RESEARCH STATEMENT

YOU QI

1. Motivation

Categorification, initiated in the seminal paper by Crane and Frenkel [CF94] is, metaphorically speaking, a way of looking at some areas of mathematics from a higher point of view, resolving complicated "shadow" problems on the ground by working with richer structures from above. Khovanov homology [Kho00], which is a categorification of the celebrated Jones polynomial invariant for knots and links, is one of the first significant examples of this phenomenon.

At this moment, there is a tremendous body of research on categorification of quantum topological invariants and representation theory of quantum groups. Their categorical liftings rely essentially on the fact that, on the decategorified level, these structures are integral over $\mathbb{Z}[q, q^{-1}]$. Multiplication by q is then categorified by grading shifts on categories. In general, constructing a categorification of some object needs a considerable amount of innovation. The subject is far reaching and has strong connections with representation theory, algebraic geometry, gauge theory and symplectic topology. However, little advance has been made regarding categorification at q equal to a root of unity.

More specifically oriented, Crane and Frenkel proposed to categorify the combinatorially-defined Kuperberg invariant for 3-manifolds [Kup96], which is associated with a (half) small quantum group $u_q^+(\mathfrak{g})$, where q is a root of unity. There they hoped to lift this invariant to a 4-dimensional topological quantum field theory (4D TQFT), which was conjectured to be related with the 4D Donaldson-Floer theory. The Kuperberg invariant is a close relative of the 3D TQFT evaluated on a closed 3-manifold constructed by Witten-Reshetikhin-Turaev [Wit89, RT90], the latter being a mathematical incarnation of the physical 3D Chern-Simons theory associated with $SU(2)$ at a certain level. The Chern-Simons theory has a combinatorial description using the representation theory of the Drinfeld double quantum group $u_q(\mathfrak{sl}_2)$, where q is a root of unity determined by the level. Therefore, of particular interest and importance is the categorification of the small quantum group $u_q^+(\mathfrak{g})$ and its double $u_q(\mathfrak{g})$, and in fact Crane and Frenkel explicitly conjectured [CF94, Conjecture 2] that:

Conjecture 1.1. *There exists a categorification of the small quantum group $u_q^+(\mathfrak{sl}_2)$ where q is a primitive N-th root of unity.*

More generally, Crane and Frenkel envisioned the program of categorification of quantum invariants at a root of unity, which can be summarized into the following:

Conjecture 1.2. *Upon categorification of the WRT 3-manifold invariant, there is a combinatorial construction of a 4D TQFT, obtainable previously only through gauge theoretic methods.*

Since 2012, I have been focusing on realizing the dream of Crane and Frenkel, and some initial progress has been made. In particular, together with Khovanov, we have provided a constructive solution to Conjecture 1.1 when q is a prime root of unity in [KQ15]. This makes the approach adopted by me and my collaborators seem promising and tractable. Since the program of realizing Conjecture 1.2 is quite broad in nature, and the techniques required are somewhat subtle, it will be divided into smaller projects that I hope to accomplish in steps. The first steps of the program will be sketched out in more detail in what follows.

2. Hopfological algebra

Categorification at prime roots of unity makes extensive use of the theory of p-differential graded (p-DG) algebras. The theory is a special case of *hopfological algebra* as developed in [Kho16, Qi14].

Homological algebra, a fundamental tool in modern mathematics and physics, is commonly regarded as the study of the equation $d^2 = 0$. It is a natural question to ask whether this equation admits non-trivial variations that yield equally beautiful and useful theories. At the discovery of simplicial homology, Mayer [May42a, May42b] observed that removing the alternating signs in the definition of the boundary maps of the simplicial chains of a topological space, and working in characteristic $p > 0$, results in a "p-chain complex" whose differential ∂ satisfies $\partial^p = 0$. Associated with these p-chain complexes are Mayer's new homology groups, which are topological invariants. However, because of the restrictions of the Eilenberg-Steenrod axioms, Spanier [Spa49] soon found out that the Mayer homology groups of topological spaces can be recovered from their usual singular homology groups over $\mathbb{Z}$. Consequently, Mayer's homology theory was ignored for many years, until Kapranov [Kap96] and Sarkaria [Sar95] independently established a quantum analogue of Mayer's generalized complexes in characteristic zero. This new construction uses an N-th root of unity q to twist the simplicially defined boundary maps, where one would usually use the second root of unity -1. In turn it gives rise to "N-chain complexes" whose differentials satisfy the equation $d^N = 0$. The subject is often referred to as *q-homological algebra* and has many applications in modern theoretical physics.

At the same time, Parageis [Par81] forged a connection between homological algebra with a certain Hopf algebra by interpreting chain complexes as (co)modules over this Hopf algebra. After applying Majid's (inverse) bosonisation procedure [Maj97], the algebra in question is $H = \Bbbk[d]/(d^2)$, a Hopf algebra in the symmetric monoidal category of super-$\Bbbk$-modules. In this language, the various algebraic constructions on chain complexes have straightforward meanings in $H-$mod: the differential on tensor products of chain complexes comes from the comultiplication of H, morphism spaces between chain complexes have natural d-actions since $H-$mod has internal hom etc. Generalizing Pargeis's interpretation, Bichon [Bic03] shows that the category of N-complexes is equivalent to (co)modules over the Borel subalgebra of the small quantum $\mathfrak{sl}_2$ at an N-th root of unity q. For such a q, this Borel subalgebra inverse-bosonizes to the Hopf algebra $H = \Bbbk[d]/(d^N)$ in the braided category of q-graded vector spaces. Likewise, Mayer's p-complexes can be explained using the Hopf algebra $\Bbbk[\partial]/(\partial^p)$ over a field of characteristic $p > 0$ in the usual category of graded vector spaces.

Of crucial importance for the Hopf algebras appearing above is the Frobenius structure on their (co)module categories. A category is said to be *Frobenius* if the class of projective objects coincides with the class of injective objects. This property allows one to pass from the module category H–fmod of finite dimensional H-modules to its associated stable category $H-\underline{\text{fmod}}$, which plays an analogous role to taking cohomology of chain complexes. In fact, a classical theorem of Sweedler states that any finite dimensional Hopf algebra H is Frobenius. To this end, Khovanov [Kho16] has generalized the above constructions to any finite dimensional Hopf algebra, bridging homological algebra and the stable category $H-\underline{\text{fmod}}$ of a finite dimensional Hopf algebra H.

Interesting algebra objects in $H-\underline{\text{fmod}}$ usually arise from H-module algebras, i. e., algebras whose structural maps are H-module homomorphisms. The study of homological properties of these H-module algebras and their module categories is known as *hopfological algebra* (terminology due to Khovanov). Hopfological algebra of Mayer's p-complexes occurs naturally in categorification at roots of unity by the following observation of Bernstein-Khovanov (see [Kho16]).

Lemma 2.1. *The Grothendieck ring of the stable category $H-\underline{\text{fmod}}$ of finite dimensional H-modules, where H is the graded Hopf algebra $\Bbbk[\partial]/(\partial^p)$ over a field of characteristic $p > 0$, is isomorphic to the ring of integers $\mathcal{O}_p$ in the pth cyclotomic field.*

In analogy with usual differential graded algebras, I have developed some basic hopfological properties of H-module algebras in [Qi14]. For instance, one has the notion of *bar resolutions* in hopfological algebra, and any hopfological module has a (simplicial) bar resolution [Qi14, Theorem 6.6]. Using these resolutions, the following theorem is proven.

Theorem 2.2. *If $\phi : A \longrightarrow B$ is a quasi-isomorphism of H-module algebras, then the derived induction and restriction functors ϕ^*, ϕ_* are equivalences of triangulated categories between $\mathcal{D}(A,H)$ and $\mathcal{D}(B,H)$ that are quasi-inverse to each other.*

The following statement summarizes some of the results of [Qi14] and shows the relationship between hopfological algebra and categorification.

Theorem 2.3. *The compact derived category $\mathcal{D}^c(A,H)$ of an H-module algebra A is a categorification of a module over the ring $K_0(H-\underline{\mathrm{fmod}})$: the derived tensor product of an A-module with a compatible H-action by a finite dimensional H-module categorifies the algebra-module multiplication map.*

More succinctly, this result can be understood via a commutative diagram encapsulating the "categorical module structure":

$$\begin{array}{ccc} H-\underline{\mathrm{fmod}} \times \mathcal{D}^c(A,H) & \xrightarrow{\otimes} & \mathcal{D}^c(A,H) \\ \Downarrow K_0 & & \Downarrow K_0 \\ K_0(H-\underline{\mathrm{fmod}}) \times K_0(\mathcal{D}^c(A,H)) & \xrightarrow{\times} & K_0(\mathcal{D}^c(A,H)). \end{array} \tag{1}$$

When $H = \Bbbk[\partial]/(\partial^p)$, an H-module algebra is also called a *p-differential graded algebra* (p-DG algebra). Combining Theorem 2.3 with Lemma 2.1, one obtains that the Grothendieck group of a p-DG algebra is a module over the ring $\mathcal{O}_p$ of cyclotomic integers. This explains the relevance of p-DG theory to categorification at prime roots of unity. However, as in the usual DG case, computing Grothendieck groups of p-DG algebras, or more generally, p-DG categories, is quite a hard problem.

I expect that, generalizing K_0, the notion of *higher algebraic K-theory* in the sense of Quillen extends naturally to the hopfological setting:

Problem 2.4. *Let A be an H-module algebra.*

(i) *Show that the category of hopfological modules over A is Waldhausen.*
(ii) *Develop the analogue of higher algebraic K-theory in hopfological algebra.*

I plan to approach the problem following the work of Thomason-Trobaugh [TT90]. Moreover, homotopy theory indicates that higher algebraic K-theory of a DG algebra is in a certain sense a decategorification of some stable infinity categories associated with the DG algebra. An upgraded version of Problem 2.4 is the following.

Problem 2.5. *Find the appropriate generalizations and properties of homotopy theory in the hopfological setting.*

This formulation of the problem is rather vague so as to avoid technicalities. Along the way towards establishing it, I expect the following subproblem of Problem 2.5 to be resolved in the near future.

Problem 2.6. *Some analogues of DG Morita theory in the sense of Toën* [Toë07] *can be generalized to hopfological algebra. In particular, hopfological functors between hopfological categories arise as Fourier-Mukai transforms associated with hopfological bimodules.*

The techniques needed for hopfological homotopy theory should also make the difficult computation of p-DG Grothendieck groups more tractable. For example, the following Künneth-type problem is a fundamental question in categorification, and one hopes it can be resolved using our homotopy-theoretic tools.

Problem 2.7. *Determine the conditions under which the following Künneth formula holds for two H-module algebras A and B:*

$$K_0(\mathcal{D}^c(A\otimes B, H)) \cong K_0(\mathcal{D}^c(A,H)) \otimes_{K_0(H\text{–}\underline{\mathrm{fmod}})} K_0(\mathcal{D}^c(B,H)).$$

Another dividend of developing enough homotopy theory in hopfological algebra is that many interesting objects coming from bordered Heegaard-Floer theory should provide examples of localizations of Fukaya categories at prime roots of unity. This phenomenon has already appeared in certain examples which I have investigated with my collaborator J. Sussan in [QS16]. The solution to such problems will exhibit interesting Koszul duality instances in hopfological algebra.

3. A Small Categorification

This section will be devoted to some initial steps towards Conjecture 1.2. Under the guidance of my former advisor M. Khovanov, I have proven Conjecture 1.1 in the affirmative when $N = p$ is prime. Categorification of the half quantum group $U_q^+(\mathfrak{sl}_2)$ at generic values of q has been constructed by Khovanov-Lauda [KL09] and Rouquier [Rou08] using nilHecke algebras. In [KQ15], we take this categorification, equip it with a p-DG algebra structure, and use the tools developed in [Qi14] to prove that it has the right Grothendieck group.

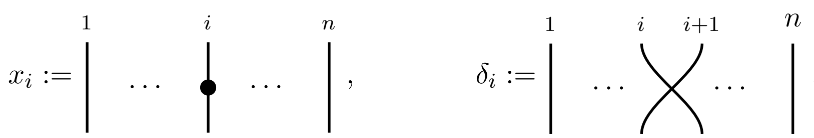
The nilHecke algebra NH_n on n letters is a $\Bbbk$-algebra with generators $x_1, \dots, x_n$ and $\delta_1, \dots, \delta_{n-1}$, subject to defining relations

$$x_ix_j = x_jx_i, \quad x_i\delta_j = \delta_jx_i \ (|i-j|>1) \quad \delta_i\delta_j = \delta_j\delta_i \ (|i-j|>1), \tag{2}$$

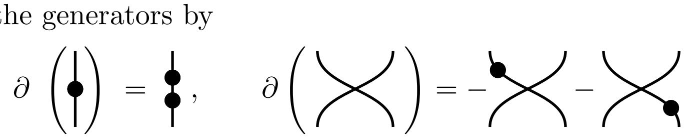
$$\delta_i^2 = 0, \quad \delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1}, \quad \delta_ix_i - x_{i+1}\delta_i = 1 = x_i\delta_i - \delta_ix_{i+1}. \tag{3}$$

Here $\Bbbk$ stands for a field, which can be assumed to be of characteristic $p > 0$ in this section.

A graphical presentation for monomials in the generators of NH_n makes clear the monoidal structure on the *nilHecke category* $\mathrm{NH} := \oplus_{n\in\mathbb{N}}\mathrm{NH}_n$. Here x_i's, respectively δ_i's, are depicted as a dot on the ith strand and the crossing of the ith and $(i+1)$st strands:



Definition 3.1. Let $\Bbbk$ be a field of positive characteristic p. Let ∂ be the differential on NH, defined diagrammatically on the generators by



and extended to the entire algebra by the Leibniz rule.

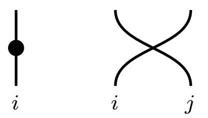
It is not hard to show that $\partial^p = 0$. The resulting *p-DG nilHecke category* is written as (NH, ∂). The following result is proven by Khovanov and myself in [KQ15, Section 3].

Theorem 3.2. *The derived p-DG nilHecke category, denoted $\mathcal{D}(\mathrm{NH},\partial)$, is triangulated monoidal, and its Grothendieck ring is isomorphic to the small quantum group $u_q^+(\mathfrak{sl}_2)$, where q is a primitive p-th root of unity.*

The KLR algebras $R(\mathfrak{g})$ defined by Khovanov-Lauda [KL09, KL11] and Rouquier [Rou08] are vast generalizations of nilHecke algebras, which categorify half of quantum Kac-Moody algebras $U_q^+(\mathfrak{g})$ at generic q-values. The KLR categorification is closely related to earlier geometric realizations of quantum groups by Lusztig [Lus93]. Therefore, one natural question extending Theorem 3.2 is:

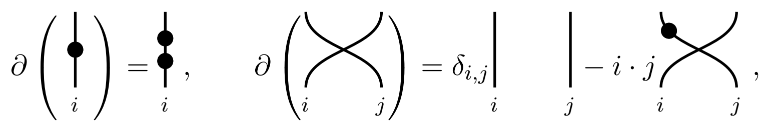
Problem 3.3. *Find the geometric/topological origin of the p-differential on* NH.

To further describe extensions of Theorem 3.2, let $\mathfrak{g}$ be given by a simply-laced, oriented Dynkin diagram whose vertex set is denoted by I. The KLR algebras over a field $\Bbbk$ are generated monoidally by dots and crossings as before, but the strands are colored by vertices of I. For any vertices $i, j \in I$, they are depicted as



In [KQ15, Section 3], we have classified all interesting local p-differentials on KLR algebras associated with a simply-laced Cartan datum, and have shown that some crucial relations in the definition of the small quantum group $u_q^+(\mathfrak{g})$ lift to the categorical setting.

Definition 3.4. Let $\Bbbk$ be a field of characteristic $p > 0$. Define ∂ to be the differential operator on $R(\mathfrak{g})$ as follows. For any vertices $i, j \in I$, set



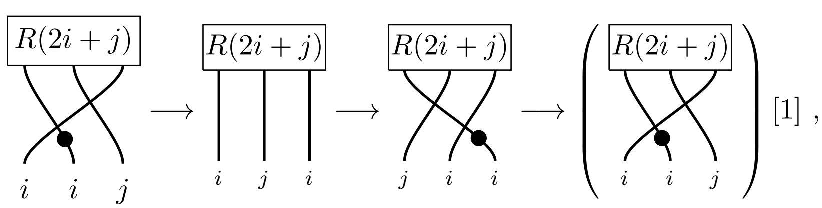
where $i \cdot j$ stands for the Cartan pairing arising from the Dynkin diagram of $\mathfrak{g}$. It is easy to show that $\partial^p = 0$, so that $(R(\mathfrak{g}), \partial)$ is a p-DG monoidal category.

Problem 3.5. *The simply-laced KLR algebras arise as* Ext*-algebras of certain sheaves on quiver varieties. Find the geometric explanation of the above p-nilpotent differential.*

This question is a natural extension of the previous Problem 3.3 about p-differentials on nilHecke algebras. The next result proved in [KQ15, Theorem 4.14] can be regarded as evidence that the approach adopted by us is on the right track.

Theorem 3.6. *The categorical quantum Serre relations, which define the small quantum group $u_q^+(\mathfrak{g})$ at a prime root of unity, hold in the triangulated category $\mathcal{D}(R(\mathfrak{g}), \partial)$, realized as certain exact triangles of p-DG $R(\mathfrak{g})$-modules.*

For instance, when $i,\ j \in I$ are vertices connected by one edge, the quantum Serre relation $E_i^{(2)} E_j + E_j E_i^{(2)} = E_i E_j E_i$ is categorified by the exact triangle

(4) 

where each term stands for a p-DG module generated by the element drawn below the boxes.

The above Theorem 3.6 leads naturally to the expectation that combining KLR algebras and hopfological algebra of p-DG algebras should give rise to a categorification of $u_q^+(\mathfrak{g})$ at a pth root of unity.

Problem 3.7 ([KQ15, Conjecture 4.18]). *The Grothendieck ring of the p-DG KLR category $\mathcal{D}(R(\mathfrak{g}), \partial)$ is isomorphic to an integral form of $u_q^+(\mathfrak{g})$ at a prime root of unity when $\mathfrak{g}$ is of ADE type.*

I believe that the development of homotopic hopfological algebra, discussed in Section 2, will provide the missing pieces towards a proof of this conjecture.

Certain quotients of the type A KLR algebras have been shown by Brundan and Kleshchev [BK09] to be isomorphic to the group algebras of the symmetric groups. It follows from their work that the representation category of all symmetric groups in characteristic $p > 0$ carries a categorical quantum affine $\widehat{\mathfrak{sl}}_p$-action coming from the KLR algebras. A natural question worth investigating is to incorporate the p-differential into symmetric groups in characteristic p.

Problem 3.8.
(i) *Compute the induced p-differential on symmetric groups in characteristic p and study its p-DG derived category.*
(ii) *Investigate the categorical quantum group action at a prime root of unity for p-DG symmetric groups.*

The second part of this problem will shed new light on this classical topic in representation theory. It is also closely related to representation theory of categorical quantum groups at a root of unity to be described in Section 4.

Another natural problem is the corresponding result for nonsimply-laced types.

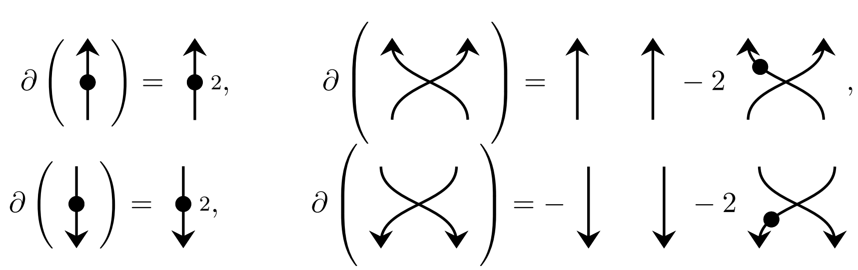
Problem 3.9. *Extend the p-DG approach to categorify non-simply-laced quantum groups at roots of unity.*

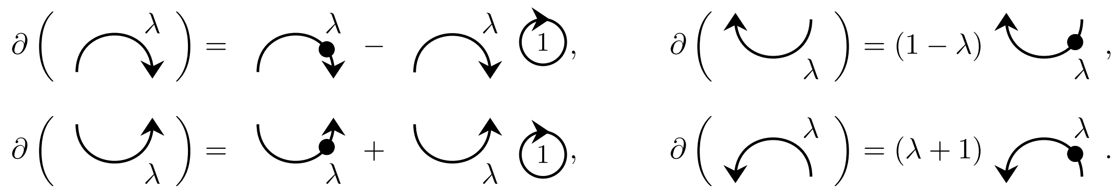
There is no obvious way to define p-differentials on non-simply laced KLR algebras. However, to approach the problem, one may take equivariant "folding" of p-DG KLR algebras in the simply-laced cases, which is more reminiscent of Lusztig's construction of non-simply laced quantum groups.

When q is a root of unity, quantum groups are no longer semisimple, and the algebra structures and their representation theory are more subtle than those at generic q values. One expects this subtlety to be reflected on the categorified level as well. The representation theory aspects will be discussed in the next section. In the remainder of this Section, categorifications of the entire quantum groups will be discussed, building on the previous categorifications of the positive halves. This is built upon Lauda's "double" construction $\mathcal{U}$ in [Lau10], a categorification of the entire quantum $\mathfrak{sl}_2$. B. Elias and myself [EQ16b] have equipped $\mathcal{U}$ with a p-DG structure over a field of characteristic $p > 0$ as below.

To define $\mathcal{U}$, Lauda takes two copies of the nilHecke category NH described by oriented string diagrams, one for the endomorphisms of E's and another for F's, and "glues" together these two copies of NH via biadjunctions, which are diagrammatically described by cups and caps. This is reminiscent of the usual "double" construction of a universal enveloping algebra from its Borel "halves."

Definition 3.10. Let $\Bbbk$ be a field of positive characteristic p. Define a p-differential on the generators of $\mathcal{U}$ by





Here the "bubble diagrams" appearing in the definition of the differential are certain closed diagrams built out of dots, cups and caps. Bubbles belong to the center of $\mathcal{U}$.

Theorem 3.11 ([EQ16b]). *The derived 2-category $\mathcal{D}(\mathcal{U},\partial)$ has its Grothendieck ring isomorphic to the Lusztig integral form of the finite-dimensional (small) quantum $\dot{u}_q(\mathfrak{sl}_2)$ at a prime root of unity.*

The next problem is a natural generalization of this result for Khovanov-Lauda's 2-category $\mathcal{U}(\mathfrak{g})$ (see [KL10] for the definition) assciated with any Kac-Moody algebra $\mathfrak{g}$. This is also a continuation of Problem 3.7.

Problem 3.12. *Generalize this result to the 2-category $\mathcal{U}(\mathfrak{g})$ for any simply-laced $\mathfrak{g}$.*

Lauda's 2-category $\mathcal{U}$ has been further extended to an idempotent complete version $\dot{\mathcal{U}}$ by Khovanov, Lauda, Mackaay and Stošić [KLMS12]. By construction, $\mathcal{U}$ and $\dot{\mathcal{U}}$ are Morita equivalent, and both categorify $\dot{U}_q(\mathfrak{sl}_2)$ at a generic q value. However, when equipped with p-differentials, these p-DG 2-categories behave quite differently! The next result of B. Elias and myself in [EQ16a] reveals this surprising fact.

Theorem 3.13. *The derived 2-category $D(\dot{\mathcal{U}},\partial)$ has its Grothendieck ring isomorphic to the infinite-dimensional divided power integral form of $\dot{U}_q(\mathfrak{sl}_2)$ at a p-th root of unity.*

There is a p-DG functor from $(\mathcal{U},\partial)$ to $(\dot{\mathcal{U}},\partial)$, which is, quite interestingly, a homotopy equivalence but not a derived equivalence under localization. On the derived category level, this functor descends to an embedding of triangulated categories, which categorifies the embedding of the small quantum $\dot{u}_q(\mathfrak{sl}_2)$ inside the divided power $\dot{U}_q(\mathfrak{sl}_2)$ at a prime root of unity. A natural categorification of the quotient map of this embedding is constructed in [Qi17], lifting Lusztig's quantum Frobenius map for quantum groups at a prime root of unity.

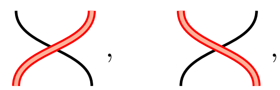
Theorem 3.14. *The restriction functor to a fully-faithful subcategory inside $D(\dot{\mathcal{U}},\partial)$ categorifies the quantum Frobenius map of Lusztig for $\dot{U}_q(\mathfrak{sl}_2)$.*

To summarize this section, the small quantum $\dot{u}_q(\mathfrak{sl}_2)$ is crucial in the construction of 3-manifold invariants in the 3D WRT TQFT. Meanwhile, the infinite dimensional form $\dot{U}_q(\mathfrak{sl}_2)$ has been studied by Beilinson-Lusztig-MacPherson [BLM90] and Kazhdan-Lusztig [KL93], and it is closely related to representation theory of $\mathfrak{sl}_2$ in characteristic $p>0$, and affine Lie algebras $\widehat{\mathfrak{sl}}_2$ at a certain level. Affine Lie algebras at the various levels are fundamental in 2D conformal field theory, which is equivalent to the WRT 3D TQFT. I expect that this relationship between quantum groups at roots of unity and affine Lie algebras in the corresponding levels should be categorified. This is also one of the background motivations to study Problem 3.8. Furthermore, the categorified version $(\dot{U}_q(\mathfrak{sl}_2),\partial)$ and its generalization to other Lie types would play a similarly important role in categorified 3D TQFT's.

4. Categorified representation theory

While the previous section has been focused on categorifying quantum groups at prime roots of unity, in this section, categorical representation theory of the quantum groups and applications towards realizing Crane and Frenkel's Conjecture 1.2 will be outlined.

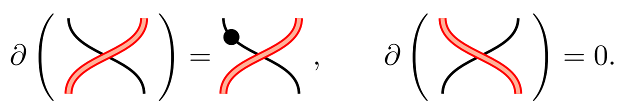
The Jones polynomial has a combinatorial description, and it can also be explained using representation theory of quantum $\mathfrak{sl}_2$ at a generic q-value. On the categorified level, Webster [Web10a, Web10b] has developed an explanation of Khovanov homology using the categorical representation theory of quantum $\mathfrak{sl}_2$. Recall that, to categorify the tensor product representation $V_1^{\otimes n}$, where V_1 stands for the standard two-dimensional defining representation of quantum $\mathfrak{sl}_2$, Webster has introduced a family of diagrammatically defined algebras with n red strands and various black strands. Just as for KLR algebras, Webster algebras are local in the sense that generators on far away strands commute with each other. The black strands are generated by KLR string diagrams, and red strands are not allowed to cross each other. The extra local generators of this algebra are depicted by



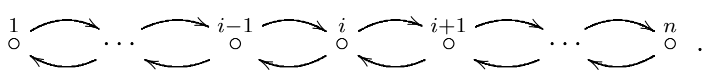
which are then subject to certain explicit relations besides the KLR relations.

In the joint work [QS16], we have classified all p-nilpotent differentials on the Webster algebra above that admit a compatible categorified small quantum $\mathfrak{sl}_2$ action (see Theorem 3.2). The differential is unique, and is given as follows.

Definition 4.1. Let $\Bbbk$ be a field of positive characteristic p. The p-DG Webster algebra over $\Bbbk$ is equipped with the p-nilpotent differential



The relevance of p-DG Webster algebras to quantum topology can already be seen through the easiest special cases. To do so, consider the quiver Q_n



Definition 4.2. The p-DG *zig-zag algebra* $(A_n^!, \partial)$ is the quotient of the path algebra $\Bbbk Q_n$ by the relations on paths

$$(i|i-1|i) = (i|i+1|i) \ (i = 2, \dots, n-1), \ (1|2|1) = 0,$$

together with the p-differential defined on the generators by

$$\partial(i|i+1) = (i|i+1|i|i+1), \qquad \partial(i|i-1) = 0.$$

The next Theorem has been established by myself and my coauthor J. Sussan in [QS16].

Theorem 4.3. (i) *The p-DG zig-zag algebra $(A_n^!, \partial)$ is isomorphic to the p-DG Webster algebra with n red strands and only one black strand.*

(ii) *The derived category $\mathcal{D}(A_n^!, \partial)$ affords a categorical n-stranded braid group action, and this action categorifies the classical Burau representation of the braid group at a pth root of unity.*

The zig-zag algebra $A_n^!$ without the differential appears quite often in representation theory and algebraic geometry. Generalizing this example, Webster algebras are related to representation categories $\mathcal{O}$ of simple Lie algebras, and derived categories of constructible sheaves on Nakajima quiver varieties [Nak01]. A fruitful parallel project to investigate is thus the following.

Problem 4.4. *Investigate the p-differential on categories $\mathcal{O}$, and on derived categories of certain constructible sheaves over quiver varieties, both in prime characteristic p.*

Another important appearance of $A_n^!$ in its Koszul dual form is in symplectic geometry, where it is used to describe the Fukaya category of a certain 4-manifold fibered over an n-punctured 2-disk. I expect that the "dual" algebra, in the p-DG sense, should be an instance of the analogous notion of A-infinity algebras for DG algebras.

Problem 4.5. *What is the Koszul dual algebra of $(A_n^!, \partial)$ in the p-DG sense?*

Together with my coauthors, I plan to investigate more complicated Webster algebras in upcoming works. The following problem plays a central role in the entire project.

Problem 4.6. *Show that the p-DG derived category of Webster algebras of n red strands categorify tensor product representation $V_1^{\otimes n}$ of quantum $\mathfrak{sl}_2$ at a pth root of unity.*

Some technical aspects for solving the problem will be discussed in the second half of this section.

On the Grothendieck group level, $V_1^{\otimes n}$ is a module over the small quantum group $\dot{u}_q(\mathfrak{sl}_2)$ at a pth root of unity. Varying n, these modules come with a Temperley-Lieb algebra (or rather, algebroid) action that intertwines the quantum $\mathfrak{sl}_2$ action. This story should also lift to the p-DG derived category level.

Problem 4.7. *Exhibit a categorical small quantum $\mathfrak{sl}_2$-action on the derived category of p-DG Webster algebras. Construct a categorical Temperley-Lieb algebra action on these derived categories.*

The Temperley-Lieb algebra action gives rise to the Jones polynomial for knots and links.

Problem 4.8. *Show that one obtains the categorical specialization of Khovanov homology at a pth root of unity from (derived) p-DG Webster algebras.*

This means that, to a link, one should obtain a p-complex with finite-dimensional p-cohomology, whose Euler characteristic equals the Jones polynomial specialized at a pth root of unity. This number is a link invariant valued in $\mathcal{O}_p$, the latter ring categorified by p-complexes (Lemma 2.1). Furthermore, this p-complex should, up to homotopy, be a functorial (projective) invariant of links with respect to link cobordisms.

Problem 4.8 constitutes an important step towards realizing Conjecture 1.2. The next step would be to investigate colored link homologies in a similar fashion. The strategy is then to construct 3-manifold invariants from state sums of various colored homology theories. Combined with the projects discussed in this proposal, this would eventually lead to an answer to Conjecture 1.2.

The rest of this section is devoted to describing three important ingredients towards resolving Problem 4.8. These will be helpful tools in analyzing p-DG Webster algebras, and will be of interest in their own right.

Schur algebras. The Webster algebras described above are Morita equivalent to some classical objects in representation theory known as (particular cases of) *Schur algebras.* The relationship between certain quotients of type A KLR algebras and graded versions of Schur algebras has been studied in detail by Hu and Mathas [HM15]. Furthermore, it is expected that Webster algebras in type A are Morita equivalent to the graded Schur algebras of the same type.

The motivation for considering Schur algebras can be summarized as follows. For simplicity, only the $\mathfrak{sl}_2$ case will be considered. One first forms the *cyclotomic quotient*

$$\mathrm{NH}^n = \oplus_{k=0}^{n} \mathrm{NH}_k^n, \tag{5}$$

of the nilHecke category, which depends on a non-negative integer n. The category NH^n is equipped with a categorical quantum $\mathfrak{sl}_2$ action. For a fixed k, each piece NH_k^n has a collection of naturally defined modules G_μ, where μ ranges over ways of writing n into k ones and $n-k$ zeros. Evidently there are $\binom{n}{k}$ many of these modules. The graded Schur algebra in this case is defined as the graded endomorphism algebra

$$S_k^n := \mathrm{END}_{\mathrm{NH}_k^n}(\oplus_\mu G_\mu). \tag{6}$$

It is then not hard to compute the Grothendieck group of S_k^n, which has rank $\binom{n}{k}$ and is equal to the rank of the weight $2n-k$ space in $V_1^{\otimes n}$.

Problem 4.9. *Equip the graded quiver Schur algebras with a p-differential in characteristic $p > 0$. Give an inductive procedure to compute its p-DG Grothendieck group.*

Together with M. Khovanov and J. Sussan, we have investigated this problem with some partial progress ([KQS17]). This approach gives an alternative yet more combinatorially tangible way for categorifying the tensor product representation $V_1^{\otimes n}$ of quantum $\mathfrak{sl}_2$ at a pth root of unity.

Problem 4.10. *In the $\mathfrak{sl}_2$ case, show that the p-DG Webster algebras are also derived equivalent to p-DG quiver Schur algebras.*

Categorical Hecke algebras. As discussed earlier, on the decategorified level, the tensor product representation $V_1^{\otimes n}$ carries a Temperley-Lieb algebra action. More generally, at a generic q value, the quantum group $\mathfrak{sl}_m$ will act on an iterated tensor product $V^{\otimes n}$ of its standard representation V, and the double commutant algebra in $\mathrm{End}(V^{\otimes n})$ will be a quotient of the Hecke algebra of the symmetric group S_n. This fact is usually known as the (quantum) *Schur-Weyl duality.* In particular, the images of Hecke algebra generators in the endormophism algebra give rise to *R-matrices* for quantum $\mathfrak{sl}_m$, which are important algebraic inputs for constructing link invariants.

The Hecke algebra at a generic q value has been categorified by Soergel [Soe92], using the category of Soergel bimodules. An explicit generators-and-relations presentation of the Soergel bimodule category has been given by Elias and Khovanov [EK10].

When specialized at a pth root of unity, one still expects the Hecke generators to give rise to R-matrices for the small quantum group. The following problems are being investigated by B. Elias and myself.

Problem 4.11. *Equip the Hecke category in characteristic $p > 0$ with a p-differential, and use it to categorify Hecke algebras in type A at a pth root of unity.*

Problem 4.12. *Construct an action of the Hecke category on Webster algebras for $V_1^{\otimes n}$. Categorically specialize this action to a pth root of unity.*

Centers of small quantum groups and their categorification. In a similar vein as for Hecke algebras, the action of small quantum groups on a tensor product representation also commutes with another natural algebra, namely, the center of the small quantum group. At a root of unity, this algebra is surprisingly large, and has not been as well studied as the Hecke algebra. One beautiful result of Andersen, Jantzen and Soergel [AJS94] about the center is that, the principle block of the center (the block that acts nontrivially on the trivial representation) is independent of the order of the root of unity.

A geometric context for understanding the principal block of the center of a small quantum group has been studied by Bezrukavnikov-Lachowska in [BL07], building on the earlier work [ABG04]. In their framework, the center of the small quantum group of type $\mathfrak{g}$ has been identified with the zeroth Hochschild cohomology ring of $\mathbb{C}^*$-equivariant coherent sheaves on the Springer resolution for $\mathfrak{g}$. By further refining the geometric techniques in [BL07], we have found some exciting connection between the principal block center and Haiman's diagonal coinvariant algebra [Hai94]:

Theorem 4.13. *When $\mathfrak{g} = \mathfrak{sl}_n$ ($n = 2, 3, 4$), the principal block of the small quantum group $u_q(\mathfrak{g})$ has the same bigraded character table as Haiman's diagonal coinvariant algebra DC_n.*

This motivates us to formulate the following conjecture in the same paper.

Problem 4.14 ([LQ17a, Conjecture 4.9])**.** *In type A, the principal block of the small quantum group has the same bigraded character table as Haiman's diagonal coinvariant algebra in the same Lie type.*

A further conjecture concerning the structure of the entire center for $u_q(\mathfrak{sl}_n)$ is formulated in [LQ17b].

Currently, I am collaborating with R. Bezrukavnikov, P. Shan and E. Vasserot on proving this conjecture, by identifying the principal block center geometrically as the cohomology ring of a certain affine Springer fiber, which points to interesting potential connections to symplectic duality.

Problem 4.15. *Find the explicit generators-and-relations presentation for the (principal block of the) center of the small quantum group.*

In the near future, I would like to have a more explicit presentation of this algebra in terms of generators and relations. Then, one would be able to categorify the explicit algebra, which would in turn act on (colored) tangle homology groups at a prime root of unity.

These problems would be interesting in view of Anderson, Jantzen and Soergel's theorem. The categorified structre for each Cartan datum should be a canonical object associated with the corresponding (quantized) Lie algebra $\mathfrak{g}$.

Problem 4.16. *Categorify the center of the small quantum group. Find its relationship with categorified quantum groups at prime roots of unity.*

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