Categorification of small quantum groups

Note Title 8/25/2012

Plan of the talk

1. Stable categories and categorification
2. Categorification of the small quantum $sl(2)$

§1. Stable Categories and Categorification

In 1994, Crane and Frenkel published their seminal paper "Four dimensional topological quantum field theory, Hopf categories, and the canonical bases". There they proposed a program called "categorification", hoping to lift the combinatorial 3D TQFT constructed by Kuperberg to a 4D TQFT

$$
\begin{tikzcd}
\text{Donaldson-Floer} \arrow[r, leftrightarrow, "?"] \arrow[d] & \boxed{\text{A 4D TQFT ?}} \\
\begin{array}{c}\text{WRT-3D TQFT} \\ \text{associated with } D(H)\end{array} \arrow[r, Leftrightarrow] & \begin{array}{l}\text{Kuperberg : a 3D TQFT} \\ \text{based on } U_q^+(sl_2) \quad (q^N=1) \\ \text{or any finite dim'l balanced} \\ \text{Hopf algebra } H.\end{array} \arrow[u, squiggly, "\text{Categorification}"']
\end{tikzcd}
$$

Homological Algebra

For simplicity assume we are working over a ground field $\Bbbk$. The usual homological algebra has the following key features.

(1). Chain complexes and their cohomology groups

$$K^\bullet = (\cdots \xrightarrow{d} K^{i-1} \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \xrightarrow{d} \cdots) \quad \text{s.t.} \quad d^2=0$$

$$H^\bullet(K^\bullet) = \mathrm{Ker}\, d / \mathrm{Im}\, d$$

(2). Direct sums of chain complexes

(3). Tensor products of chain complexes

$$(K^\bullet \otimes L^\bullet)^i := \bigoplus_{k+\ell=i} K^k \otimes L^\ell, \quad d(x\otimes y) = d(x)\otimes y + (-1)^{|x|} x \otimes d(y)$$

$$K^\bullet \otimes \Bbbk \cong \Bbbk \otimes K^\bullet \cong K^\bullet$$

(4). Inner homs

$$\mathrm{HOM}^\bullet(K^\bullet, L^\bullet), \qquad \mathrm{HOM}^i(K^\bullet, L^\bullet) := \{ f: K^\bullet \to L^\bullet \mid f(K^k) \subseteq L^{k+i} \}$$

$$d(f) = d_L \circ f - (-1)^{|f|} f \circ d_K$$

with the property

$$\mathrm{HOM}^\bullet(K^\bullet \otimes L^\bullet, M^\bullet) \cong \mathrm{HOM}^\bullet(K^\bullet, \mathrm{HOM}^\bullet(L^\bullet, M^\bullet))$$

$$H^0(\mathrm{HOM}^\bullet(K^\bullet, L^\bullet)) = \mathrm{Hom}_{C(\Bbbk)}(K^\bullet, L^\bullet)$$

(5). Triangular structures: homological grading shifts / the cone construction / distinguished triangles coming from s.e.s. / compatibility axioms etc.

Homological algebra is an important tool in categorification since it gives a systematic lift of operations in $\mathbb{Z}$:

$$\begin{array}{ccc} D^b(\Bbbk) & \xrightarrow{\quad K_0 \quad} & \mathbb{Z} \\ K^\bullet & \mapsto & \sum (-1)^i \dim_{\Bbbk} K^i \\ \oplus & \mapsto & \text{addition} \\ \otimes & \mapsto & \text{multiplication} \\ \text{tensor unit} & \mapsto & 1 \\ [1] & \mapsto & \text{multiplication by } (-1). \end{array}$$

Rmk: If we take graded vector spaces, we get a categorification of $\mathbb{Z}[q, q^{-1}]$:

$$K_0(D^b(\Bbbk\text{-gvect})) \cong \mathbb{Z}[q, q^{-1}]$$

The grading shift $\{1\}$ becomes multiplication by q.

Observation: Properties (2), (3), (4) are reminiscent of some familiar constructions in representation theory: Take a group G. $H := \Bbbk G$ is a Hopf algebra and the category of H-mod has

(2') $V \oplus W \in H\text{-mod}$

(3') $V \otimes W \in H\text{-mod}$ $\quad h \cdot (v \otimes w) := h_{(1)} v \otimes h_{(2)} w$

(4') $\mathrm{HOM}(V, W) \in H\text{-mod}$ $\quad h \cdot f(v) := h_{(2)} f(S^{-1}(h_{(1)}))$, right adjoint to $\otimes$.

The usual homological algebra can thus be regard as for the graded Hopf super algebra $H = \Bbbk[d]/(d^2)$

Question: Are there analogues of the other features displayed by the usual homological algebra ? For instance, what is cohomology?

Any $K^\bullet$ decomposes "uniquely" into direct sums of

$$\oplus\,(0 \longrightarrow \underbrace{\Bbbk}_{\text{hom. deg } i} \longrightarrow 0) \;\oplus\; \oplus(0 \longrightarrow \underbrace{\Bbbk}_{\text{hom deg } j} \longrightarrow \Bbbk \longrightarrow 0)$$

Taking cohomology just kills the second factor. Note that the second factor consists of projective $\Bbbk[d]/(d^2)$ modules.

Less obvious: $(0 \to \Bbbk \to \Bbbk \to 0)$ is also injective. In fact, $\Bbbk[d]/(d^2)$ is a Frobenius algebra.

Thm (Radford - Larson, Sweedler) Any finite dim'l Hopf (super) algebra is Frobenius. In particular, the class of projective modules coincide with the class of injective modules.

What's the systematic way of killing projectives/injectives ?

The stable category H-$\underline{\text{mod}}$

Intuitively, H-$\underline{\text{mod}}$ is the categorical quotient of H-mod by the class of projective/injective objects.

Def. H-$\underline{\text{mod}}$ consists of the same objects as H-mod, while the morphism space between two objects K, L are given by

$$\mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(K,L) := \mathrm{Hom}_{H\text{-}\mathrm{mod}}(K,L) \Big/ \left(\begin{array}{l}\text{morphisms that factor}\\ \text{through projectives}\end{array}\right)$$

The notion of stable categories makes sense for any Frobenius algebra, not necessarily those coming as finite dim'l Hopf algebras.

Thm. (Happel) If H is a Frobenius algebra, then $H\text{-}\underline{\mathrm{mod}}$ is triangulated.

In general, the morphism space between objects in an arbitrary stable category is hard to compute. But for $H\text{-}\underline{\mathrm{mod}}$, the morphism spaces can be computed explicitly. To do this, we need the notion of integrals for Hopf algebras.

Def. Let H be a Hopf algebra. An element $\Lambda \in H$ is called a left integral of H if $\forall\, h \in H$,

$$h\Lambda = \epsilon(h)\Lambda$$

Thm (Radford-Larson, Sweedler) Any finite dim'l H has a non-zero integral Λ, unique up to a non-zero constant.

Example. (1). $\Bbbk G$ $\quad \Lambda = \Sigma_{g \in G}\, g$ is a left integral

(2). $\Bbbk[d]/(d^2)$, d is a left integral.

(3). More generally $\Lambda^* V$ $\quad v_1 \wedge \cdots \wedge v_n$

(4) $\Bbbk[\partial]/(\partial^p)$ $(\mathrm{char}\,\Bbbk = p > 0)$ $\quad \partial^{p-1}$ is a left integral.

Prop. Let H be a finite dim'l Hopf algebra, and K, L be H-modules. Then

$$\begin{aligned}\mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(K, L) &\cong \mathrm{Hom}_{H\text{-}\mathrm{mod}}(K, L)/\Lambda\cdot \mathrm{HOM}(K, L)\\ &\cong \mathrm{HOM}(K, L)^H/\Lambda\cdot \mathrm{HOM}(K, L)\end{aligned}$$

Example. Note that H acts on $\mathrm{HOM}(K,L)$ by $h\cdot(f)(k) := h_{(2)} f(S^{-1}(h_{(1)})k)$

$$\left.\begin{array}{ll} (2) & \Lambda\cdot(f) = df + (-1)^{|f|} f d \\ (4) & \Lambda\cdot(f) = \partial^{p-1}(f) = \sum_i \partial^{p-1-i} f \partial^i \end{array}\right\} \text{null-homotopies}$$

Relation to categorification

Def. $H = \Bbbk[\partial]/(\partial^p)$ $\deg\partial = 1$. We call the category H-gmod the category of p-complexes while H-$\underline{\text{gmod}}$ the homotopy category of p-complexes

The first consideration of p-complexes in history was by Mayer (1942). In the def. of simplicial homology theory, $d = \sum (-1)^i d_i$ where d_i are some simplicial face maps. Mayer proposed that, if we work over a field of char $p>0$, and define $\partial := \sum d_i$, then $\partial^p = 0$ and there are notions of homology groups. But Spanier soon showed that the homology groups can be recovered from the usual homology groups, and thus are not that interesting.

Then why do we care about p-complexes ?

This was because of an observation of Bernstein-Khovanov:
If $H = \Bbbk[\partial]/(\partial^p)$, where $\deg(\partial) = 1$, then both H-gmod and H-$\underline{\text{gmod}}$ are symmetric monoidal. Furthermore,

$$K_0(H\text{-gmod}) \cong \mathbb{Z}[q, q^{-1}]$$

$$K_0(H\text{-}\underline{\text{gmod}}) \cong \mathbb{Z}[q]/(1+q+\cdots+q^{p-1}) := \mathbb{O}_p.$$

Indeed K_0 is generated by the symbol of $[\Bbbk]$ subject to the only relation:

$$\begin{aligned} 0 = [H] &= [\Bbbk] + [\Bbbk\{1\}] + \cdots + [\Bbbk\{p-1\}] \\ &= (1+q+\cdots+q^{p-1})[\Bbbk] \end{aligned}$$

To utilize this "categorical cyclotomic integers" H-$\underline{\text{gmod}}$, we need to find interesting "algebra" objects in this category. Then the Grothendieck group of this algebra object will naturally be $\mathcal{O}_p$-modules. Recall that an algebra object in the category of chain complexes is given by a differential graded algebra.

Def. A DG algebra A consists of a graded algebra $A \cong \oplus_{n \in \mathbb{Z}} A_n$ together with a differential d such that, $\forall\, a, b \in A$

$$d(ab) = d(a) \cdot b + (-1)^{|a|} a \cdot d(b)$$
$$d^2(a) = 0$$

Def. A p-DG algebra A over a field of char $p > 0$ is a graded algebra together with a differential ∂ such that $\forall\, a, b \in A$

$$\partial(ab) = \partial(a) b + a \partial(b)$$
$$\partial^p(a) = 0$$

More generally, one has the notion of an H-module algebra, which gives rise to algebra objects in H-$\underline{\text{mod}}$. We refer to the study of homological properties of such algebra objects in H-$\underline{\text{mod}}$ as "hopfological algebra".

In analogy with the usual DG case, one has the notion of the abelian category of p-DG modules, homotopic morphisms, quasi-isomorphisms, homotopy categories and derived categories

Thm. (Khovanov, Qi). The homotopy and derived categories of p-DG algebras are module categories over H-$\underline{\text{gmod}}$. Under taking Grothendieck groups (in some appropriate sense), $K_0(\mathcal{D}(A, \partial))$ has the structure of an $\mathcal{O}_p$-module.

$$\begin{tikzcd} H\text{-}\underline{\mathrm{gmod}} \times \mathcal{D}(A,\partial) \arrow[r, "\otimes"] \arrow[d, Rightarrow, "K_0"] & \mathcal{D}(A,\partial) \arrow[d, Rightarrow, "K_0"] \\ \mathcal{O}_p \times K_0(A,\partial) \arrow[r, "\mathrm{mult}"] & K_0(A,\partial) \end{tikzcd}$$

Meta-Question (Khovanov): Are there other symmetric monoidal categories whose Grothendieck ring are isomorphic to other rings of integers in number fields?

§2. Categorification of $\mathcal{U}_q^+(\mathfrak{sl}_2)$

Crane and Frenkel's conjecture

In their 1994 paper, Crane and Frenkel conjectured that, if q is a root of unity, there is a categorification of $U_q^+(\mathfrak{sl}_2)$ (conjecture 2). Furthermore, one should use this categorification to lift Kuperberg's 3D TQFT to a 4D TQFT.

$\mathcal{U}_q^+(\mathfrak{sl}_2)$ has the following integral form over $\mathcal{O}_N \cong \mathbb{Z}[\zeta_N]$ (Lusztig)

$$\mathcal{U}^+ \cong \mathcal{O}_N[E]/(E^N)$$

where it is equipped with the twisted bialgebra structure as follows:

$$\Delta : \mathcal{U}^+ \longrightarrow \mathcal{U}^+ \otimes \mathcal{U}^+ \ : \ E \mapsto E \otimes 1 + 1 \otimes E$$

$$m : \mathcal{U}^+ \otimes \mathcal{U}^+ \longrightarrow \mathcal{U}^+ \ : \ (1 \otimes E^r)\cdot(E^s \otimes 1) = \zeta_N^{rs} E^s \otimes E^r.$$

There is a more refined divided power structure on $\mathcal{U}^+$ given as follows. Let $E^{(r)} := E^r/[r]!$, where $[r] = \sum_{i=1}^{r} \zeta_N^{r+1-2i}$. Then

$$E^{(r)} E^{(s)} = \begin{cases} \begin{bmatrix} r+s \\ r \end{bmatrix} E^{(r+s)} & \text{if } (r+s) \le N-1 \\ 0 & \text{otherwise} \end{cases}$$

We will describe an approach to prove this conjecture when N is a prime number p.

Review of the categorification of $U_q^+(\mathfrak{sl}_2)$

We briefly review how the categorification of $U_q^+(\mathfrak{sl}_2)$ is done at generic values of q, as done by Khovanov-Lauda.

By categorification of the algebra U_q^+ we mean that to find a monoidal category $\mathcal{U}^+$ whose Grothendieck group is naturally isomorphic to U_q^+. In particular

$$\begin{array}{ccc} \mathcal{U}^+ & \xrightarrow{\quad K_0 \quad} & U_q^+ \\ \mathcal{E}, \mathcal{E}^i, \mathcal{E}^{(i)} & \longmapsto & E, E^i, E^{(i)} \end{array}$$

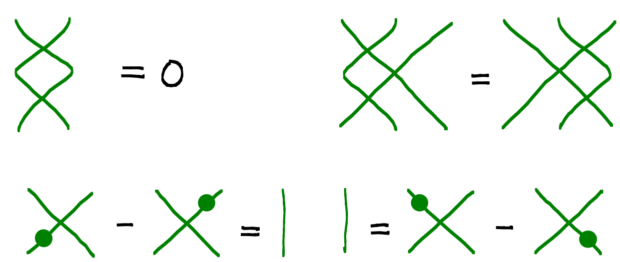
The algebra structure on U_q^+ also requires there is a map

$$\mathcal{U}^+ \otimes \mathcal{U}^+ \longrightarrow \mathcal{U}^+$$

so that $\mathcal{E}^i \mathcal{E}^j$, $\mathcal{E}^{(i)}\mathcal{E}^{(j)}\mathcal{E}^{(k)}$ etc. make sense. The hard part of the game is to pin down the morphism spaces in $\mathcal{U}^+$, which we give the answer of Khovanov-Lauda now.

Consider the nilHecke algebra on n-strands NH_n which has the following diagrammatic description. It is generate by:

subject to the local relations



More intrinsically, there is an inclusion of rings $Sym_n := \Bbbk[x_1, \cdots, x_n]^{S_n} \subseteq Pol_n := \Bbbk[x_1, \cdots, x_n]$. The rank one free Pol_n-module P_n is free of rank $n!$ over Sym_n. NH_n is the endomorphism ring of P_n over Sym_n.

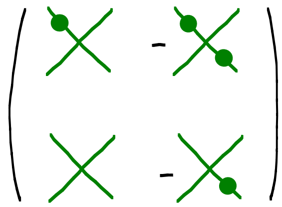
$$NH_n \cong End_{Sym_n}(P_n)$$

where dots act by multiplication by x_i, and crossings act by divided difference operators. From this one can see that

$$NH_n \cong Mat(n!, Sym_n)$$

and it follows that $K_0(NH_n) \cong \mathbb{Z}[q, q^{-1}]$.

Example. When $n=2$,

$NH_2 \cong$ 

The diagrammatic presentation gives rise to an inclusion of algebras

$$\iota_{n,m} : NH_n \otimes NH_m \longrightarrow NH_{n+m}$$

by putting pictures sideways next to each other. Summing over all n,m we get

$$\iota : NH \otimes NH \longrightarrow NH$$

where $NH := \bigoplus_{n \in \mathbb{N}} NH_n$. The map ι induces induction and restriction functors between module categories.

Thm. (Khovanov-Lauda) NH categorifies $\mathcal{U}^+_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{sl}_2)$

$$\begin{array}{ccc} NH & \xrightarrow{K_0} & \mathcal{U}^+(\mathfrak{sl}_2) \\ NH_n & \longmapsto & E^n \\ P_n & \longmapsto & E^{(n)} \\ \mathrm{Ind}/\mathrm{Res} & \longmapsto & \text{multiplication/comultiplication} \end{array}$$

p-DG structure on NH_n

Fix a field $\Bbbk$ of char $p>0$. We want to build a monoidal p-DG category whose Grothendieck group is isomorphic to the small quantum group u^+. The best one can hope is that NH carries a p-DG structure so that the theorem of KL "specializes" to roots of unity directly. This is indeed the case.

Recall that from before a p-DG algebra (A, ∂) is a graded algebra equipped with an endomorphism of degree one such that $\forall\, a, b \in A$

$$\partial(ab) = \partial(a)\cdot b + a\,\partial(b)$$
$$\partial^p(a) = 0$$

A p-DG module M is an A-module with a compatible ∂-action. As with the usual DG algebras, one can pass from

$$\begin{array}{ccccc} (A,\partial)\text{-mod} & \longrightarrow & \mathcal{C}(A,\partial) & \longrightarrow & \mathcal{D}(A,\partial) \\ \| & & \text{the homotopy} & & \text{the derived} \\ A\#\Bbbk[\partial]/(\partial^p)\text{-mod} & & \text{category} & & \text{category} \end{array}$$

Many properties of the usual DG algebra holds in this context.

Example. $NH_1 \cong \Bbbk[x]$. $[NH_1] = E$. If we set $\partial(x) = x^2$, then $\Bbbk \hookrightarrow NH_1$ is a quasi-isomorphism $\Rightarrow K_0(NH_1,\partial) \cong \mathcal{O}_p$. This is an application of the following theorem.

Thm (Qi) If $A \longrightarrow B$ is a quasi-isomorphism of p-DG algebras, then there is a derived equivalence $\mathcal{D}(A,\partial) \cong \mathcal{D}(B,\partial)$.

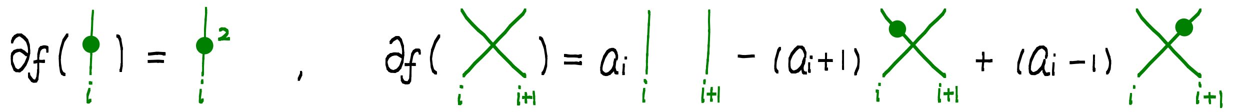
To go further, we have to utilize the polynomial module P_n of Pol_n. If we let $Pol_n = \Bbbk[x_1,\cdots,x_n]$ be the p-DG algebra with $\partial(x_i) = x_i^2$, then the rank one free P_n-module has many p-DG module structures: pick f homogeneous of degree 1 in Pol_n. define a $\partial = \partial_f$-action on P_n by

$$\partial_f(h\cdot 1) := \partial(h)\cdot 1 + f\cdot h$$

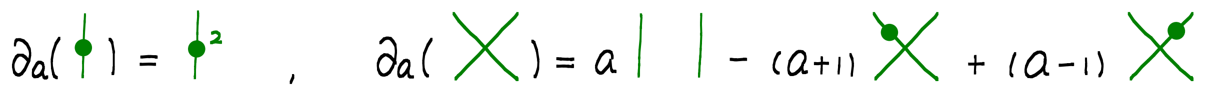
Then $\partial_f^p \equiv 0$ iff $f \in \sum \mathbb{F}_p x_i$. $(Sym_n,\partial) \subseteq (Pol_n,\partial)$ naturally as a subalgebra since the ∂-action on Pol_n commutes with permutations. Thus

$$NH_n(f) \cong End_{Sym_n}(P_n(f))$$

inherits a differential ∂_f:



where $f = \alpha_1 x_1 + \cdots + \alpha_n x_n$, $a_i = \alpha_{i+1} - \alpha_i$. We want ∂_f to be local, so assume all $a_i = a$ and we study the p-DG algebra NH_n with the induced differential:



Thm. (Khovanov-Qi) When $a = \pm 1$,

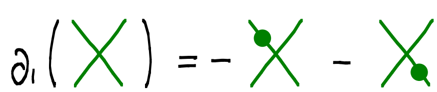
$$(NH, \partial_{\pm 1}) \xrightarrow{K_0} \mathcal{U}^+_{\zeta_p}(\mathfrak{sl}_2)$$

$$\left.\begin{array}{lcl} (NH_n, \partial_{\pm 1}) & \longmapsto & E^n \\ (P_n, \partial_{\pm 1}) & \longmapsto & E^{(n)} \end{array}\right\}$$

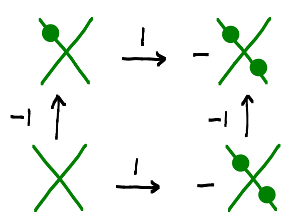
$$\begin{array}{lcl} \text{Derived induction} & \longmapsto & \text{multiplication/comultiplication} \\ \text{/restriction} & & \end{array}$$

Categorifying $E^2 = [2]! E^{(2)}$

Under ∂_1, we have



from which we compute the differential on the matrix algebra $\mathrm{Mat}(2, \mathrm{Sym}_2)$ is given by



Therefore the left regular representation (NH_2, ∂_1) fits into a s.e.s. of p-DG modules

$$0 \longrightarrow P_2\{1\} \longrightarrow NH_2 \longrightarrow P_2\{-1\} \longrightarrow 0$$

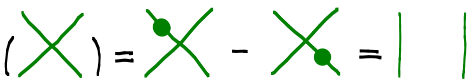
which results in a distinguished triangle in $D(NH_2, \partial_1)$

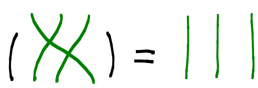
$$P_2\{1\} \longrightarrow NH_2 \longrightarrow P_2\{-1\} \longrightarrow P_2\{1\}[1]$$

$\Rightarrow$ In K_0, $[NH_2] = \zeta_p [P_2] + \zeta_p^{-1} [P_2]$.

Categorifying $E^p = 0$

We illustrate this with two examples.

Examples (i). $p=2$. $\partial_1($  $\Rightarrow (NH_2, \partial_1)$ is acyclic as a DG algebra $\Rightarrow D(NH_2, \partial_1) \cong 0 \Rightarrow K_0(NH_2, \partial_1) = 0$.
$\Rightarrow E^2 = 0$

(ii). $p=3$ $\partial_1^2($  $\Rightarrow (NH_3, \partial_1)$ is acyclic as a 3-DG algebra $\Rightarrow D(NH_3, \partial_1) \cong 0 \Rightarrow K_0(NH_3, \partial_1) = 0 \Rightarrow E^3 = 0$.

Lemma. $(NH_n, \partial_1)_{n \geq p}$ are all acyclic when $\mathrm{char}\, \Bbbk = p$.

The relation $E^p = 0$ is categorified by this lemma and the following general characterization of acyclic p-DG algebras.

Prop. TFAE.

(i). A is acyclic

(ii). $D(A, \partial) \cong 0$

(iii). $\exists\, x \in A$ s.t. $\partial^{p-1}(x) = 1$

(iv). $\exists\, y \in A$ s.t. $\partial(y) = 1$

Outline of future works

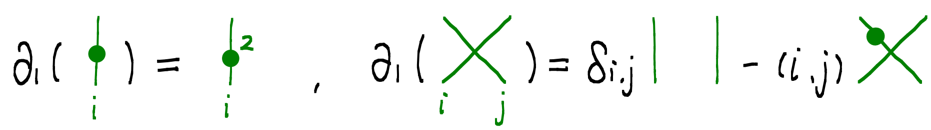
The Khovanov-Lauda-Rouquier algebra $R(\Gamma)$ associated with any Cartan datum Γ is a natural generalization of NH to categorify the quantum group $\mathcal{U}_q^+(\mathfrak{g}_\Gamma)$ associated with Γ.

Thm. (Khovanov-Lauda) $K_0(R(\Gamma)) \cong \mathcal{U}_q^+(\mathfrak{g}_\Gamma)$

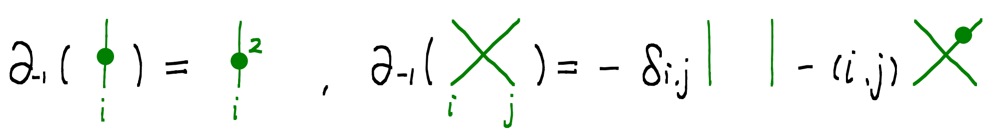
Now if Γ is simply-laced.

Thm/Def. (i). $R(\Gamma)$ admits a multi-parameter family of p-nilpotent derivations

(ii). The quantum Serre relations hold on the Grothendieck group level iff ∂ is given by either



or



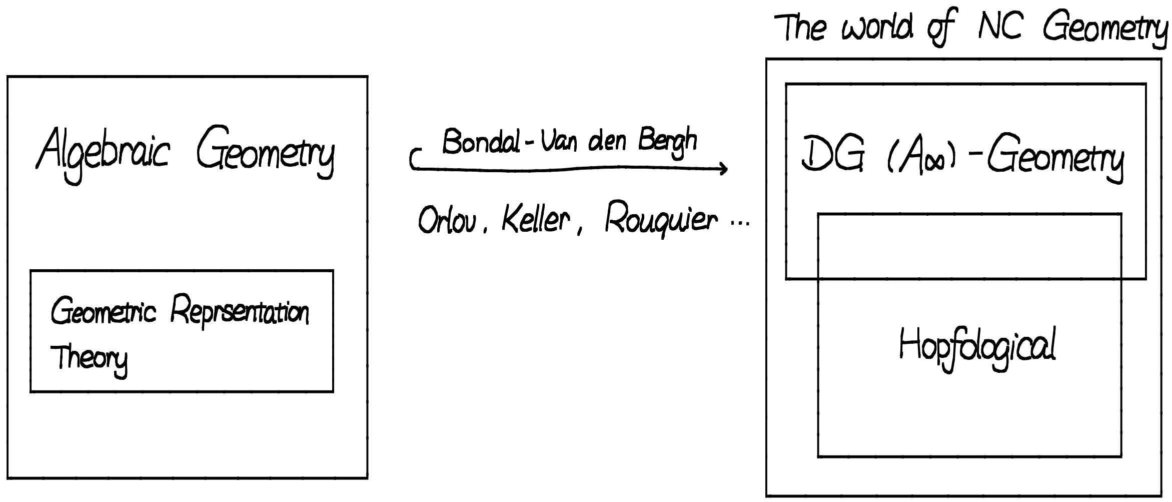
Here $(,)$ denotes the Cartan pairing associated with Γ.

Conjecture: When $\Gamma = A, D, E$, $(R(\Gamma), \partial)$ categorifies an integral form of the corresponding small quantum group.

Many algebraic structures arising from higher representation theory naturally afford p-derivations. Careful choices of parameters allow one to categorify the corresponding structures at prime roots of unity. Such examples include:

(1). Lauda's category $\mathcal{U}$
(2). The thick category $\dot{\mathcal{U}}$
(3). Webster's algebras etc.

Meta question: What's the geometric interpretation?



What's the connection to Bezrukaunikov - Mirkovic - Ruminin's work on localization in char p?