On the center of small quantum groups

- Based on joint work with A. Lachowska

• The small quantum group

The small quantum group $u_q(\mathfrak{g})$ at a root of unity $q^\ell = 1$ is the Hopf algebra defined by Lusztig as the Frobenius kernel

$$0 \longrightarrow u_q(\mathfrak{g}) \longrightarrow U_q(\mathfrak{g}) \xrightarrow{Fr} U(\mathfrak{g}) \otimes \mathbb{Z}[q,q^{-1}] \longrightarrow 0$$

$$E_i^{(m)} \longmapsto \begin{cases} E_i^{(m/p)} & p \mid m \\ 0 & \text{otherwise} \end{cases} \quad \cdots$$

The center of $u_q(\mathfrak{g})$ is a very interesting object to me since they are used to parametrize 3D TQFTs that behave like Witten-Reshetikhin-Turaev theories.

Surprisingly, up until this year, not much about the center $z_q(\mathfrak{g})$ is known beyond the case of $\mathfrak{sl}_2$. But the following beautiful theorem of Andersen-Jantzen-Soergel emphasizes that $z_q(\mathfrak{g})$ is a canonical commutative algebra attached to any Lie algebra $\mathfrak{g}$:

Thm (Andersen-Jantzen-Soergel) The algebra structure of $z_q^0(\mathfrak{g})$ is independent of the order q. The same algebra, base changed to an algebraically closed field $\mathbb{k}$ of char $p>0$, is isomorphic to the center of the restricted universal enveloping algebra over $\mathbb{k}$.

This talk will aim at reporting some recent progress beyond the $\mathfrak{sl}_2$ case. We will mostly focus on the example of $\mathfrak{sl}_3$.

• Geometric interpretation

Recall the celebrated work of Beilinson-Bernstein. Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and G the corresponding (adjoint type) algebraic group. Denote by $\mathcal{O}_0(\mathfrak{g}) :=$ the principal block of the BGG category $\mathcal{O}$ for $\mathfrak{g}$, and

$X := G/B$ the flag variety, $\tilde{\mathcal{N}} := T^*X$ the Springer variety.

$$\begin{tikzcd} D^b(\mathcal{O}_0(\mathfrak{g})) \cong D^b_c(\mathcal{D}_{G/B}\text{-mod}) \arrow[r, dashed, "\text{ass.gr}"] & D^b(\mathrm{Coh}\,\tilde{\mathcal{N}}) \arrow[d, "\mathrm{Ext}^*_{\tilde{\mathcal{N}}}(\mathcal{O}_X,-)"] \\ \circlearrowright \quad C_W := \dfrac{\mathbb{C}[\mathfrak{h}]}{\mathbb{C}[\mathfrak{h}]^W} \cong H^*(X,\mathbb{C}) \cong & \mathrm{Ext}^*_{\tilde{\mathcal{N}}}(\mathcal{O}_X,\mathcal{O}_X)\text{-mod} \end{tikzcd}$$

Soergel showed that C_W is the center of $\mathcal{O}_0(\mathfrak{g})$.

This story has been pushed further by Arkhipov-Bezrukavnikov-Ginzburg for the big quantum group $U_q(\mathfrak{g})$ at a root of unity:

Thm. (Arkhipov-Bezrukavnikov-Ginzburg) There is an equivalence of triangulated categories

$$F: D^b(\mathrm{Coh}_{G\times\mathbb{C}^*}\tilde{\mathcal{N}}) \longrightarrow D^b(\mathrm{Bl}_0(U_q(\mathfrak{g}))) \quad (\text{mixed version})$$

where G acts on $\tilde{\mathcal{N}} = T^*X$ by the natural symplectomorphisms, while $\mathbb{C}^*$ rescales the fibers of $T^*X \longrightarrow X$ by the character $z \longmapsto z^{-2}$.

This thm "geometrizes" the quantum Frobenius map, since $\mathrm{Rep}(G)$ ($\approx U(\mathfrak{g})$-mod) acts on both sides and intertwines F. As a consequence, we have, by forgetting the G-equivariance:

Thm (Bezrukavnikov-Lachowska) There is an equivalence of triangulated categories:

$$F_u: D^b(\mathrm{Coh}_{\mathbb{C}^*}(\tilde{\mathcal{N}})) \longrightarrow D^b(u_q^0\text{-mod}) \quad (\text{mixed version})$$

Cor. (Bezrukavnikov-Lachowska) $Z_q^0(\mathfrak{g}) \cong HH^0_{\mathbb{C}^*}(\tilde{\mathcal{N}}) \cong \bigoplus_{i+j+k=0} H^i(\tilde{\mathcal{N}}, \wedge^j T\tilde{\mathcal{N}})^k$

Our goal is to use $HH^0_{\mathbb{C}^*}(\tilde{\mathcal{N}})$ to help us compute $Z_q^0(\mathfrak{g})$.

• The first observations

From now on, we will use the case of $u_q(\mathfrak{sl}_3)$ to illustrate the general theory. In this case, $\dim X = 3$, $\dim \tilde{N} = 6$. Write the Hochschild cohomology groups into the following table

$H^0(\Lambda^0 T\tilde{N})^0$			
$H^1(\Lambda^1 T\tilde{N})^{-2}$	$H^0(\Lambda^2 T\tilde{N})^{-2}$		
$H^2(\Lambda^2 T\tilde{N})^{-4}$	$H^1(\Lambda^3 T\tilde{N})^{-4}$	$H^0(\Lambda^4 T\tilde{N})^{-4}$	
$H^3(\Lambda^3 T\tilde{N})^{-6}$	$H^2(\Lambda^4 T\tilde{N})^{-6}$	$H^1(\Lambda^5 T\tilde{N})^{-6}$	$H^0(\Lambda^6 T\tilde{N})^{-6}$

Cor. The left-most column and bottom row are isomorphic to $C_W = \dfrac{\mathbb{C}[\mathfrak{h}]}{\mathbb{C}[\mathfrak{h}]^W_+}$ as graded vector spaces.

This is true essentially for degree reasons: $T\tilde{N}$ fits into a s.e.s.

$$0 \longrightarrow \underset{\substack{\text{tangents in}\\ \text{fiber direction}\\ (\deg -2)}}{\pi^*\Omega_X} \longrightarrow T\tilde{N} \longrightarrow \underset{\substack{\text{tangents in}\\ \text{horizontal direction}\\ (\deg 0)}}{\pi^* TX} \longrightarrow 0, \qquad (*)$$

Thus, for instance,

$$\begin{cases} H^1(\Lambda^1 T\tilde{N})^{-2} \cong H^1(\pi_* T\tilde{N})^{-2} \cong H^1(X, \Omega_X) \cong \mathfrak{h}. \\ H^1(\Lambda^5 T\tilde{N})^{-6} \cong H^1(\pi_* \Lambda^5 T\tilde{N})^{-6} \cong H^1(X, \Omega_X^3 \otimes \Lambda^2 T_X) \cong H^1(X, \Omega_X) \cong \mathfrak{h}. \end{cases}$$

Rmk: For $u_q^0(\mathfrak{sl}_2)$, this subspace is the entire center by the work of Kerler.

We next seek to enlarge this subspace (actually subalgeba).

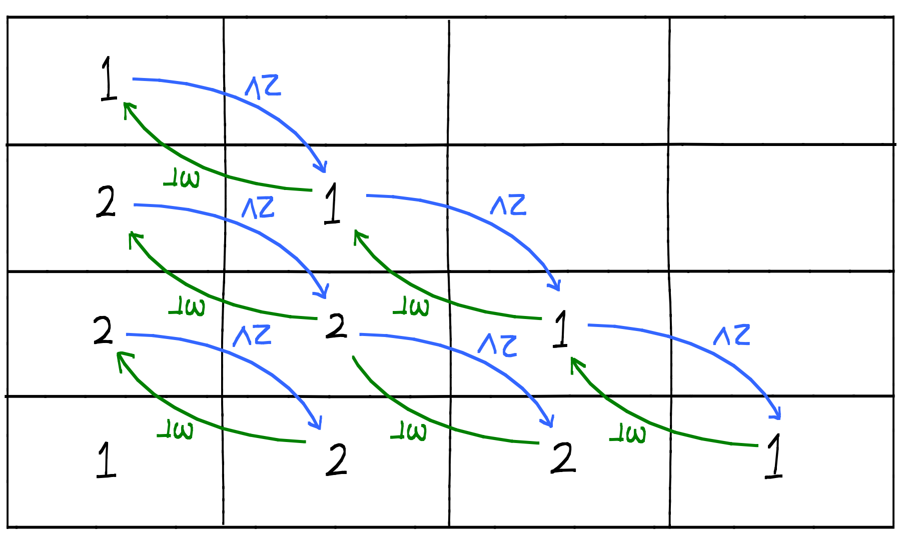
Lemma. If G is simple, then $H^0(\Lambda^2 T\tilde{N})^{-2}$ is 1-dim'l and spanned by the Poisson bivector field τ (dual to the symplectic form ω on $\tilde{N}$)

Recall that τ can locally be written as $\tau = \sum \frac{\partial}{\partial y_i} \wedge \frac{\partial}{\partial x_i}$, with $\{x_i\}$ the local coordinate system on X, and $\{y_i\}$ the fiber coordinates. By the choice of the $\mathbb{C}^*$-action earlier, $\deg \frac{\partial}{\partial y_i} = -2$, $\deg \frac{\partial}{\partial x_i} = 0$.

Thm (Lachowska-Q.) There is an $\mathfrak{sl}_2(\mathbb{C})$-action on $HH^0(\tilde{N})$ induced from the action on sheaves

$$\tau \wedge (-) : \Lambda^\bullet T\tilde{N} \longrightarrow \Lambda^{\bullet+2} T\tilde{N}$$
$$\omega \lrcorner (-) : \Lambda^\bullet T\tilde{N} \longrightarrow \Lambda^{\bullet-2} T\tilde{N}$$

From the $\mathfrak{sl}_2$ action, we immediately conclude that the left-most column and τ generates a big larger subalgebra:



$\boxed{\text{Question}}$ Does this subalgebra agree with $\mathfrak{z}^0_q(\mathfrak{sl}_3)$?

Thm (Lachowska-Q.) The degree-0 Hochschild cohomology ring $HH^0(\tilde{N})$ has the following bigraded character table

1			
2	1		
2	2+1 = 3	1	
1	2	2	1

We'll sketch a proof of this result, or rather, how we computed these dimension numbers if time permits. But we first make some observations.

- A conjecture

Haiman's diagonal coinvariant algebra is the commutative algebra ($\mathfrak{h} = \mathfrak{h}(\mathfrak{gl}_n)$)

$$DC_n := \frac{\mathbb{C}[\mathfrak{h}\times\mathfrak{h}]}{\mathbb{C}[\mathfrak{h}\times\mathfrak{h}]^W_+} = \frac{\mathbb{C}[x_1,\cdots,x_n,y_1,\cdots,y_n]}{\mathbb{C}[x_1,\cdots,x_n,y_1,\cdots,y_n]^{S_n}_+}$$

where any $\sigma\in S_n$ acts diagonally by $\sigma\cdot x_i = x_{\sigma(i)}$, $\sigma\cdot y_i = y_{\sigma(i)}$, $i=1,\cdots,n$. If we put $\deg(x_i)=(1,0)$, $\deg(y_i)=(0,1)$, then DC_n is bigraded. It further carries a natural $(S_n, \mathbb{Z}_2)$-action.

Eg. The bigraded dim table for DC_2 and DC_3:

DC_2:

1	
1	1

DC_3 :

1			
2	1		
2	3	1	
1	2	2	1

Conjecture (Lachowska-Q). Let $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$. There is a natural S_n action on $Z_q^0(\mathfrak{sl}_n) \cong HH^0(\tilde{N})$ which commutes with the $\mathfrak{sl}_2$ action. Furthermore, as S_n-modules,

$$\boxed{H^i(\tilde{N}, \wedge^j T\tilde{N})^{-i-j} \cong (DC_n \otimes sgn)^{\binom{n}{2} - \frac{i+j}{2}, \frac{j-i}{2}}}$$

We have checked the conjecture up to $\mathfrak{sl}_4$, and the boundary spaces for any $\mathfrak{sl}_n(\mathbb{C})$.

- Singular block analogue

The proof of Bezrukavnikov-Lachowska can also be easily generalized to singular blocks of $u_q(\mathfrak{g})$. Roughly speaking, if $\lambda \in \mathfrak{h}^*$ is a singular weight, $W_P \subseteq W$ is the stablizer subgroup, $P \subseteq G$ the corresponding parabolic, then

$$Z_q^\lambda(\mathfrak{g}) = HH^0(\tilde{N}_P) \cong \bigoplus_{i+j+k} H^i(\tilde{N}_P, \wedge^j T\tilde{N}_P)^{-i-j}$$

where $\tilde{N}_P = T^*(G/P)$ with a similar $G \times \mathbb{C}^*$-action.

Eg. For $\mathfrak{sl}_3$, the only nontrivial singular blocks correspond to $S_2 \subseteq S_3$, so that $G/P \cong \mathbb{P}^2$. In this case, we have

$HH^\circ(T^*\mathbb{P}^2)$:

1		
1	1	
1	1	1

The unique Steinberg block, corresponding to $S_3 \subseteq S_3$, has 1-dim'l center. Thus we have

Cor. $q^\ell = 1$ and ℓ odd. Then

$$\dim Z_q(\mathfrak{U}_3) = 1 + (\ell-1)\cdot 6 + \frac{(\ell-1)(\ell-2)}{6}\cdot 16$$

At $\ell = 5$, Cor $\Rightarrow$ dim $= 1 + 24 + 32 = 57$. Around 2008, Steve Jackson wrote a computer algorithm confirming the correctness of this number. He is also developing his own approach finding the Hochschild cohomology groups of $\tilde{N}$, which he calls "local Čech complexes."

- Sketch of proof

If we want to compute coherent cohomology on $\tilde{N}$, a very useful tool to use is the Borel-Weil-Bott theorem. It requires us to have a good understanding of $\pi_*(\Lambda^\bullet T\tilde{N})$ on X as G-equivariant sheaves, where $\pi: \tilde{N} = T^*X \longrightarrow X = G/B$ is the natural G-equivariant projection map.

We look at the sequence $(*)$ earlier, which can be identified with

$$0 \longrightarrow \pi^*(G\times_B \mathfrak{n}) \longrightarrow T\tilde{N} \longrightarrow \pi^*(G\times_B \mathfrak{u}) \longrightarrow 0$$

where $\mathfrak{n} = [\mathfrak{b}, \mathfrak{b}]$ is the nilpotent radical of $\mathfrak{b} = Lie(B)$, and $\mathfrak{u} := \mathfrak{n}^*$. On X, we also have the natural

$$0 \longrightarrow G\times_B \mathfrak{b} \longrightarrow G\times_B \mathfrak{g} \cong \mathfrak{g}\times X \longrightarrow G\times_B \mathfrak{u} \longrightarrow 0.$$

which pulls back to $\tilde{N}$ to be:

$$0 \longrightarrow \pi^*(G\times_B \mathfrak{b}) \longrightarrow \pi^*(G\times_B \mathfrak{g}) \cong \mathfrak{g}\times\tilde{N} \longrightarrow \pi^*(G\times_B \mathfrak{u}) \longrightarrow 0 \qquad (**)$$

Notice that $(**)$ extends to a map of s.e.s of vector bundles

$$\begin{tikzcd}
0 \arrow[r] & \pi^*(G\times_B \mathfrak{b}) \arrow[r] \arrow[d, dashed] & \mathfrak{g}\times\tilde{N} \arrow[r] \arrow[d, "\text{action}"] & \pi^*(G\times_B \mathfrak{u}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \pi^*(G\times_B \mathfrak{n}) \arrow[r] & T\tilde{N} \arrow[r] & \pi^*(G\times_B \mathfrak{u}) \arrow[r] & 0
\end{tikzcd} \qquad (***)$$

since π is G-, and thus $\mathfrak{g}$-, equivariant:

$$\begin{tikzcd}
G\times\tilde{N} \arrow[r, "A_1"] \arrow[d, "Id\times\pi"'] & \tilde{N} \arrow[d, "\pi"] \\
G\times X \arrow[r, "A_1"] & X
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
\mathfrak{g}\times\tilde{N} \arrow[r, "DA_1"] \arrow[d] & T\tilde{N} \arrow[d, "D\pi"] \\
\mathfrak{g}\times X \arrow[r, "DA_2"] & TX
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
\mathfrak{g}\times\tilde{N} \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \pi^*TX \arrow[d, equal] \\
T\tilde{N} \arrow[r] & \pi^*TX
\end{tikzcd}$$

Notice that $\mathfrak{b}$ acts on the fiber $\mathfrak{n}$ by the adjoint action:

$$ad: \mathfrak{b} \longrightarrow End(\mathfrak{n}) \cong \mathfrak{u}\otimes\mathfrak{n}$$
$$x \longmapsto ad\,x = \sum y_i \frac{\partial}{\partial x_i}$$

Pushing $(***)$ onto X we get

$$\begin{tikzcd}
0 \arrow[r] & G\times_B(S^{\bullet}(\mathfrak{u})\otimes\mathfrak{b}) \arrow[r, "\iota"] \arrow[d, "ad"] \arrow[dr, phantom, "\#"] & G\times_B(S^{\bullet}(\mathfrak{u})\otimes\mathfrak{g}) \arrow[r] \arrow[d] & G\times_B(S^{\bullet}(\mathfrak{u})\otimes\mathfrak{u}) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & G\times_B(S^{\bullet}(\mathfrak{u})\otimes\mathfrak{n}) \arrow[r] & \pi_*(T\tilde{N}) \arrow[r] & G\times_B(S^{\bullet}(\mathfrak{u})\otimes\mathfrak{u}) \arrow[r] & 0
\end{tikzcd}$$

The $\#$ must be a Cartesian square, and thus

Thm. (Lachowska–Q.) The bundle $\pi_*T\tilde{N}$ has the equivariant structure $G\times_B V$ where V is the $B\#S^{\bullet}(\mathfrak{u})$-module

$$V := \frac{S^{\bullet}(\mathfrak{u})\otimes\mathfrak{g}\oplus S^{\bullet}(\mathfrak{u})\otimes\mathfrak{n}}{\Delta(S^{\bullet}(\mathfrak{u})\otimes\mathfrak{b})}.$$

Here Δ is the composition map

$$S^{\bullet}(\mathfrak{u})\otimes\mathfrak{b} \xrightarrow{(\iota,\, ad)} S^{\bullet}(\mathfrak{u})\otimes\mathfrak{g}\oplus S^{\bullet}(\mathfrak{u})\otimes\mathfrak{u}\otimes\mathfrak{n} \longrightarrow S^{\bullet}(\mathfrak{u})\otimes\mathfrak{g}\oplus S^{\bullet+1}(\mathfrak{u})\otimes\mathfrak{n}$$