Categorification of small $sl(2)$

In 1994, Crane and Frenkel published their seminal paper "Four dimensional topological quantum field theory, Hopf categories, and the canonical bases". There they proposed a program called "categorification", hoping to lift the combinatorial 3D TQFT constructed by Kuperberg to a 4D TQFT:

$$\begin{array}{ccc} \text{Donaldson-Floer} & \xleftrightarrow{\ ?\ } & \boxed{\text{A 4D TQFT ?}} \\ \Big\uparrow \text{Categorification} & & \Big\uparrow \\ \begin{array}{l}\text{Kuperberg: a 3D TQFT} \\ \text{based on } u_q^+(sl_2) \quad (q^N=1) \\ \text{or any finite dim'l balanced} \\ \text{Hopf algebra } H.\end{array} & \sim & \begin{array}{l}\text{WRT-3D TQFT} \\ \text{associated with } D(H)\end{array} \end{array}$$

Homological Algebra

For simplicity assume we are working over a ground field $\Bbbk$. The usual homological algebra has the following key features.

(0). Chain complexes and their cohomology groups

$$K^\bullet = (\cdots \xrightarrow{d} K^{i-1} \xrightarrow{d} K^i \xrightarrow{d} K^{i+1} \xrightarrow{d} \cdots) \quad \text{s.t. } d^2=0$$

$$H^\bullet(K^\bullet) = \operatorname{Ker} d / \operatorname{Im} d$$

(1). Direct sums of chain complexes

(2). Tensor products of chain complexes

$$(K\otimes L)^\bullet := \bigoplus_{k+l=i} K^k \otimes L^l, \quad d(x\otimes y) = d(x)\otimes y + (-1)^{|x|} x\otimes d(y)$$

$$K^\bullet \otimes \Bbbk \cong \Bbbk \otimes K^\bullet \cong K^\bullet$$

(3). Inner homs $\mathrm{HOM}^\bullet(K^\bullet, L^\bullet)$

$$\mathrm{HOM}^i(K^\bullet, L^\bullet) := \{ f: K^\bullet \to L^\bullet \mid f(K^k) \subseteq L^{k+i} \}$$

$$d(f) = d_L \circ f - (-1)^{|f|} f \circ d_K$$

with the property

$$HOM^\bullet(K^\bullet \otimes L^\bullet, M^\bullet) \cong HOM^\bullet(K^\bullet, HOM^\bullet(L^\bullet, M^\bullet))$$

$$H^0(HOM^\bullet(K^\bullet, L^\bullet)) = Hom_{C(\Bbbk)}(K^\bullet, L^\bullet)$$

(4). Triangular structures: homological grading shifts / the cone construction / distinguished triangles coming from s.e.s. / compatibility axioms etc.

Homological algebra is an important tool in categorification since it gives a systematic lift of operations in $\mathbb{Z}$:

$$\begin{array}{ccc}
D^b(\Bbbk\text{-vec}) & \xrightarrow{\quad \chi \quad} & \mathbb{Z} \\
K^\bullet & \mapsto & \sum (-1)^i \dim_i K^i \\
\oplus, \otimes & \mapsto & +, \times \\
\text{tensor unit} & \mapsto & 1 \\
[1] & \mapsto & \text{multiplication by } (-1).
\end{array}$$

Rmk: If we take graded vector spaces, we get a categorification of $\mathbb{Z}[q, q^{-1}]$:

$$K_0(D^b(\Bbbk\text{-gvec})) \cong \mathbb{Z}[q, q^{-1}]$$

The grading shift $\{1\}$ becomes multiplication by q.

Observation: Properties (2), (3), (4) are reminiscent of some familiar constructions in representation theory: Take a group G, $H := \Bbbk G$ is a Hopf algebra and the category of H-mod has

(1') $V \oplus W \in H\text{-mod}$

(2') $V \otimes W \in H\text{-mod}$ $\quad h \cdot (v \otimes w) := h_{(1)} v \otimes h_{(2)} w$

(3') $HOM(V, W) \in H\text{-mod}$ $\quad h \cdot f(v) := h_{(2)} f(S^{-1}(h_{(1)}))$, right adjoint to $\otimes$.

The usual homological algebra can thus be regard as for the graded Hopf super algebra $H = \Bbbk[d]/(d^2)$

Question: Are there analogues of the other features displayed by the usual homological algebra? For instance, what is cohomology?

Any $K^\bullet$ decomposes "uniquely" into direct sums of

$$\oplus(0 \to \underbrace{\mathbb{k}}_{\text{hom. deg } i} \to 0) \oplus \oplus(0 \to \underbrace{\mathbb{k}}_{\text{hom deg } j} \to \mathbb{k} \to 0)$$

Taking cohomology just kills the second factor. Note that the second factor consists of projective $\mathbb{k}[d]/(d^2)$ modules.

Less obvious: $(0 \to \mathbb{k} \to \mathbb{k} \to 0)$ is also injective. In fact, $\mathbb{k}[d]/(d^2)$ is a Frobenius algebra $(\cong H^*(S^1, \mathbb{k}))$.

Thm (Radford - Larson, Sweedler) Any finite dim'l Hopf (super) algebra is Frobenius. In particular, the class of projective modules coincide with the class of injective modules.

What's the systematic way of killing projectives/injectives?

The stable category H-$\underline{\text{mod}}$

Intuitively, H-$\underline{\text{mod}}$ is the categorical quotient of H-mod by the class of projective/injective objects.

Def. H-$\underline{\text{mod}}$ consists of the same objects as H-mod, while the morphism space between two objects K, L are given by

$$\mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(K,L) := \mathrm{Hom}_{H\text{-mod}}(K,L) \Big/ \left(\begin{array}{l}\text{morphisms that factor}\\ \text{through projectives}\end{array}\right)$$

The notion of stable categories makes sense for any Frobenius algebra, not necessarily those coming as finite dim'l Hopf algebras.

Thm. (Heller) If H is a Frobenius algebra, then H-<u>mod</u> is triangulated.

In general, the morphism space between objects in an arbitrary stable category is hard to compute. But for H-<u>mod</u>, the morphism spaces can be computed explicitly. To do this, we need the notion of integrals of Hopf algebras.

Def. Let H be a Hopf algebra. An element $\Lambda \in H$ is called a left integral of H if $\forall\, h \in H$,

$$h\Lambda = \epsilon(h)\Lambda.$$

Thm (Radford-Larson, Sweedler) Any finite dim'l H has a non-zero integral Λ, unique up to a non-zero constant.

Example. (1). $\Bbbk G$ $\quad \Lambda = \sum_{g \in G} g$ is a left integral

(2). $\Bbbk[d]/(d^2)$, $\Lambda = d$ is a left integral.

(3). $u_{\mathbb{N}}^{+}(\mathfrak{sl}_2)$, $\Lambda = (\sum_{i=0}^{N-1} K^i)\, d^{N-1}$ is a left integral

(4) $\Bbbk[\partial]/(\partial^p)$ (char $\Bbbk = p > 0$), $\Lambda = \partial^{p-1}$ is a left integral.

Prop. Let H be a finite dim'l Hopf algebra, and K, L be H-modules. Then

$$\begin{aligned} \mathrm{Hom}_{H\text{-}\underline{mod}}(K, L) &\cong \mathrm{Hom}_{H\text{-}mod}(K, L) / \Lambda \cdot \mathrm{HOM}(K, L) \\ &\cong \mathrm{HOM}(K, L)^{H} / \Lambda \cdot \mathrm{HOM}(K, L) \end{aligned}$$

Example. Note that H acts on $\mathrm{HOM}(K, L)$ by $h \cdot (f)(k) := h_{(2)} f(S^{-1}(h_{(1)})k)$

(2) $\Lambda \cdot (f) = df + (-1)^{|f|} f d$

(3). f homogeneous of K-degree $1-N$, $\Lambda \cdot (f) = \sum_{i=0}^{N-1} d^{N-1-i} f d^{i}$

(4). $\Lambda \cdot (f) = \sum_{i=0}^{p-1} \partial^{p-1-i} f \partial^{i}$

Relation to categorification

Def. $H = \Bbbk[\partial]/(\partial^p)$ $\deg \partial = 1$. We call the category H-gmod the category of p-complexes while H-$\underline{\mathrm{gmod}}$ the homotopy category of p-complexes.

Why do we care about p-complexes ?

This was an observation of Bernstein-Khovanov : If $H = \Bbbk[\partial]/(\partial^p)$, where $\deg(\partial) = 1$, then both H-gmod and H-$\underline{\mathrm{gmod}}$ are symmetric monoidal. Furthermore,

$$K_0(H\text{-gmod}) \cong \mathbb{Z}[q, q^{-1}]$$

$$K_0(H\text{-}\underline{\mathrm{gmod}}) \cong \mathbb{Z}[q]/(1+q+\cdots+q^{p-1}) := \mathcal{O}_p .$$

To utilize this "categorical cyclotomic integers" H-$\underline{\mathrm{gmod}}$, we need to find interesting "algebra" objects in this category. Then the Grothendieck group of this algebra object will naturally be $\mathcal{O}_p$-modules. Recall that an algebra object in the category of chain complexes is given by a differential graded algebra. This motivates the following.

Def. A p-DG algebra A over a field of char $p > 0$ is a graded algebra together with a degree-one endomorphism ∂ s.t. $\forall\, a, b \in A$.

$$\partial(ab) = \partial(a)b + a\,\partial(b)$$

$$\partial^p(a) = 0$$

More generally, one has the notion of an H-module algebra, which gives rise to algebra objects in H-$\underline{\mathrm{mod}}$. We refer to the study of homological properties of such algebra objects in H-$\underline{\mathrm{mod}}$ and their module categories as "hopfological algebra".

In analogy with the usual DG case, one has, for a p-DG algebra

$$(A,\partial)\text{-mod} \longrightarrow \mathcal{C}(A,\partial) \longrightarrow \mathcal{D}(A,\partial)$$

Thm. (Khovanov, Qi). The homotopy and derived categories of p-DG algebras are module categories over H-$\underline{\mathrm{gmod}}$. Under taking Grothendieck groups (in some appropriate sense), $K_0(\mathcal{D}(A,\partial))$ has the structure of an $\mathcal{O}_p$-module.

$$\begin{tikzcd} H\text{-}\underline{\mathrm{gmod}} \times \mathcal{D}(A,\partial) \arrow[r, "\otimes"] \arrow[d, Rightarrow, "K_0"] & \mathcal{D}(A,\partial) \arrow[d, Rightarrow, "K_0"] \\ \mathcal{O}_p \times K_0(A,\partial) \arrow[r, "\times"] & K_0(A,\partial) \end{tikzcd}$$

Categorification of small $\mathfrak{sl}(2)$

The small quantum group $U_q^+(\mathfrak{sl}_2)$, where $q=\zeta_N$ is a primitive N-th root of unity, is defined by Lusztig as the twisted bialgebra $\mathcal{O}_N[E]/(E^N)$, equipped with:

$$\Delta: \mathcal{U}^+ \longrightarrow \mathcal{U}^+\otimes\mathcal{U}^+ \;:\; E \mapsto E\otimes 1 + 1\otimes E$$

$$m: \mathcal{U}^+\otimes\mathcal{U}^+ \longrightarrow \mathcal{U}^+ \;:\; E^r\otimes E^s \mapsto E^{r+s}$$

Here twisted means that $\mathcal{U}^+\otimes\mathcal{U}^+$ has the product structure:

$$(E^{r_1}\otimes E^{s_1})\cdot(E^{r_2}\otimes E^{s_2}) = \zeta_N^{s_1r_2}\, E^{r_1+r_2}\otimes E^{s_1+s_2}.$$

We briefly review how the categorification of $U_q^+(\mathfrak{sl}_2)$ is done at generic values of q, as done by Khovanov-Lauda.

By categorification of the algebra U_q^+ we mean that to find a monoidal category $\mathcal{U}^+$ whose Grothendieck group is naturally isomorphic to U_q^+. In particular

$$\begin{array}{ccc} \mathcal{U}^+ & \xrightarrow{K_0} & U_q^+ \\ \mathcal{E},\ \mathcal{E}^i,\ \mathcal{E}^{(i)} & \longmapsto & E,\ E^i,\ E^{(i)} \end{array}$$

The hard part of the story is to pin down the morphism spaces between the objects. We give the answer due to Khovanov-Lauda.

The nilHecke algebra on n-strands $NH_n := \mathrm{End}_{Sym_n}(P_n)$, which has the following Coxeter presentation. It's generated by $x_1, \cdots, x_n, \delta_1, \cdots, \delta_{n-1}$, subject to relations that the x_i's commute and

$$x_i\delta_i - \delta_i x_{i+1} = 1 = \delta_i x_i - x_{i+1}\delta_i$$

$$\delta_i x_j = x_j\delta_i, \ \delta_i\delta_j = \delta_j\delta_i \quad \text{if } |i-j|>1$$

$$\delta_i^2 = 0, \quad \delta_i\delta_{i+1}\delta_i = \delta_{i+1}\delta_i\delta_{i+1}.$$

By definition $NH_n \cong \mathrm{Mat}(n!, Sym_n)$ so that $K_0(NH_n) \cong \mathbb{Z}[q,q^{-1}]$. There is an inclusion of algebras

$$\iota_{n,m} : NH_n \otimes NH_m \hookrightarrow NH_{n+m}$$

which gives rise to induction and restriction functors on module categories.

Thm. (Khovanov-Lauda)

$$\begin{array}{ccc} \bigoplus_{n\geq 0} NH_n\text{-mod} & \xrightarrow{K_0} & U_q^+(\mathfrak{sl}_2) \\ P_n & \longmapsto & E^{(n)} \\ NH_n & \longmapsto & E^n = [n]!\,E^{(n)} \\ \mathrm{Ind}/\mathrm{Res} & \longmapsto & \text{multiplication/comultiplication} \end{array}$$

To utilize this beautiful 2-quantum group in our current work, the most naive guess might be to put a p-DG structure on NH and study the associated p-DG categories. The miracle is that this naive guess works!

Define on NH_n the following differential

$$\partial(x_i) := x_i^2 \qquad (i=1,\cdots,n)$$
$$\partial(\delta_i) := -x_i\delta_i - \delta_i x_{i+1} \quad (i=1,\cdots,n-1)$$

It's easy to check that $\partial^p \equiv 0$.

Thm. (Khovanov - Qi)

$$\begin{array}{ccc}
\bigoplus_{n\geq 0} \mathcal{D}(NH_n,\partial) & \xrightarrow{K_0} & U^+_{\mathbb{O}_p}(\mathfrak{sl}_2) \\
(P_n,\partial) & \longmapsto & E^{(n)} \\
(NH_n,\partial) & \longmapsto & E^n = [n]!\, E^{(n)} \\
\text{derived induction/ restriction} & \longmapsto & \text{multiplication/ comultiplication}
\end{array}$$

Examples.

(1). $n=0$, $(NH_0,\partial)\cong(\Bbbk,\partial)$, so that $\mathcal{D}(NH_0,\partial)\cong H\text{-}\underline{gmod}$. Therefore $K_0(NH_0,\partial_0)\cong\mathbb{O}_p$, as a gift from Bernstein.

(2). $n=1$, $NH_1\cong\Bbbk[x]$, $\partial(x)=x^2$ so that NH_1 decomposes

$$(\Bbbk[x],\partial) \cong \Big(\underset{1}{\bullet} \qquad \underset{x}{\circ} \xrightarrow{\partial} \underset{x^2}{\circ} \xrightarrow{\partial} \cdots \; \underset{x^p}{\circ} \qquad \underset{x^{p+1}}{\circ} \xrightarrow{\partial} \underset{x^{p+2}}{\circ} \xrightarrow{\partial} \cdots \; \underset{x^{2p}}{\circ} \quad \cdots \Big)$$

Therefore the inclusion of the ground field $\Bbbk \hookrightarrow NH_1$ is a qis, which induces an equivalence of derived derived categories.

$$\Rightarrow \quad K_0(NH_1,\partial)\cong K_0(\Bbbk,\partial)\cong\mathbb{O}_p.$$

Thm. (Qi). If $A\xrightarrow{\varphi}B$ is a map of p-DG algebras, then

$$\varphi_*,\varphi^*: \mathcal{D}(B,\partial)\longrightarrow\mathcal{D}(A,\partial)$$

are equivalences of triangulated categories that are quasi-inverses of each other.

(3). $n=p$. (NH_p,∂) is acyclic. The above thm $\Longrightarrow$ $0\hookrightarrow NH_p$ is a quasi-isomorphism $\Longrightarrow$ $\mathcal{D}(NH_p,\partial)\cong 0$ $\Longrightarrow$ $E^p=[(NH_p,\partial)]=0$.