_1 v \neq 0$ as well), then v generates a copy of $P^{i,j}$ and we can split it off from $V^{\bullet,\bullet}$:

$$V^{\bullet,\bullet} \cong P^{i,j} \oplus V'^{\bullet,\bullet}.$$

That v generates a copy of $P^{i,j}$ is readily seen, so that it's a projective module (free). To show that it's actually a direct summand, we shall show that $P^{i,j}$ is injective as well. To do this it's worthwhile to be slightly more general:

Lemma 2. Let A be a Frobenius algebra/$\mathbb{k}$ (i.e. a finite dimensional, unital, associative algebra equipped with a bilinear, non-degenerate, pairing $\varepsilon: A \otimes_{\mathbb{k}} A \longrightarrow \mathbb{k}$ s.t. $\varepsilon(ab, c) = \varepsilon(a, bc)$, $\forall a, b, c \in A$). Then as a module over itself, the free module A is also injective.

Example: Frobenius algebras.

(1). $M_n(\Bbbk)$: the matrix algebra with $\varepsilon(A,B) \triangleq Tr(AB)$, $\forall A,B \in M_n(\Bbbk)$.
(2). $\Bbbk[G]$: the group algebra of a finite group G, with ε given by:

$$\varepsilon(g) \triangleq \begin{cases} 1 & g=1 \\ 0 & g \neq 1 \end{cases}$$

(3). $H^*(M,\Bbbk)$: cohomology rings of compact, $\Bbbk$-orientable manifolds, where ε is given by, $\forall a,b \in H^*(M,\Bbbk)$:

$$\varepsilon(a,b) \triangleq \int_{[M]} a \cup b$$

and $[M]$ denotes a chosen $\Bbbk$-fundamental class. The non-degeneracy of ε is guanranteed by Poincaré duality.

The rings we are considering are of this type:

$$H^*(S^1,\Bbbk) \cong \Bbbk[d]/(d^2)$$

$$H^*(S^1 \times S^1, \Bbbk) \cong \Lambda_2 = \Bbbk[d_1,d_2]/(d_1^2, d_2^2, d_1d_2+d_2d_1)$$

Pf of lemma 2.

For any finite dimensional $\Bbbk$-algebra A, $A^* = Hom_{\Bbbk}(A,\Bbbk)$ becomes an A module if we define: $\forall f \in A^*$, $(a \cdot f)(x) \triangleq f(xa)$. If A is also Frobenius,

$$\begin{aligned} \varepsilon: A &\xrightarrow{\sim} A^* \\ a &\longmapsto \varepsilon(-,a) \triangleq \varepsilon_a \end{aligned}$$

is an isomorphism of A-modules: it's a map of A-modules since $\forall a,b,x \in A$,

$$\varepsilon_{a \cdot b}(x) = \varepsilon(x,ab) = \varepsilon(xa,b) = (a \cdot \varepsilon_b)(x),$$

and it's an isomorphism since ε is nondegenerate. It follows that A is injective since

$$\begin{aligned} Hom_A(-,A) &\cong Hom_A(-,A^*) \\ &\cong Hom_{\Bbbk}(A \otimes_A (-), \Bbbk) \end{aligned}$$

is a composition of exact functors so that it's exact. The last step follows from the general tensor-hom adjunction: if A is a B-algebra, then, for any A-module M and B-module N,

$$Hom_B(A \otimes_A M, N) \cong Hom_A(M, Hom_B(A,N))$$

□

It follows that we can split off all vectors v with $d_2 d_1 v \neq 0$. Thus we may assume that $\forall\, v \in V^{\bullet,\bullet}$, $d_2 d_1 v = 0$. Again, if we set

$$D = d_1 + d_2,$$

we can obtain a decomposition, $\forall\, i,j \in \mathbb{Z}$:

$$V^{i,j} \cong (\ker D)^{i,j} \oplus \mathrm{Comp}^{i,j}$$

where $\mathrm{Comp}^{i,j}$ is an arbitrary vector space complement to $(\ker D)^{i,j}$. Under our assumption, the module $V^{\bullet,\bullet}$:

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r, dash] & V^{i,j+1} \arrow[r] \arrow[u, dash] & V^{i+1,j+1} \arrow[r, dash] \arrow[u, dash] & \cdots \\
\cdots \arrow[r, dash] & V^{i,j} \arrow[r] \arrow[u] \arrow[d, dash] & V^{i+1,j} \arrow[r, dash] \arrow[u] \arrow[d, dash] & \cdots \\
 & \vdots & \vdots &
\end{tikzcd}$$

decomposes as:

$$\begin{tikzcd}
 & \vdots & \vdots & \\
\cdots \arrow[r] & (\mathrm{Ker} D)^{i,j+1} \oplus \mathrm{Comp}^{i,j+1} \arrow[r] \arrow[u] & (\mathrm{Ker} D)^{i+1,j+1} \oplus \mathrm{Comp}^{i+1,j+1} \arrow[r] \arrow[u] & \cdots \\
\cdots \arrow[r] & (\mathrm{Ker} D)^{i,j} \oplus \mathrm{Comp}^{i,j} \arrow[r] \arrow[u] & (\mathrm{Ker} D)^{i+1,j} \oplus \mathrm{Comp}^{i+1,j} \arrow[r] \arrow[u] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

i.e. it's decomposed into "zig-zag" types (the red part):

$$\cdots \longrightarrow (\ker D)^{i,j+2} \longleftarrow \mathrm{Comp}^{i,j+1} \longrightarrow (\ker D)^{i+1,j+1} \longleftarrow \mathrm{Comp}^{i+1,j} \longrightarrow \cdots$$

Modules of this type are no other than modules over the type A path algebras we introduced in the previous section:

$$\cdots\bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \cdots \qquad \text{type } A \text{ quiver}$$

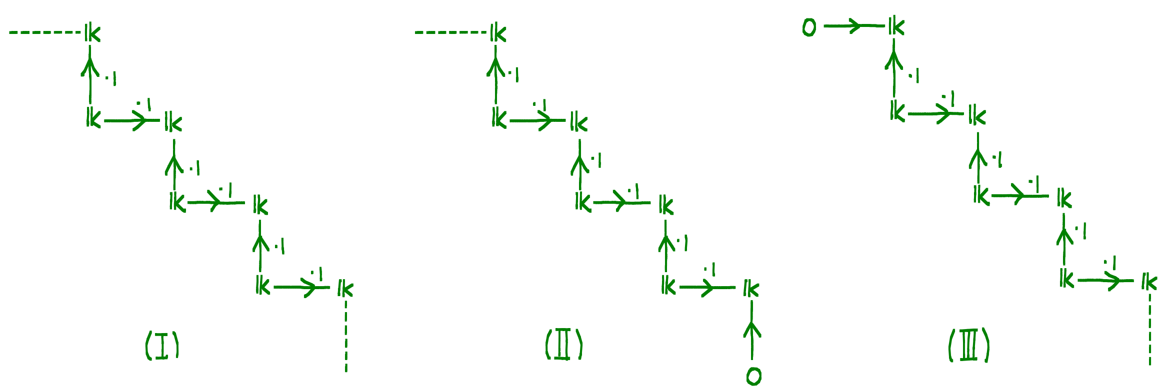
Moreover, bounded modules of this quiver, i.e. modules over some A_n for $n \gg 0$, are direct sums of indecomposables (even infinite dimensional ones), which were classified to be in bijection with the positive roots of A_n, and of the form:

$$\cdots 0 \longrightarrow 0 \longleftarrow \Bbbk \xrightarrow{\cdot 1} \Bbbk \text{-------} \Bbbk \xrightarrow{\cdot 1} \Bbbk \longleftarrow 0 \longrightarrow 0 \cdots$$

These are precisely the "zig-zag" and simples described in thm 1, and we are done.

Finally, we remark that if we are considering unbounded bicomplexes, we obtain 3 more types of unbounded modules, coming as module over "A_∞", which are unbounded on both ends, or bounded on one end:

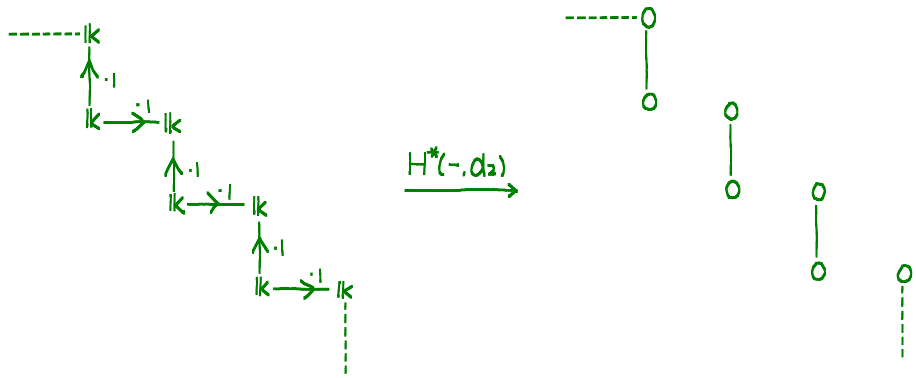
$$\cdots\bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \rightarrow \bullet \leftarrow \bullet \cdots \qquad \text{"}A_\infty\text{" quiver}$$



Note that if a bicomplex contains some of these infinite length modules, the spectral sequences constructed from it need not converge. Let's look at, for instance, the unbounded case (I):

On the E_1 page,

$$E_1^{i,j} = H^i(V_I^{\bullet,j}, d_2) = 0$$



However, the D cohomology is non-zero:

$$\cdots \longrightarrow 0 \longrightarrow \bigoplus_{i+j=r} \Bbbk \xrightarrow{D} \bigoplus_{i+j=r+1} \Bbbk \longrightarrow 0 \longrightarrow \cdots$$

D is injective, but $\operatorname{Im} D$ consists of $(a_{ij})_{i+j=r+1}$ with

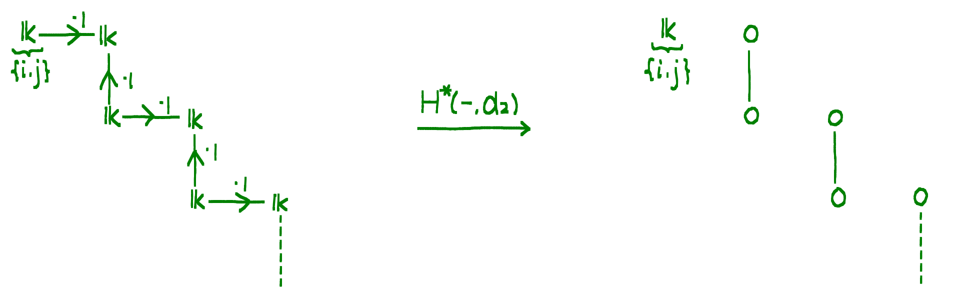
$$\sum (-1)^j a_{ij} = 0,$$

which is of codimension 1 in $\bigoplus_{i+j=r+1} \Bbbk$, and thus

$$H^*(\operatorname{Tot}(V_{\mathrm{I}}^{\bullet,\bullet}), D) = \begin{cases} \Bbbk & * = 0 \\ 0 & \text{otherwise} \end{cases}$$

and these discrepancies couldn't be compensated throughout E_r.

Similarly, the E_1 page of V_{II}:



This copy of $\Bbbk$ is never killed in the spectral sequence, and will contribute a copy of $\Bbbk$ to degree $r = i+j$ if we collapse the bigrading. But the D cohomology is again $\Bbbk$ in $r+1$!

Problem: Try to work out what we did for filtered complexes.