Categorified Quantum $\mathfrak{sl}(2)$ at Prime Roots of Unity

- **Why do we want to categorify $\mathcal{U}_q(\mathfrak{sl}_2)$?**
  - Reshetikhin-Turaev-Witten:

    $U_q(\mathfrak{sl}_2)$ is the quantized gauge group of 3d Chern-Simons theory.

  - Crane-Frenkel:

    Categorify 3d Chern-Simons to a 4d-TQFT.

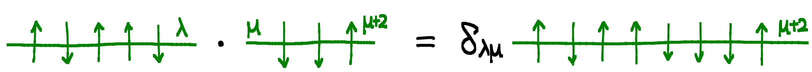
    $\mathcal{U}_q(\mathfrak{sl}_2)$: quantized 2-gauge group?

- **Quantum $\mathfrak{sl}(2)$ at roots of unity.**

We are interested in the idempotented version of $U_q(\mathfrak{sl}_2)$. It is generated over $\mathbb{Z}[q,q^{-1}]$ by pictures of the form

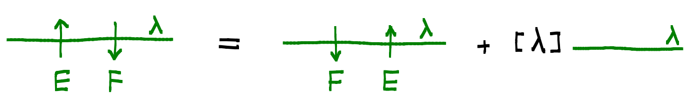
$(\lambda \in \mathbb{Z})$

with the algebra structure

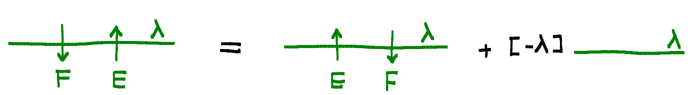


(etc)

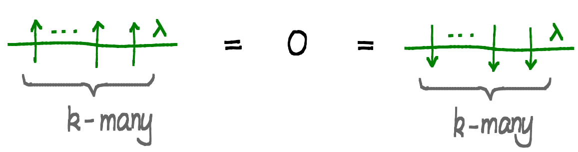
Modulo relations (at a $2k$-th root of unity, k odd)



$(\lambda \geq 0)$



$(\lambda \leq 0)$



(Nilpotency relation)

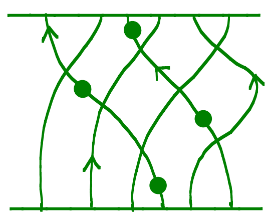
• Categorification of $U_q(\mathfrak{sl}_2)$

Below we present Lauda's diagrammatic calculus for $U_q(\mathfrak{sl}_2)$ at a generic q-value.
The rough idea is that:

- Pictures = Isomorphism class / symbol of some modules
- Sum of pictures = symbol of direct sum of modules
- Equalities of pictures = isomorphisms of modules.

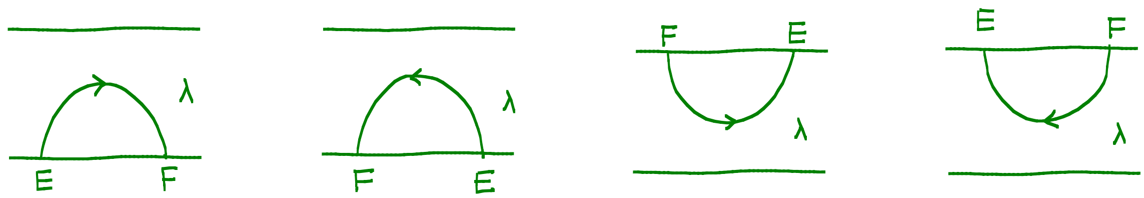
In general, isomorphisms are rare between modules. Instead, study homomorphisms between them. Intuitively, homomorphisms = evolution of pictures, which is not necessarily reversible.

- Maps just among E's (or F's) (Khovanov - Lauda - Rouquier)



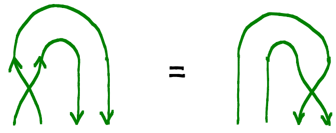
(Nil-Hecke algebra)

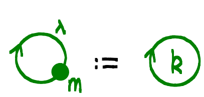
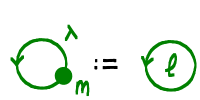
- To categorically Drinfeld-double E's, Lauda introduces cups and caps



Together with the nilHecke algebra generators, cups and caps satisfy certain relations

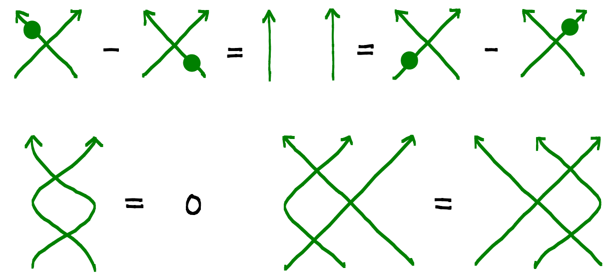
(i) Biadjointness E.g.



(ii) Bubble positivity (degrees of   must be ≥ 0)

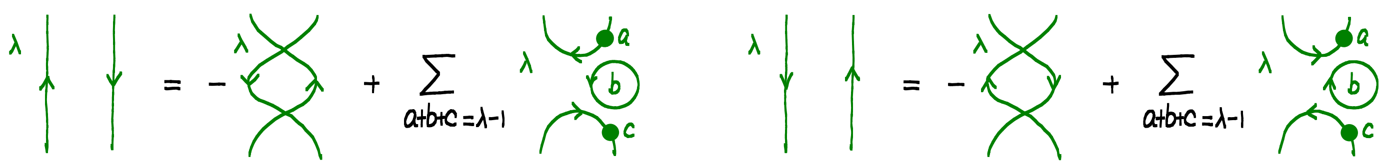
$k = \frac{1}{2}(m+1-\lambda) \geq 0$ $\qquad$ $\ell = \frac{1}{2}(m+1+\lambda) \geq 0$

(iii). NilHecke relations



(iv) Reduction to bubbles

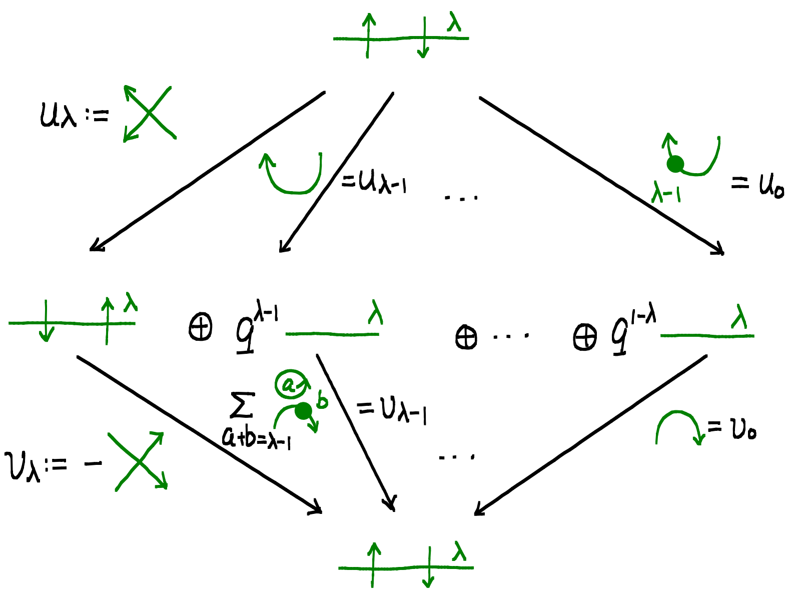
(v). Identity decomposition



Thm. (Lauda) This graphical calculus is non-degenerate and categorifies $\dot{U}_q(\mathfrak{sl}_2)$ at a generic q-value.

Rmk: Lauda's calculus is a 2-dim'l idempotented algebra, i.e. it has two compatible multiplication structures (vertical and horizontal). Such idempotented algebras are also known as a 2-category)

To see the plausibility of this categorification, we consider how $EF\mathbb{1}_\lambda$ can "evolve" into $FE\mathbb{1}_\lambda \oplus \mathbb{1}_\lambda^{\oplus[\lambda]}$



These elements $\{u_\lambda\}$, $\{v_\lambda\}$ satisfy

$$\begin{cases} u_i v_i u_i = u_i \\ v_i u_i v_i = v_i \\ v_i u_j = 0 \quad (i \neq j) \ , \end{cases}$$

which follows from the identity decomposation relation. Consequently $\{u_i v_i \mid i = 0, \cdots, \lambda\}$ form an orthogonal set of idempotents in $\mathrm{End}_{\dot{\mathcal{U}}}(\mathcal{E}\mathcal{F}\mathbb{1}_\lambda)$

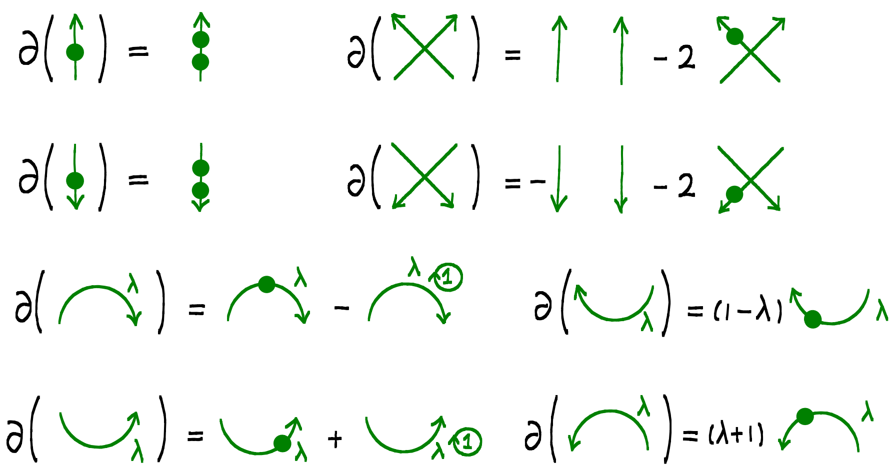
(Factorization of idempotents)

• Enhancing $\dot{\mathcal{U}}$ with a p-differential

As we have heard from Mikhail's talk, if A is a p-DG algebra, then the derived category of p-DG modules over A is a module-category over the homotopy category of p-complexes.

$$\begin{tikzcd}
\Bbbk[\partial]/(\partial^p)\text{-}\underline{gmod} \arrow[d, Rightarrow, "K_0"] & \times & \mathcal{D}(A,\partial) \arrow[r, "\otimes"] \arrow[d, Rightarrow, "K_0"] & \mathcal{D}(A,\partial) \arrow[d, Rightarrow, "K_0"] \\
\mathcal{O}_p & \times & K_0(A,\partial) \arrow[r, "\times"] & K_0(A,\partial)
\end{tikzcd}$$

Def. Let $(\mathcal{U}, \partial)$ be Lauda's 2-dimensional algebra equipped with the differential ∂-action on generators given by



Lemma. The above ∂ preserves all relations of $\mathcal{U}$, and it is p-nilpotent over a field of characteristic $p>0$.

Thm. (Elias-Q.) The derived module category $\mathcal{D}^b(\mathcal{U}, \partial)$ is Karoubian, and it categorifies $\dot{u}_{\mathbb{q}}(\mathfrak{sl}_2)$ at a p-th primitive root of unity.

$$K_0(\mathcal{U}, \partial) \cong \dot{u}_{\mathbb{q}}(\mathfrak{sl}_2)$$

• Decomposition v.s. filtration.

In Lauda's abelian categorification, the relations in $\dot{U}_q(\mathfrak{sl}_2)$ are usually realized as different ways of decomposing projective $\mathcal{U}$-modules.

In the realm of triangulated categories, direct sum decompositions are very rare. Instead, a short exact sequence of p-DG $\mathcal{U}$-modules gives rise to a distinguished triangle in $\mathcal{D}(\mathcal{U}, \partial)$.

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0 \quad \text{in} \quad (\mathcal{U}, \partial)\text{-mod}$$

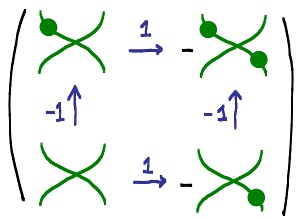
$$\rightsquigarrow$$

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1] \quad \text{in} \quad \mathcal{D}(\mathcal{U}, \partial) \Longrightarrow [B] = [A] + [C] \quad \in \quad K_0(\mathcal{U}, \partial)$$

More generally, a filtered p-DG module $(M, F^\bullet)$ presents M as a convolution (Postnikov tower) of $\mathrm{gr} F^\bullet$.

Example In the nilHecke algebra NH_2:

$$NH_2 \cong Sym_2 \cdot$$



$\Rightarrow$ $0 \longrightarrow P_2\{1\} \longrightarrow NH_2 \longrightarrow P_2\{1\} \longrightarrow 0$ is a s.e.s. of $(\mathcal{U}, \partial)$-modules.

$\Rightarrow$ In $K_0(\mathcal{U}, \partial)$, $E^2 = [(NH_2, \partial)] = \underline{q}[P_2] + \underline{q}^{-1}[P_2] = (q + q^{-1}) E^{(2)}$

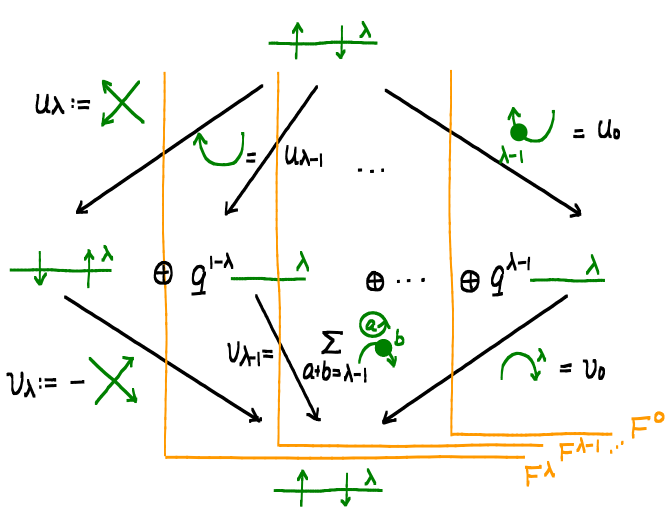
Prop. Let $\{(u_i, v_i) \mid i \in I\}$ be factorization of idempotents in a p-DG algebra R. If there is a total ordering on I such that

$$\begin{cases} v_i\, \partial(u_i) = 0 \\ u_i\, \partial(v_i) \equiv 0 \quad \text{(modulo lower order terms)} \end{cases}$$

Then if $\varepsilon = \sum_{i \in I} u_i v_i$, then the p-DG module $R\varepsilon$ admits a filtration $F^\bullet$ whose subquotients are isomorphic to $R v_i u_i$'s

Cor. (Fantastic!) In the situation of the Prop. $[R\varepsilon] = \sum_{i \in I} [R v_i u_i]$.

Cor. Under the differential defined earlier on $\mathcal{U}$, there is a filtration on $\mathcal{E}\mathcal{F}1_\lambda$



- **Uniqueness: a small surprise!**

Lauda's factorization of idempotents, in general, is not unique.

However, in the presence of a diagrammatically local differential (not necessarily the differential we defined here, but any ∂ compatible with the local relations of $\mathcal{U}$), we have, up to conjugation by diagrammatic automorphisms

- The differential we defined here is the unique differential such that the modules $\mathcal{E}\mathcal{F}\mathbb{1}_\lambda$ $(\lambda \geq 0)$ admit filtrations whose subquotients are isomorphic to $\mathcal{F}\mathcal{E}\mathbb{1}_\lambda$, $\mathbb{1}_\lambda\{1-\lambda\}, \cdots$ $\mathbb{1}_\lambda\{\lambda-1\}$.

- Lauda's factorization of idempotents is the unique choice that is compatible with the differential. (Fantastic Filtration)