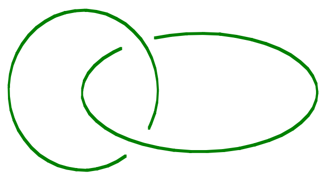
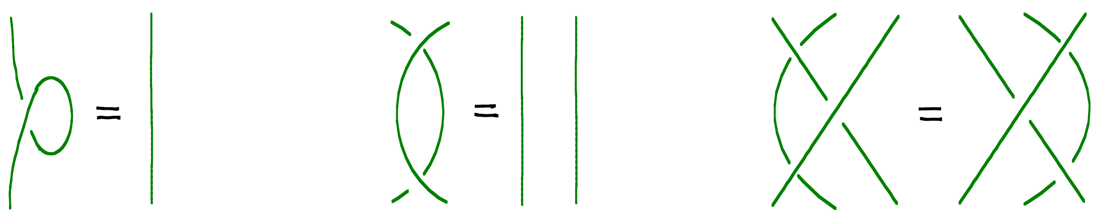
§3. Categorification of the Jones Polynomial

A link in $\mathbb{R}^3$ or S^3 is a finite collection of smoothly embedded circles. Via a deep result stating that any smooth structure of a smooth manifold has a unique piecewise linear structure, links admit very combinatorial descriptions by their projections to $\mathbb{R}^2$:



Two link projections onto $\mathbb{R}^2$ give isotopic links iff they are related via a finite number of isotopies and Reidemeister moves RI - RIII:



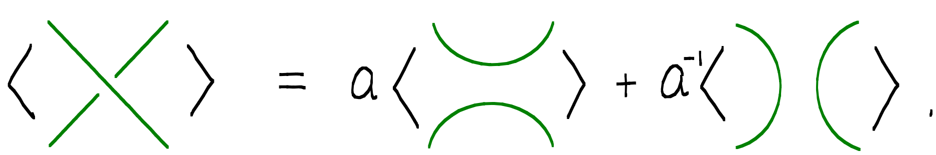
RI   RII   RIII

The Kauffman bracket

The Kauffman bracket is an assignment from $\mathbb{R}^2$-link projections to the ring of Laurent polynomials $\mathbb{Z}[a, a^{-1}]$:

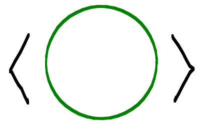
$$\langle \quad \rangle : \text{Link diagrams} \longrightarrow \mathbb{Z}[a, a^{-1}],$$

subject to the local relation:



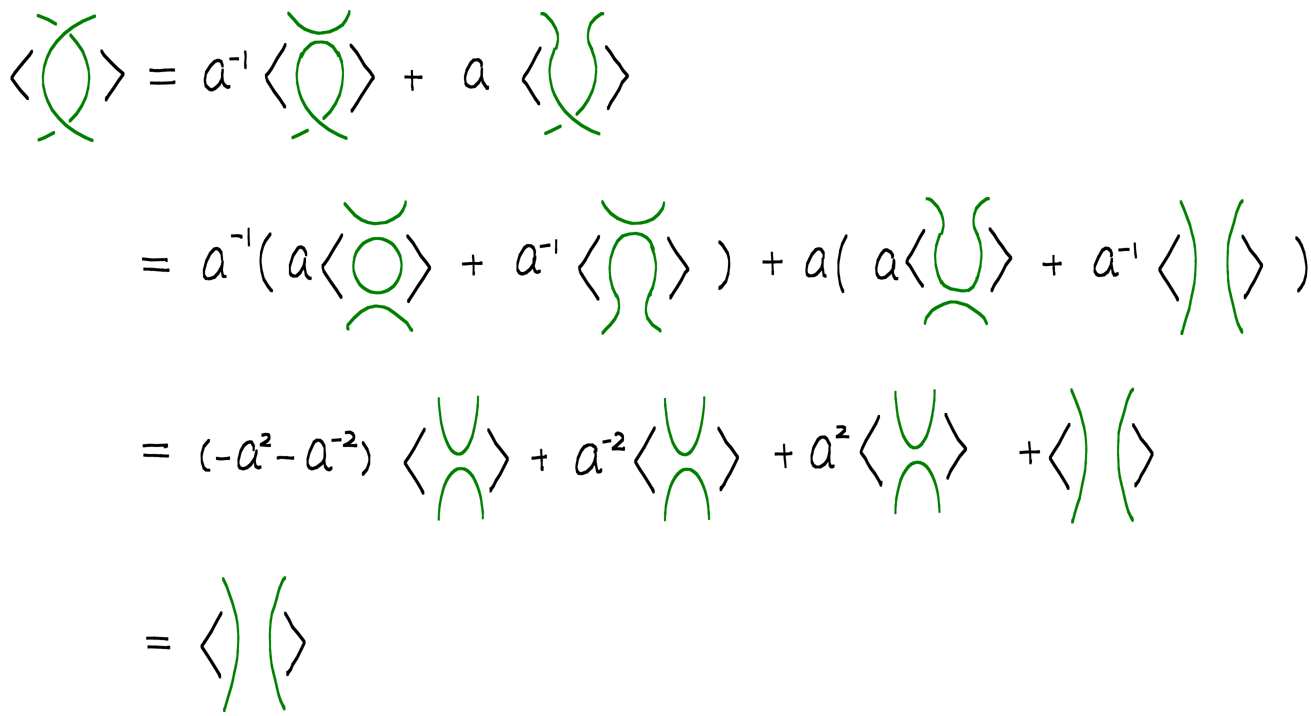
(*)'

and the normalization condition that

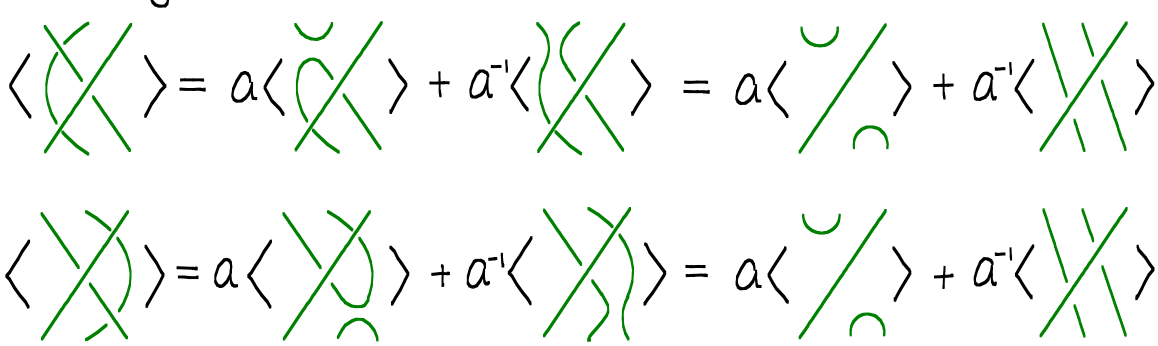
 $= -a^2 - a^{-2}$

This is almost a link invariant:

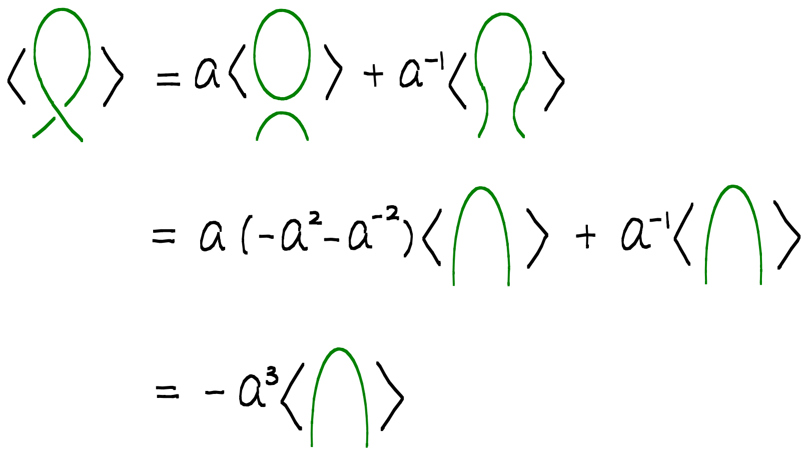
RII:

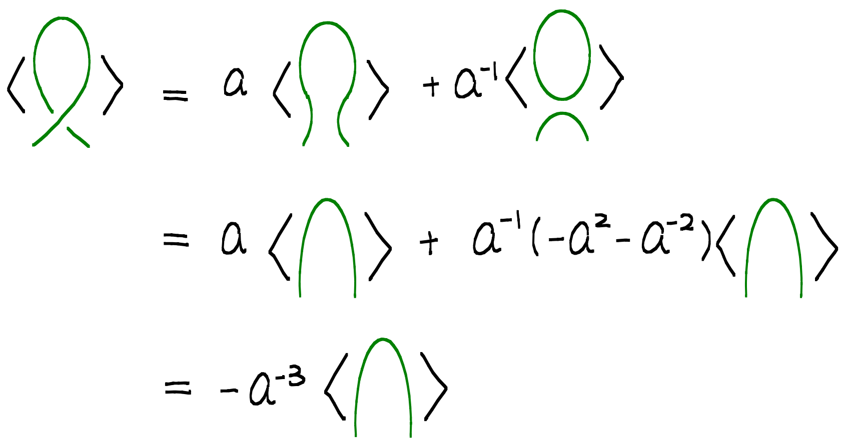


RIII: (using the invariance under RII).



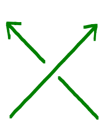
However, it's not invariant under RI:

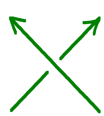




Thus it means that the Kauffman bracket is an invariant of unoriented framed links.

To obtain an invariant of unframed links out of $\langle \ \rangle$, we have to put orientations on links to differentiate the two ways of resolving RI. and balance the powers $a^{\pm 3}$ coming from them. Introduce ± 1 crossings:

 : +1 crossing

 : −1 crossing

and define the writhe of an oriented link projection to be

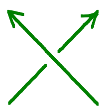
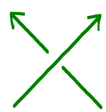
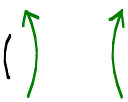
$$w(D) \triangleq \#(+1 \text{ crossing}) - \#(-1 \text{ crossing})$$

The computation above shows that we may introduce:

Def. The Kauffman polynomial of an oriented link L in $\mathbb{R}^3$ or S^3 is the Laurent polynomial

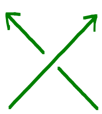
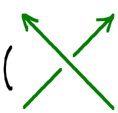
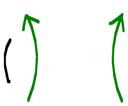
$$K(L) \triangleq (-a^3)^{-\omega(D)} \langle D \rangle \in \mathbb{Z}[a, a^{-1}].$$

Ex. Check that $K(L)$ satisfies the skein relation:

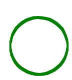
$q^2 K($  $) - q^{-2} K($  $) = (q - q^{-1}) K($  $)$ (**)'

where $q = -a^2$.

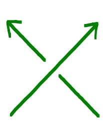
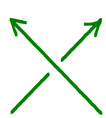
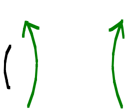
(**)' says that the Kauffman polynomial agrees with the celebrated oriented link polynomial invariant discovered by Jones when studying von Neumann algebras. It's charactered by the skein relation:

$q^2 J($  $) - q^{-2} J($  $) = (q - q^{-1}) J($  $)$. (**)

We also fix the normalization condition of the Jones polynomial:

$K($  $) = q + q^{-1}$.

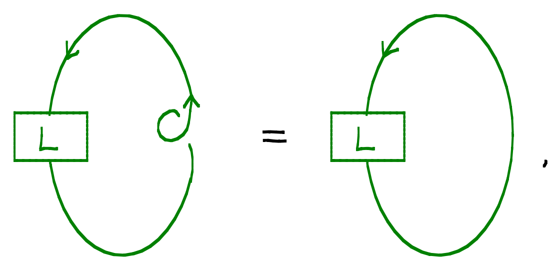
There are also variations of the Jones polynomial, defined by replacing $q^{\pm 2}$ on the left hand side of the above formula by $q^{\pm n}$, denoted $P_n(q) \in \mathbb{Z}[q, q^{-1}]$:

$q^n P_n($  $) - q^{-n} P_n($  $) = (q - q^{-1}) P_n($  $)$

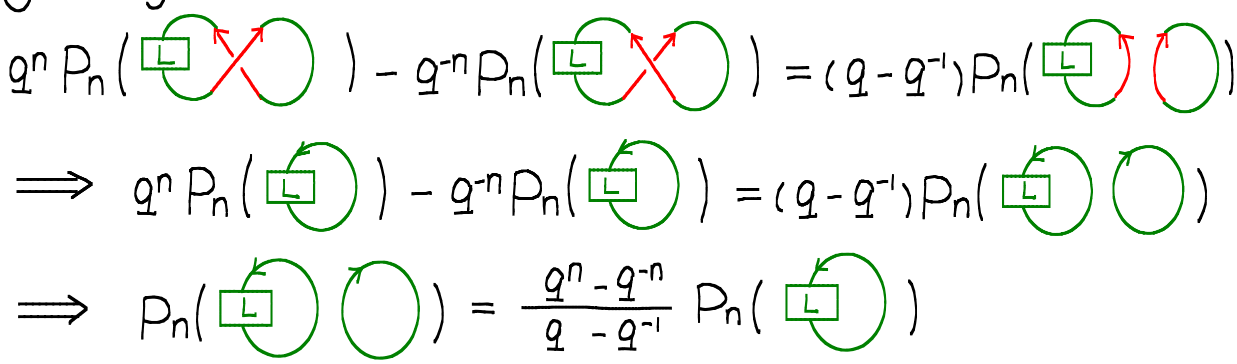
P_0 is the Alexander polynomial. $P_1 \equiv 1$. P_2 is the Jones polynomial.

For $n \geq 2$, the polynomial becomes harder to compute (#P) while the Alexander polynomial can be computed in polynomial time (P).

Rmk: About the normalization.
Note that, since



by the formula above, we have,



This says that, if we want P_n to be a tensor functor, we'd better set

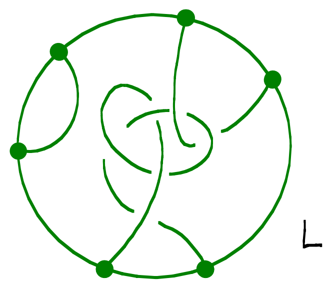
$P_n($$) = \dfrac{q^n - q^{-n}}{q - q^{-1}}$

Notice that $(q^n - q^{-n})/(q - q^{-1}) = q^{n-1} + q^{n-2} + \cdots + q^{1-n}$ is the shifted Poincaré polynomial $1 + t + \cdots + t^{n-1}$ of $\mathbb{CP}^{n-1}$. The occurrence of $\mathbb{CP}^{n-1}$ will be explained later. These polynomials in q and t are also refered to as the quantum integers, since they are "deformations" of the usual integer n.

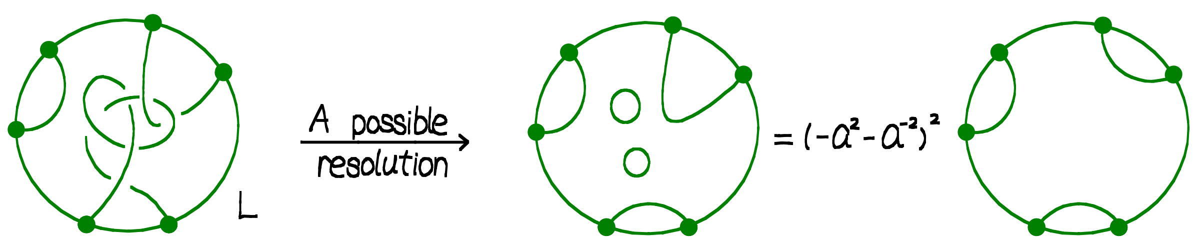
Localizing the Jones polynomial and Kauffman bracket

Before categorifying the Jones polynomial, we need a local or relative description of the Jones polynomial of links with fixed boundary points.

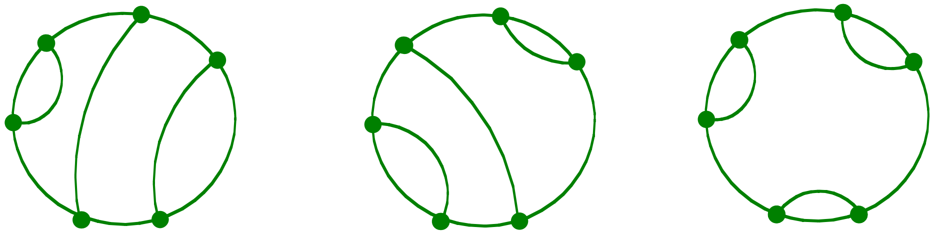
We regard links with some fixed boundary points as living inside the closed unit 3-ball D^3, with all its boundary points fixed on $\partial D^3 = S^2$. Similar as before, these links admit combinatorial descriptions via projecting them onto the unit 2-disk, and boundary points lying on $\partial D^2 = S^1$.



The Kauffmann bracket is defined for these links, by taking all possible resolutions of crossings, with appropriate coefficients inserted according to $(*)'$, and closed circles evaluated to be $-a^2-a^{-2}$.



It follows from this definition that the Kauffman bracket $\langle L \rangle$ of a link projection L on D^2 is a Laurent polynomial in $\mathbb{Z}[a, a^{-1}]$. Notice that links with $2n$ boundary points have all their possible resolutions, with circles evaluated, in 1-1 bijection with matchings of the boundary points:

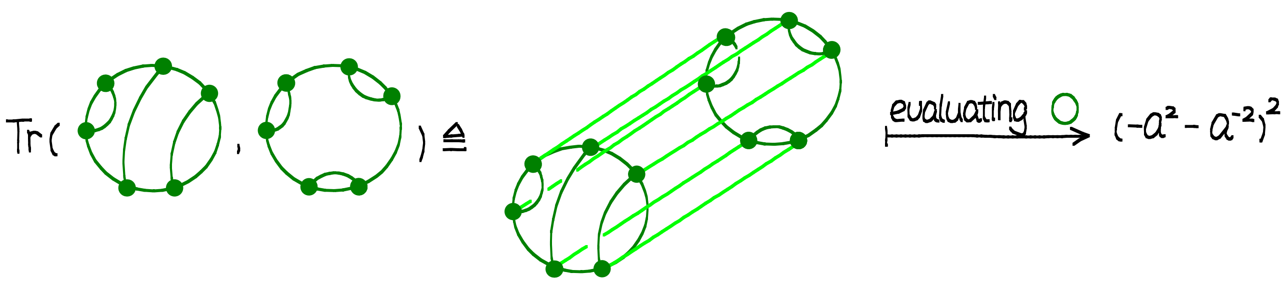


some possible matchings

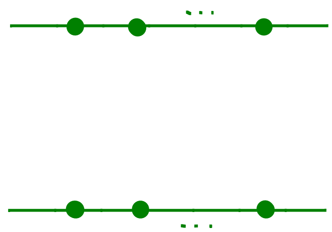
Ex. Prove that the number of all possible matchings as above of $2n$ points on S^1 is the Catalan number $\frac{1}{n+1}\binom{2n}{n}$.

Thus by the exercise, $\langle L \rangle$ takes value in the free $\mathbb{Z}[a,a^{-1}]$-module of rank $N = \frac{1}{n+1}\binom{2n}{n}$, with $\langle$ matchings $\rangle$ as basis. We denote this module by I^N.

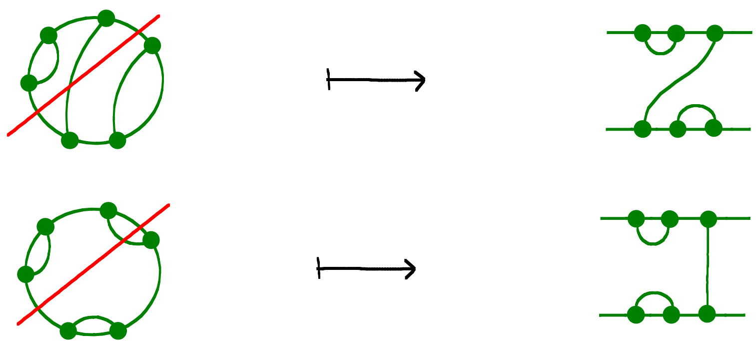
Introduce the following trace pairing on I^N. It's the unique $\mathbb{Z}[a,a^{-1}]$ bilinear map extended from pairing of basis elements, which is defined by joining the corresponding boundary points, and evaluating cirles so obtained:



Another equivalent way of representing the same datum (I^N, tr) is to break the symmetry of this circle: We put the $2n$ boundary points on 2 parallel line segments, n many points on each. Each point corresponds to a fixed point on the circle:

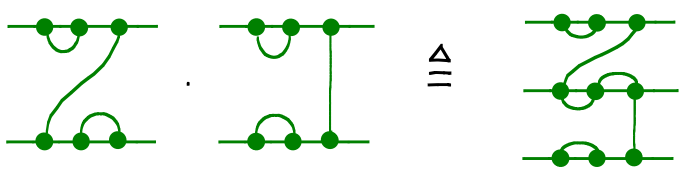


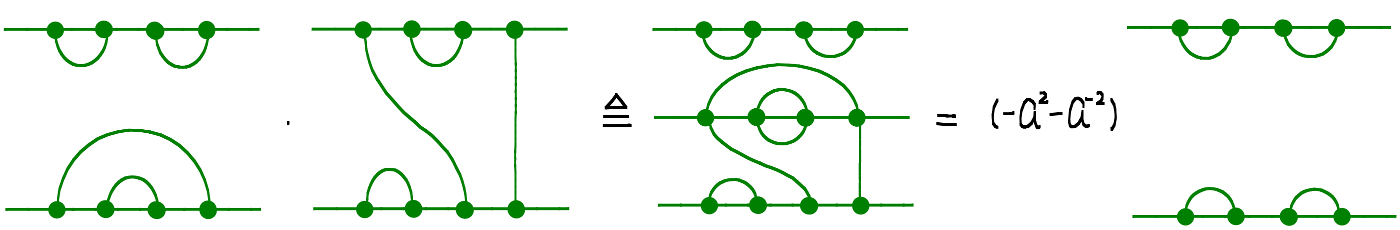
and draw the matchings accordinly:



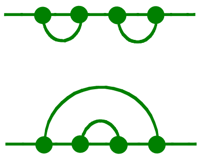
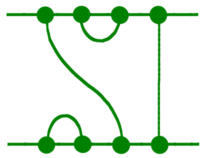
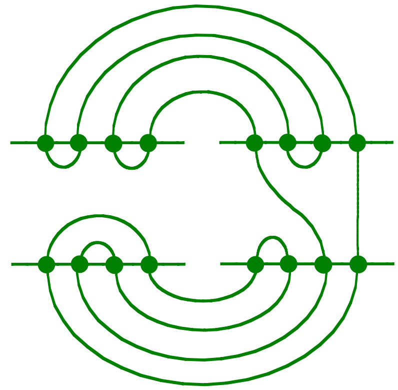
What we gain from this point of view is an algebra structure on I^N, whose multiplication is given by stacking pictures, and evaluating circles:

E.g.





The trace form now is given by connecting corresponding points on the 2 boundary segments, and evaluating circles:

$\mathrm{Tr}\Big($  ,  $\Big) =$  $= (-a^2 - a^{-2})$

This algebra is no other than the famous Temperley-Lieb algebra, which Jones used to define his polynomial.

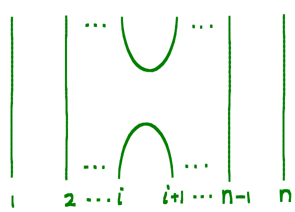
Def. (Temperley-Lieb algebra TL_n). For each $n \geq 0$, TL_n is the $\mathbb{Z}[a, a^{-1}]$ algebra generated by U_i, $i = 1, \cdots, n-1$, modulo relations:

(1). $U_i^2 = (-a^2 - a^{-2})\, U_i$

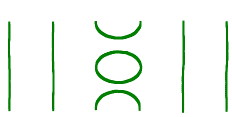
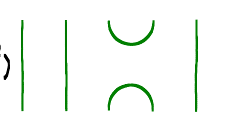
(2). $U_i U_{i\pm1} U_i = U_i$

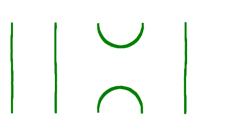
(3). $U_i U_j = U_j U_i \quad |i-j| > 1$

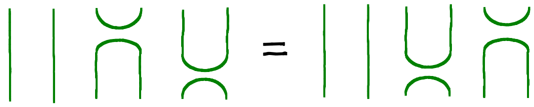
Using the graphical calculus above,

$U_i =$ 

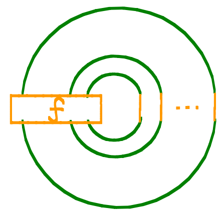
Then (1) – (3) above become evaluation of circles and isotopy relations on the graphs.

 $= (-a^2 - a^{-2})$ 

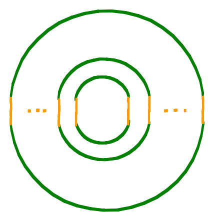
 $=$ 



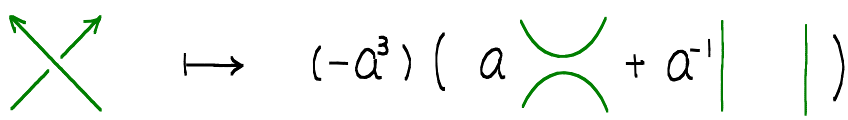
The Temperley-Lieb algebra inherits the trace form defined above by closing the diagrams : $\forall f \in TL_n$

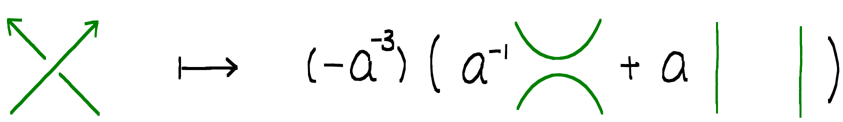
$Tr(f) \triangleq \langle$  $\rangle$

Note that :

$Tr(1_n) = \langle$  $\rangle = (-a^2 - a^{-2})^n$.

So how do we get the Jones polynomial out of TL_n ? We use the fact that every link can be obtained as the closure of an element of the braid group Br_n for some n. Then to the generators, namely, local oriented crossings, we assign:





One can then check that the Jones polynomial of the braid closure is given by taking trace of the element obtained according to this assignment in TL_n.

Rmk: The two variable Laurent polynomial invariant of links HOMFLYPT polynomial can be similarly defined with TL_n replaced by the Hecke algebra H_n.

Categorification in general

The moral of categorification is to consistently convert integers into vector spaces / free abelian groups.

For instance, to natural numbers, we can assign to them vector spaces / free abelian groups with the corresponding dim / rank (which is unique up to isomorphism, but not canonical isomorphism!) Then, the operations on integers are upgraded into:

$\mathbb{N}$	Categorification
$n \in \mathbb{N}$	V_n: $\dim V / \mathrm{rk} V_n = n$
$n+m$	$V_n \oplus W_m$
$n \cdot m$	$V_n \otimes W_m$
$n-m$	?

To categorify "n-m", we are forced to introduce complexes of vector spaces or free abelian groups, whose Euler characteristic is the alternating sum of dim / rank:

$$\chi(0 \to V_n \xrightarrow{d_0} W_m \to 0) = n-m.$$

Moreover, tensor products of complexes can be defined:

$$V^\bullet: (\cdots \to V^i \xrightarrow{d} V^{i+1} \to \cdots)$$
$$W^\bullet: (\cdots \to W^i \xrightarrow{d} W^{i+1} \to \cdots)$$

Then the tensor product of $V^\bullet$ and $W^\bullet$ is the complex $T^\bullet$

$$T^p \triangleq \bigoplus_{k\in\mathbb{Z}} V^k \otimes W^{p-k}$$

whose differential is given by:

$$d(v^i \otimes w^j) = (dv^i)\otimes w^j + (-1)^i v^i \otimes dw^j.$$

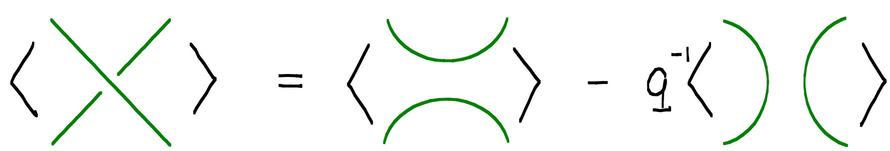
It also satisfies that:

$$\chi(V^\bullet \otimes W^\bullet) = \chi(V^\bullet)\cdot\chi(W^\bullet)$$

In fact, it's well-known that the category of (complexes) of vector spaces / free abelian groups is an additive, symmetric monoidal category, whose Grothendieck ring is $\mathbb{Z}$.

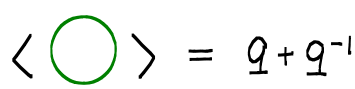
Categorification of the Jones polynomial

We slightly change notation from above, and define the Kauffman bracket by a modification of (*)':



(*)

(morally, $a = \sqrt{-q}$). And we normalize it by:



For an oriented link diagram D, we denote by $x(D)$ the number of negative crossings on it, and $y(D)$ the number of positive crossings:

The Kauffman polynomial is modified as:

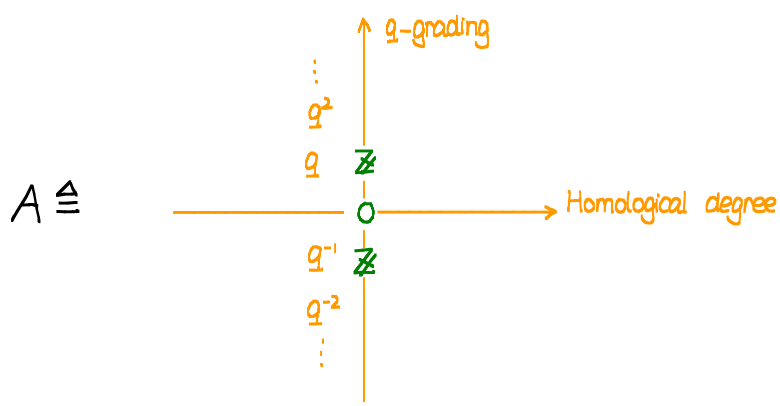
$$K(D) \triangleq (-1)^{x(D)}\, q^{2x(D)-y(D)} \langle D\rangle.$$

As discussed above, what we would like to do here is to lift these polynomials with integer coefficients, in a "consistent" way, into complexes of vector spaces or free abelian groups, so that when we take the Euler characteristic of it, we recover $K(D)$. Now in the presence of q, what we shoud lift each link to is a bigraded "homology" theory of links $H^{i,j}(D)$, so that when taking its graded Euler characteristic by collapsing the homological grading, we could recover $K(D)$:

$$K(D) = \sum_{i,j} (-1)^i q^j \dim H^{i,j}(D)$$

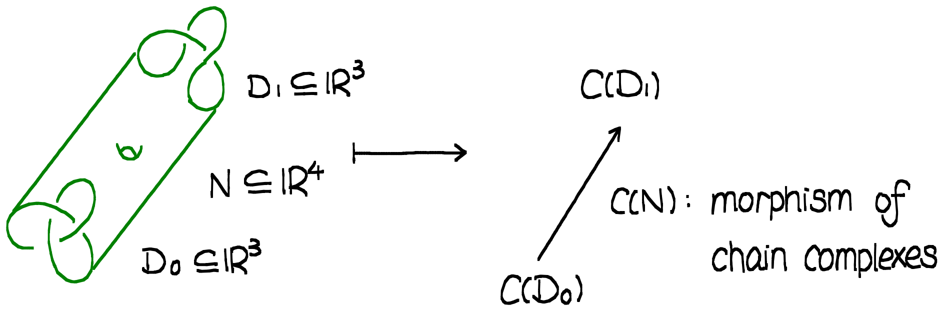
Here "homology" theory will in fact be complexes, which we know is more fundamental than homology groups. So what we will do is to assign to each D a chain complex of graded vector spaces / free abelian groups.

E.g. By the normalization requirement, the most obvious graded chain complex for ◯ is:

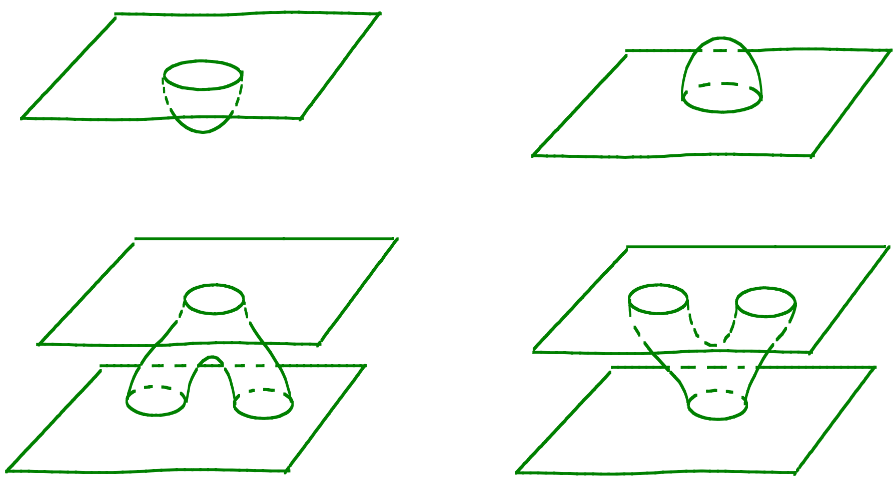


so that $\mathrm{gr}\dim A = q + q^{-1}$.

Here we need to say more about the word "consistent", which is in fact a requirement of functoriality. In the presence of higher structures in the symmetric monoidal category, namely, morphisms between complexes of graded modules, we should naturally expect to have liftings of cobordisms to these morphisms:



Thus from §1, we know that we must assign to the unknot a commutative Frobenius algebra (differential graded Frobenius algebra)

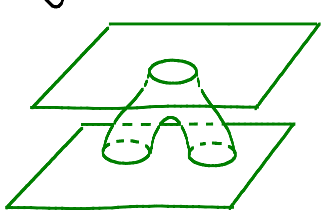


These cobordisms of various copies of the unknot ($\subseteq \mathbb{R}^2$) lie in $\mathbb{R}^2 \times I \subseteq \mathbb{R}^3 \times I$

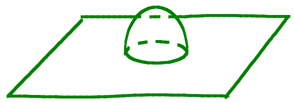
Here, the obvious way is to make

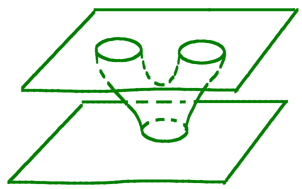
$$A = \mathbb{Z}q^{-1} \oplus \mathbb{Z}q \cong \mathbb{Z}[X]/(X^2) \cong H^*(\mathbb{P}^1),$$

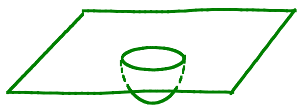
the latter ring has the usual Frobenius structure:



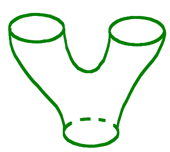
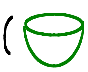
$$\longmapsto \quad m: \begin{cases} 1\cdot 1 = 1 \quad 1\cdot X = X \\ X^2 = 0 \end{cases}$$

 $\longmapsto$ $\varepsilon : \begin{cases} \varepsilon(1) = 0 \\ \varepsilon(x) = 1 \end{cases}$

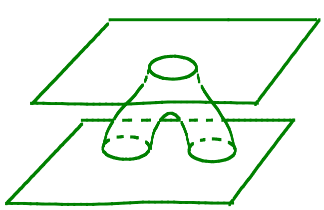
 $\longmapsto$ $\Delta : \begin{cases} \Delta(1) = 1 \otimes x + x \otimes 1 \\ \Delta(x) = x \otimes x \end{cases}$

 $\longmapsto$ $i : 1 \longmapsto 1$

Notice that these maps are not homogeneous : our $1 \in A$ sits in q-deg -1, but $\Delta(1) = 1 \otimes x + x \otimes 1$ sits in q-deg 1 , $i(1)$ sits in q-deg -1. However, observe that the topological Euler characteristic

$\chi($  $) = -1$ $\chi($  $) = 1$

If we require these cobordisms to take into account of the grading shift by $q^{-\chi(S)}$, where $\chi(S)$ is the topological Euler characteristic of the cobordism surface, the q-deg match now :

 $\longmapsto$

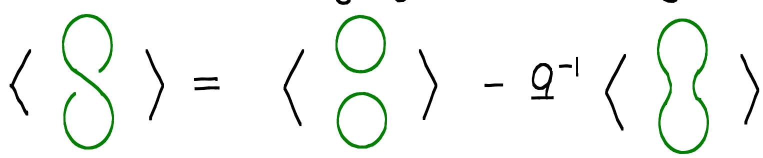
$A = \{\mathbb{Z}x \; (q), \; \mathbb{Z} \; (q^{-1})\}$

$\uparrow m\{1\}$

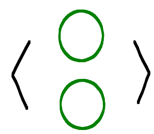
$A^{\otimes 2} = \{\mathbb{Z}x \otimes x \; (q^2), \; \mathbb{Z}x \otimes 1 \oplus \mathbb{Z}1 \otimes x \; (q^0), \; \mathbb{Z}1 \otimes 1 \; (q^{-2})\}$

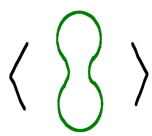
where we denote q-shift by $\{1\}$, in contrast with homological degree shift $[1]$.

E.g. Let's look at the categorified version of the equation:



First of all, we have the assignement

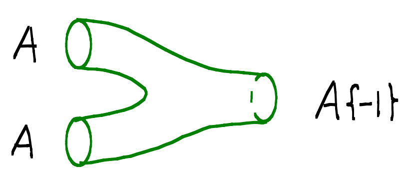
 $\mapsto A^{\otimes 2}$

 $\mapsto A$

The negative sign in the formula indicates that these modules should sit in different homological degrees. As the coefficient of the first factor is positive, we place it in homological degree 0, while the second term in homological degree 1. (This will be our convention in the future as well.) The q^{-1} should correspond to q degree shift by -1:

$$0 \longrightarrow A^{\otimes 2} \longrightarrow A\{-1\} \longrightarrow 0$$

But what map should we put in the middle? Recall that the multiplication $m\{1\}$ is a q-deg 1 map between $A^{\otimes 2}$ and A. Thus we may take this map to be m.



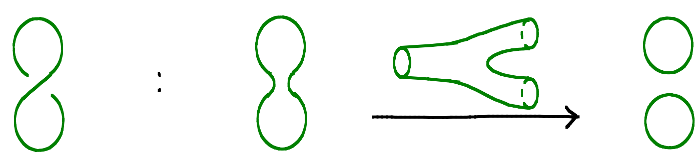
The homology of this complex is easy to compute:

$$0 \longrightarrow A^{\otimes 2} \xrightarrow{m} A\{-1\} \longrightarrow 0$$

$$\Longrightarrow \begin{cases} H_0 \cong \mathbb{Z}\underbrace{(1\otimes x - x\otimes 1)}_{\deg 0} \oplus \mathbb{Z}\underbrace{x\otimes x}_{\deg 2} \\ H_1 = 0 \end{cases}$$

which up to a grading shift is isomorphic to A (which will be taken care of once we assign orientations to links and deal with $K(D)$). This is what we wanted since the "kink" above is just the circle.

The opposite kink can be similarly computed using the comultiplication:



$$A \xrightarrow{\Delta} A^{\otimes 2}\{-1\}.$$

To procede, we recall the "cone" construction from homological algebra (See Gelfand & Manin, Methods of homological algebra).

Let $f: M^\bullet \longrightarrow N^\bullet$ be a map of chain complexes on some additive category. The cone of f, denoted $C(f)^\bullet$, is the complex

$$C(f)^\bullet = M[1]^\bullet \oplus N^\bullet$$

whose differential is given by

$$d_{C(f)^\bullet} = \begin{pmatrix} d_{M[1]^\bullet} & f \\ 0 & d_{N^\bullet} \end{pmatrix}$$

where $d_{M^\bullet[1]} = -d_{M^\bullet}$.

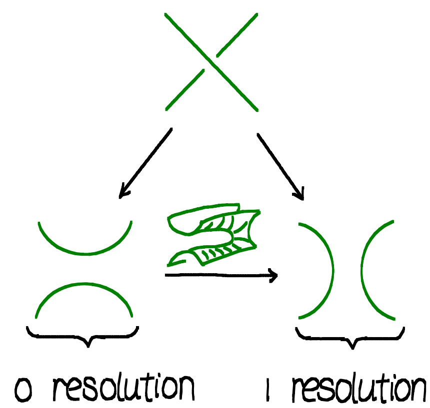
The cone construction always gives rise to a long exact sequence in cohomology :

$$\cdots \longrightarrow H^i(M^\bullet) \longrightarrow H^i(N^\bullet) \longrightarrow H^i(C(f)^\bullet) \longrightarrow H^{i+1}(M^\bullet) \longrightarrow \cdots$$

$$\Longrightarrow \quad \chi(C(f)^\bullet) = \chi(N^\bullet) - \chi(M^\bullet)$$

The cone construction works over the categories $Kom(\mathcal{A})$, $Comp(\mathcal{A})$ and $D(\mathcal{A})$ (for $D(\mathcal{A})$ to be defined, $\mathcal{A}$ has to be abelian).

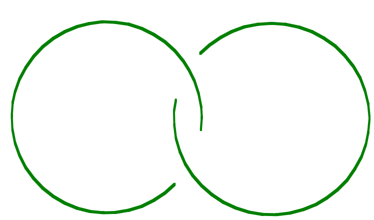
The process of the example above can be generalized immediately to any oriented link diagram with n crossings. First of all, we temporarily forget about the orientation on the link, and take its complete resolution as before. We shall always take the (local) resolution of a crossing to be:



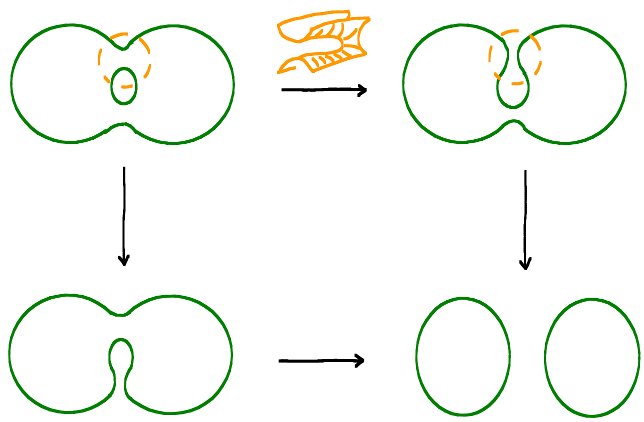
(we may regard these resolutions as objects in some additive category sitting in corresponding homological degrees)

The n crossings after resolution become 2^n resolution diagrams, each consisting a certain number of circles. This is better seen through an example:

E.g. The Hopf link.



It's resolution is:



Here locally we are just doing the cobordism

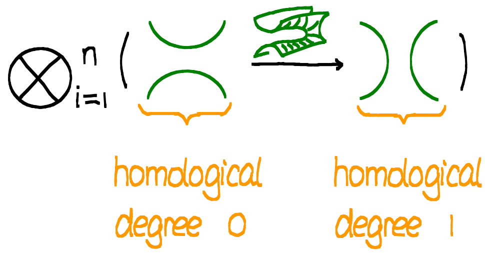
Now we apply the oriented TQFT we constructed by assigning to a circle the Frobenius algebra A (technically we need to orient each circle to apply the TQFT), assigning to merging of circles the multiplication and separating circles the comultiplication:

$$\begin{array}{ccc} A^{\otimes 2} & \xrightarrow{m} & A\{-1\} \\ \downarrow m & & \downarrow \Delta \\ A\{-1\} & \xrightarrow{\Delta} & A^{\otimes 2}\{-2\} \end{array}$$

Notice that terms at two ends of one arrow have their numbers of copies of A differ by ± 1, since arrows correspond to either merging two circles into one or dividing one circle into two.

The above diagram of abelian groups obviously commutes. This is also true in general. This is because, different paths of resolutions give far apart local cobordisms done in different order, which are in fact the same cobordism, so that functoriality of TQFT says that the diagram is commutative.

To get a complex out of the commutative diagram, we need to insert some negative signs in appropriate places. Here we can borrow the standard convention from homological algebra by regarding the complex as constructed by taking $\otimes$ products of the "formal local complexes" of resolutions:



In the previous example, we just take:

$$\begin{array}{ccc} A^{\otimes 2} & \xrightarrow{m} & A\{-1\} \\ \downarrow{\scriptstyle m} & & \downarrow{\scriptstyle -\Delta} \\ A\{-1\} & \xrightarrow{\Delta} & A^{\otimes 2}\{-2\} \end{array}$$

Finally, we form the total complex of the "tensor product" complex. (This is just the cone construction!) It's denoted $C'(D)$ of the original oriented link projection D. By this construction, we clearly have:

$$\chi(C'(D)) = \langle D \rangle .$$

The complex $C(D)$ is then constructed by taking into account signs and degree shifts in the Kauffman polynomial:

$$\boxed{C(D) \triangleq C'(D)[x(D)]\{2x(D)-y(D)\}}$$

which satisfies

$$\chi(C(D)) = K(D).$$

Thm. $C(D)$ in the homotopy category of graded free abelian groups is an invariant of oriented links.

We will sketch the proof of the theorem in the following. What we will show is that the Reidemeister moves give rise to homotopic complexes. The theorem implies that the homology of $C(D)$:

$$H(D) = \oplus_{i,j} H^{i,j}(D)$$

is an oriented link invariant, and

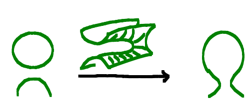
$$\begin{aligned} K(D) &= \chi(C(D)) \\ &= \chi(H(D)) \\ &= \textstyle\sum_{i,j\in\mathbb{Z}} (-1)^i q^j \operatorname{rank} H^{i,j}(D). \end{aligned}$$

Sketch of proof of theorem

RI. We need to check that

in Comp(gr.Ab).

We shall show the first isomorphism. We forget about the orientation momentarily since it only provides degree and grading shifts for $C(D)$. Then

$C'(D) = \text{Cone}($  $)[-1]$

$$= (0 \longrightarrow A \otimes C'(\cap) \xrightarrow{m} C'(\cap) \longrightarrow 0)$$

But m decomposes:

$$m:\ A\otimes C'(\cap)\cong \begin{array}{c}\mathbb{Z}\cdot 1\otimes C'(\cap)\\ \oplus \\ \mathbb{Z}\cdot x\otimes C'(\cap)\end{array} \begin{array}{c} \xrightarrow{1\otimes a\ \mapsto\ 1\cdot a} \\ \\ \xrightarrow{x\otimes a\ \mapsto\ x\cdot a}\end{array} C'(\cap)$$

We can split the sequence as a direct sum of a contractible complex and a copy of $\mathbb{Z}\cdot x\otimes C'(\cap)$, as follows: we take elements $x\otimes a$ and twist them by $-1\otimes xa\in\mathbb{Z}\cdot 1\otimes C'(\cap)$. The image of these elements $x\otimes a-1\otimes xa\in A\otimes C'(\cap)$ under m is 0, and they constitute a copy of $\mathbb{Z}\cdot x\otimes C'(\cap)$. Together with $\mathbb{Z}\cdot 1\otimes C'(\cap)$, they span $A\otimes C'(\cap)$ and thus the complex is isomorphic to

$$\begin{array}{c} 0\longrightarrow \mathbb{Z}\cdot 1\otimes C'(\cap)\longrightarrow C'(\cap)\longrightarrow 0\\ \oplus\\ 0\longrightarrow \mathbb{Z}\cdot x\otimes C'(\cap)\longrightarrow 0\end{array}$$

The top subcomplex being obviously contractible.

Thus in Comp(gr.Ab),

$C'($  $)\cong C'(\cap)\{1\}$.

When we take orientation back into account, we get:

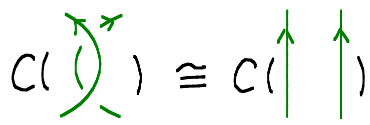
$C($  $) = C'($  $)\{-1\}$ $(x(D)=0,\ y(D)=1)$

$$\cong C'(\cap)$$

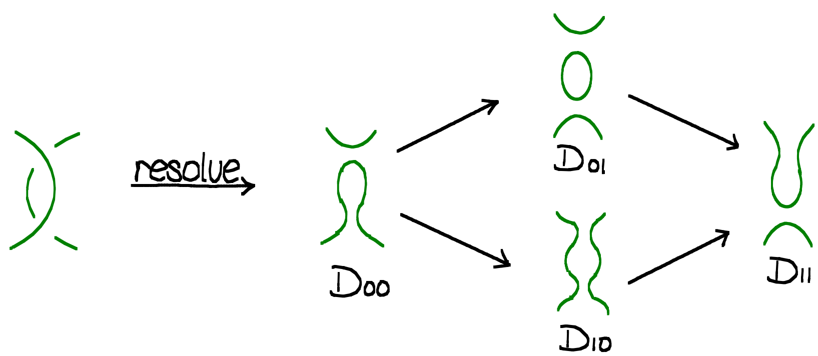
$$= C(\cap).$$

□ of RI

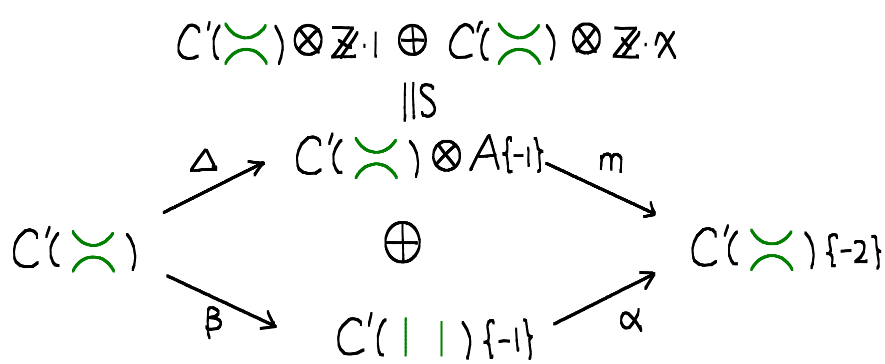
RII. Now we check one of the oriented RII moves, and the rest is similar.



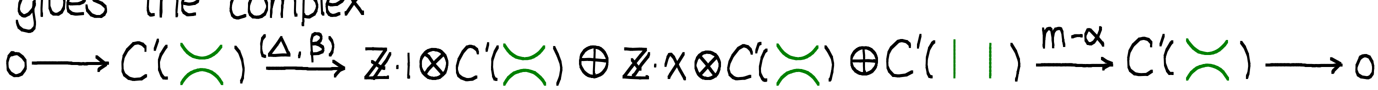
We again ignore the orientations first, and take the resolution of the left hand side:



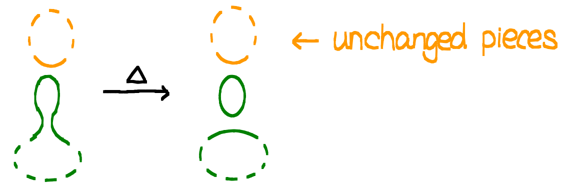
Notice that D_{10} is the same as the right hand side. Thus we will try to show that after applying C', the whole complex is isomorphic to a direct sum of $C'(D_{10})$ and a contractible complex. Also note that D_{00} is the same as D_{11} while D_{01} is just D_{00} disjoint union with a circle. Hence,



gives the complex



Since Δ is given by:



$$a \otimes 1 \mapsto a \otimes \Delta(1) = a \otimes 1 \otimes x + a \otimes x \otimes 1$$
$$a \otimes x \mapsto a \otimes \Delta(x) = a \otimes x \otimes x$$

we see that $(\Delta, \beta)(t) = t \otimes x \oplus \Delta'(t) \otimes 1 \oplus \beta(t) \in C'(\asymp) \otimes A \oplus C'(|\ |)$ is an injective map and so that

$$0 \longrightarrow C'(\asymp) \xrightarrow{(\Delta,\beta)} \mathrm{Im}(\Delta,\beta) \longrightarrow 0$$
$$\cong$$
$$C'(\asymp)$$

is a subcomplex of the total complex. Next, the multiplication map m induces an isomorphism $m: C'(\asymp) \otimes \mathbb{Z} \cdot 1 \longrightarrow C'(\asymp)$. Lastly, we "twist" $C'(|\ |)$ to be the isomorphic submodule $\{(\alpha(t) \otimes 1, t) \mid t \in C'(|\ |)\}$ so that it maps to 0 under $(m, -\alpha)$. We have thus obtained a direct sum decomposition of the total complex:

$$0 \longrightarrow C'(\asymp) \xrightarrow{id} C'(\asymp) \longrightarrow 0$$
$$\oplus$$
$$0 \longrightarrow C'(\asymp) \xrightarrow{id} C'(\asymp) \longrightarrow 0$$
$$\oplus$$
$$0 \longrightarrow \underbrace{C'(|\ |)\{-1\}}_{\text{homological degree } 1} \longrightarrow 0 \, ,$$

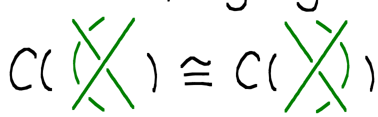
where all but the last direct summand are contractable, as desired. Taking orientation back into account, we have $x(D) = y(D) = 1$. Hence

$$C(\text{RII tangle}) \cong (C'(|\ |)[-1]\{-1\})[1]\{1\} = C'(|\ |) = C(\uparrow\ \uparrow)$$

□ of RII

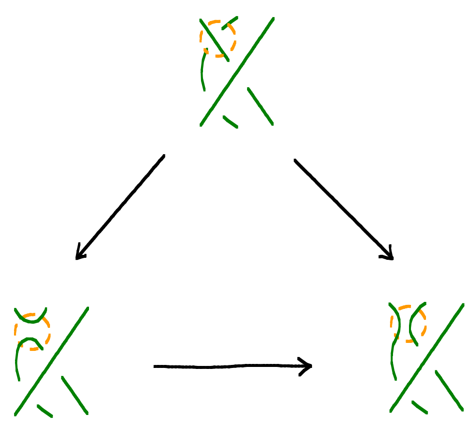
Rmk: The above method is also used in algebraic geometry, symplectic geometry, and geometric representation theory when people are constructing braid group actions on some category using the so called "spherical objects".

RIII. Lastly we sketch the proof of:

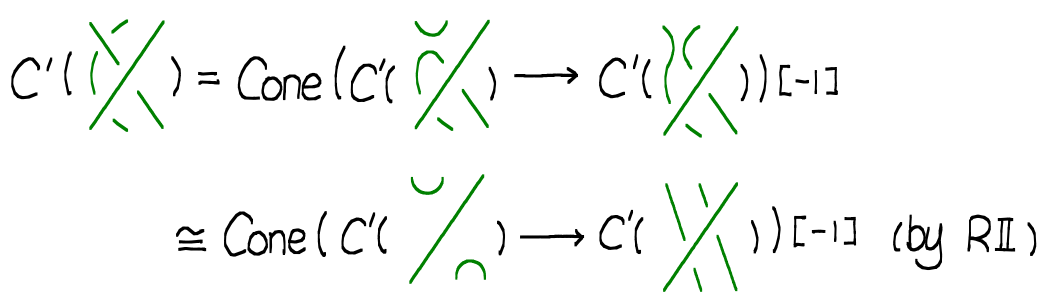


The same trick of resolving the crossings as we did at the beginning of the section to deal with Kauffman bracket can be applied here as well. We shall only work out $C'(D)$'s for the unoriented case. The oriented case follows by a degree/sign shift counting.

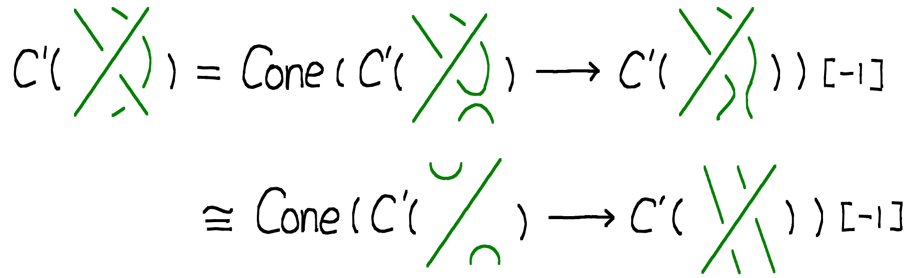
For the left hand side, we resolve the upper left crossing first:



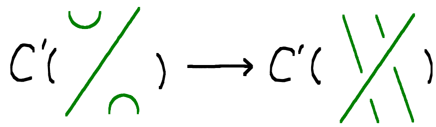
Thus



One similarly checks that the right hand side, after resolving the lower right crossing, becomes:



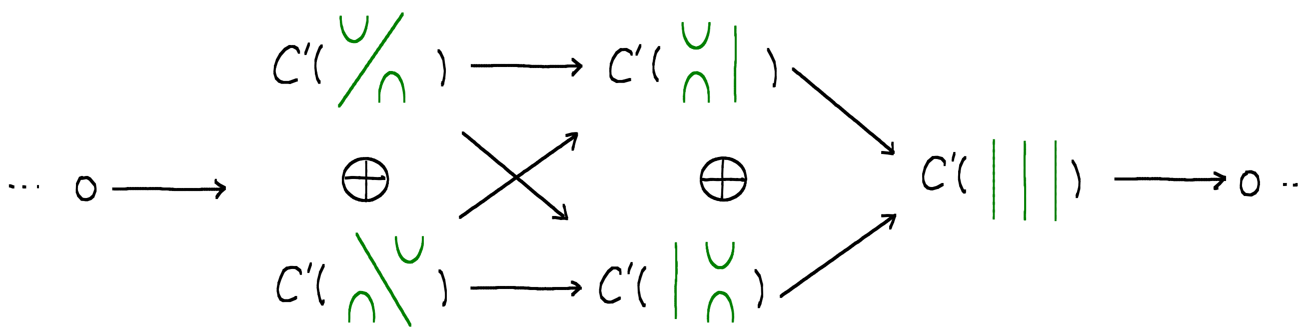
One can then check that the two maps:



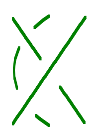
can be taken to be the same and the result follows.

□ of RIII.

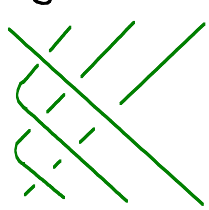
Rmk: One can take the full resolution of the above and find out that in both cases the complexes are of the form:



It's an interesting phenomenon that all elements of the Temperley-Lieb algebra TL_3 appear in the resolution of



Ex. Check if all elements of TL_4 appear in the resolution of:

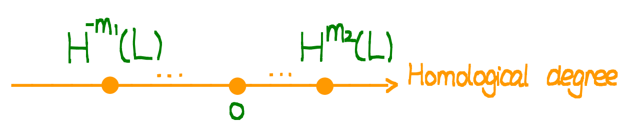


Application : Tait conjecture

Recall that from our construction, the complex $C'(D)$ of graded abelian groups for any link projection diagram D is obtained by tensoring complexes coming from 0 and 1 resolutions of crossings. The construction immediately implies that, $C'(D)$ is a bounded complex whose non-zero terms sit in homological degrees between 0 and the total number of crossings in D, i.e. $[0, x(D)+y(D)]$. It follows that

$$C(D) = C'(D)[x(D)]\{2x(D)-y(D)\} \in Comp^{[-x(D), y(D)]}(gr.Ab)$$

By the thm. above, the homology of $C(D)$ is an oriented link invariant. Thus if the homology $H(L)$ of an oriented link L has non-trivial terms H^{-m_1}, H^{m_2} for some $m_1, m_2 \in \mathbb{Z}_{\geq 0}$,



then any diagram D of L cannot have less than m_1 negative crossings and less than m_2 positive crossings.

Cor.

$$x(D) \geq m_1 = \max\{k \mid H^{-k}(L) \neq 0\}$$
$$y(D) \geq m_2 = \max\{\ell \mid H^{\ell}(L) \neq 0\}.$$

□

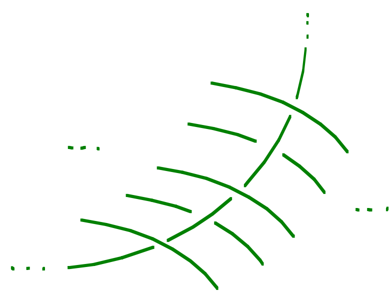
The smallest number of total crossings of projection diagrams of a link L is called its crossing number $c(L)$. The discussion above gives a lower bound for $c(L)$ to be m_1+m_2. In general, it's an open conjecture that, for any two links K and L,

$$c(K \# L) = c(K) + c(L).$$

("≤" is trivially true).

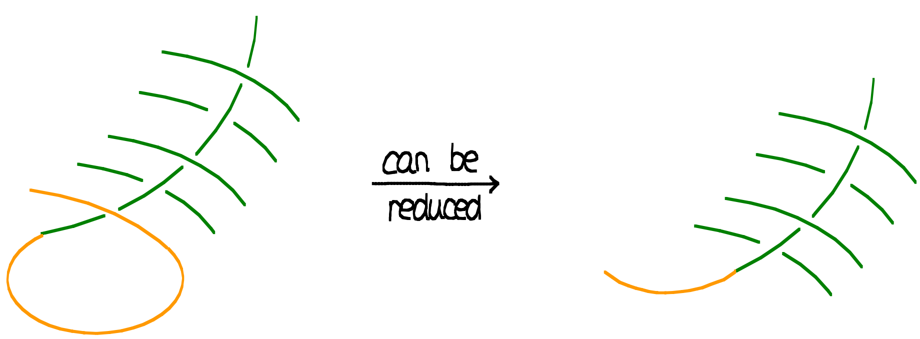
As an application of the homology theory, we give a new proof of the Tait conjecture stating that any minimal alternating diagram of L is minimal for L (i.e. the number of crossings attains $c(L)$). We recall the definitions involved:

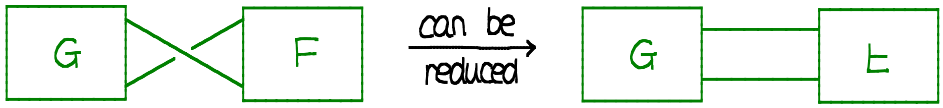
Def. A link L is called alternating if all its crossings can be rearranged together to look like:



It's known that links with $c(L) \leq 7$ are alternating.

A projection diagram for an alternating link L is called minimal alternating if it's alternating and has the least number of crossings in all the alternating projections of L.





The Tait conjecture (theorem) states that a minimal alternating diagram for L is minimal among all its projection diagrams

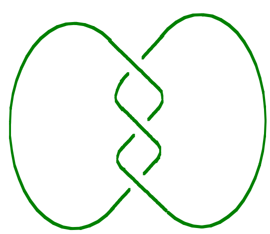
To prove the conjecture, we review some basic facts about H(L).

First off, observe that if $H^{-\chi(D)}(L) \neq 0$, then in C(D):

$$\cdots 0 \longrightarrow C^{-\chi(D)}(D) \xrightarrow{d} C^{-\chi(D)+1}(D) \longrightarrow \cdots$$

d is not injective.

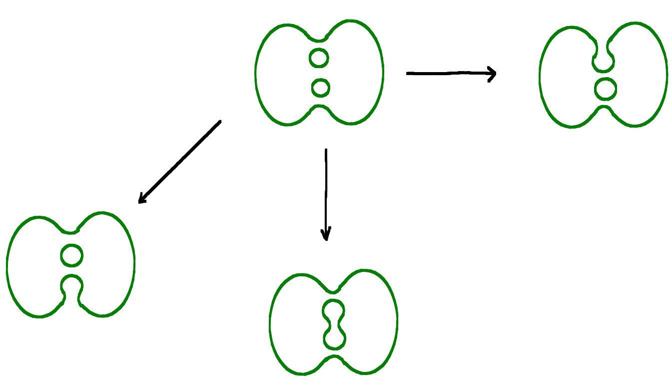
E.g. Consider the trefoil knot:



One can see that the left end of C(D) looks likes

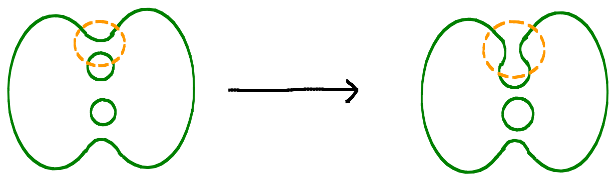
$$0 \longrightarrow A^{\otimes 3} \xrightarrow{d} \oplus_3 A^{\otimes 2} \longrightarrow \cdots ,$$

which comes from the upper left corner of the complete resolution:



$$\begin{array}{ccc} & A^{\otimes 3} \xrightarrow{m_{12}} A^{\otimes 2} \\ \swarrow m_{13} & \downarrow m_{23} \\ A^{\otimes 2} & A^{\otimes 2} \end{array}$$

where m_{ij} is the multiplication coming from merging the i, j circles



$$m_{12}: A\otimes A\otimes A \longrightarrow A\otimes A$$
$$x\otimes y\otimes z \longmapsto x\cdot y\otimes z$$

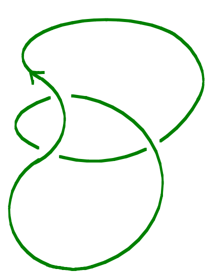
Thus the element $x^{\otimes 3}$ lies in the kernel of

$$d = (m_{12}, m_{23}, m_{13}) : A^{\otimes 3} \longrightarrow \oplus_3 A^{\otimes 2}$$

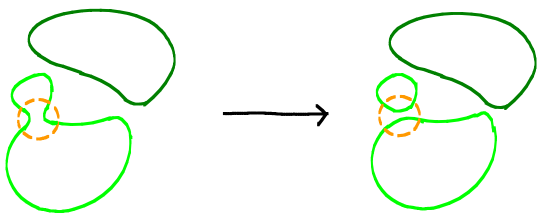
since $x^2 = 0$.

However, this example is not the general case: It's not true that if $C^{-\chi(D)}(D) = A^{\otimes k}$, where k = # circles in the 0-resolution of all crossings, then $x^{\otimes k}$ lies in the kernel of the differential. For instance, we have the following:

E.g. The trivial knot



With the above orientation, it has two negative crossings so that the left most term in $C(D)$ is $C^{-2}(D)$. But we know that $C(D) \cong \underline{A}$, the free abelian group A sitting in homological deg. 0. Thus there can be no $H^{-2}(D)$. One can easily check that $d: C^{-2}(D) \longrightarrow C^{-1}(D)$ is injective. For instance, we look at a component of d:



$$A^{\otimes 2} \xrightarrow{\Delta \otimes Id_A} A^{\otimes 3}$$
$$a \otimes b \mapsto \Delta(a) \otimes b \, ,$$

which is clearly injective.

The injective map Δ appeared since the number of circles on the complete 0-resolution is less than that of one of its neighboring resolution, which is obtained by replacing one of the 0-resolutions by a 1-resolution. This motivates:

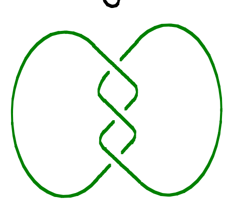
Def. (Adequete diagrams)

(1). A link projection diagram is called "−" adequete if its complete 0-resolution has k-circles, and if we replace any one of the 0-resolutions by a 1-resolution we get $k-1$ circles.

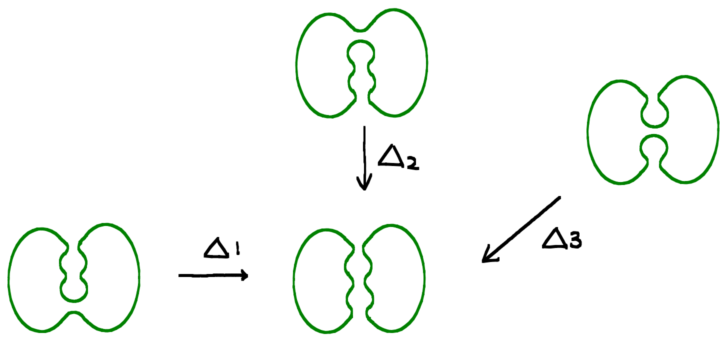
(2). A link projection diagram is called "+" adequete if its complete 1-resolution has k-circles, and if we replace any one of the 1-resolutions by a 0-resolution we get $k-1$ circles.

(3). A link projection diagram is called adequete if it's both "+" and "−" adequete.

The following projection of the trefoil knot is adequete:



D



The trefoil knot is also + adequete. The term $1 \otimes 1$ couldn't be hit under the differential $d = \Sigma \Delta_i$

It follows from our discussion that if D is an adequete diagram, then $H^{-x(D)}(L) \neq 0$, $H^{y(D)}(L) = 0$, so that

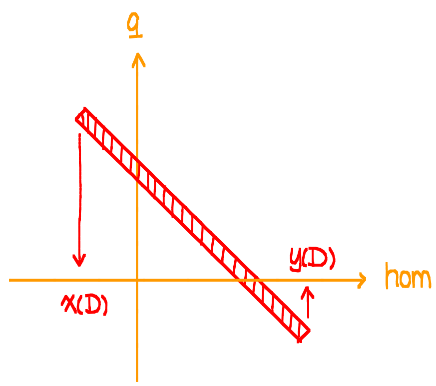
$$c(L) = x(D) + y(D) = \# \text{ of crossings in } D,$$

and D is minimal

Now the Tait conjecture follows easily from:

Ex. Show that any minimal alternating diagram is adequete.

Rmk: The Tait conjecture also follows from a reduced version of Khovanov homology. The alternating links have the reduced homology complex sitting anti-diagonally in the q-deg / hom. deg. plane:



so that one sees directly that all terms survie in homology.

The reduced complex can be viewed as a categorification of the Jones polynomial subject to the normalization condition that $J(\text{unknot}) = 1$. This is done by fixing a marked point on a link diagram D and when constructing $\tilde{C}(D)$ we set any circle in the complete resolution containing that marked point to be $\mathbb{Z}$ instead of A. Alternatively,

$$\tilde{C}(D) = C(D) \otimes_A \mathbb{Z}$$

with $x \in A$ acting trivially on $\mathbb{Z}$. Thus $\tilde{C}(D)$ can be constructed by tensoring $C(D)$ over the s.e.s:

$$0 \longrightarrow \mathbb{Z}x \longrightarrow A \longrightarrow \mathbb{Z} \longrightarrow 0$$

$$\Longrightarrow \quad 0 \longrightarrow \tilde{C}(D)\{1\} \longrightarrow C(D) \longrightarrow \tilde{C}(D)\{-1\} \longrightarrow 0$$