§11. Hochschild Homology and Applications to Link Homology II

A toy model

Let $\Bbbk$ be a field of char $\Bbbk \neq 2$. Let $A = \Bbbk[x]$ be the polynomial ring in x, and $A_0 = \Bbbk[x^2] \subseteq A$. We will assume $\deg x = 2$. Then:

$$A \cong A_0 \cdot 1 \oplus A_0 \cdot x .$$

Define $B \triangleq A \otimes_{A_0} A$. Then B is a free rank 2 left A module, with a basis $\{1\otimes 1, 1\otimes x\}$. The multiplication map:

$$\begin{aligned} B &\xrightarrow{m} A \\ a\otimes b &\longmapsto ab \end{aligned}$$

is surjective and has a kernel $\ker m = \langle x\otimes 1 - 1\otimes x\rangle \subseteq B$. $\ker m$ is a copy of A where the right action of x on it is the same as the left action of $-x$ on it. We will denote it by A^-. In general if $\varphi: A \longrightarrow A$ is an endomorphism, we will denote by A^φ the A-bimodule where the right action is twisted by φ: $a \circ x = a \cdot \varphi(x)$. ${}_A(A^\varphi)_A \ncong {}_A A_A$ in general, and if φ, ψ are two such endomorphisms, $A^\varphi \otimes_A A^\psi \cong A^{\varphi\psi}$. In particular $A^- \otimes_A A^- \cong A$.

Thus we have a s.e.s. of A-bimodules:

$$0 \longrightarrow A^- \longrightarrow B \longrightarrow A \longrightarrow 0$$

Similarly, we also have the s.e.s. the other way around:

$$\begin{aligned} 0 \longrightarrow A &\longrightarrow B \longrightarrow A^- \longrightarrow 0 \\ f &\longmapsto f\cdot(x\otimes 1 + 1\otimes x) \end{aligned}$$

Now, consider the triangulated categories:

$$\begin{cases} \text{I. } \mathrm{Com}(A\text{-mod}) \\ \text{II. } D(A\text{-mod}), \end{cases}$$

and let R be the complex of bimodules:

$$0 \longrightarrow B \longrightarrow A \longrightarrow 0$$

where B sits in homological degree 0. Tensoring with R over A is an endo-functor in both categories I and II. But in II it's a boring action since,

$$\begin{tikzcd} 0 \arrow[r] & A^- \arrow[r] \arrow[d, "\gamma"] & 0 & \\ 0 \arrow[r] & B \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

is a qis, so that $R \otimes_A M$ is qis to $A^- \otimes_A M$, and

$$\gamma : A^- \otimes_A - \Longrightarrow R \otimes_A -$$

is a canonical isomorphism of functors.

Likewise, we introduce the complex R':

$$0 \longrightarrow A \longrightarrow B \longrightarrow 0$$

where B also sits in homological degree 0. It's also true that in case II, $R' \otimes_A - \cong A^- \otimes_A -$.

However, the functors are more interesting in I. For instance, if we take $M = \Bbbk[x]/(x) \cong \Bbbk$, we get

$$\begin{tikzcd} A^- \otimes_A \Bbbk : & 0 \arrow[r] & \Bbbk \arrow[r] \arrow[d, "\gamma"] \arrow[dl, "0"'] & 0 \arrow[dl, "0"'] & \\ R \otimes_A \Bbbk : & 0 \arrow[r] & \Bbbk^2 \arrow[r] & \Bbbk \arrow[r] & 0 \end{tikzcd}$$

two non-homotopic complexes.

Lemma 1. R and R' are mutually inverse functors on $\mathrm{Com}(A\text{-mod})$.
Pf: We will show that, as complexes of bimodules, $R \otimes_A R' \cong {}_A A_A \cong R' \otimes_A R$. In fact, both $R \otimes_A R'$ and $R' \otimes_A R$ are isomorphic to the total complex of the cube:

$$\begin{tikzcd} B \otimes_A B \arrow[r] & B \otimes_A A \\ A \otimes_A B \arrow[r] \arrow[u] & A \otimes_A A \arrow[u] \end{tikzcd},$$

and notice that

$$\begin{aligned} B \otimes_A B &\cong A \otimes_{A_0} A \otimes_{A_0} A \\ &\cong A \otimes_{A_0} 1 \otimes_{A_0} A \oplus A \otimes_{A_0} X \otimes_{A_0} A \\ &\cong B \oplus B\{2\} \end{aligned}$$

One checks readily that the total complex decomposes into:

$$\begin{array}{ccccccccc} & & 0 & \longrightarrow & B & \longrightarrow & B & \longrightarrow & 0 \\ & & & & \oplus & & & & \\ 0 & \longrightarrow & B & \longrightarrow & B & \longrightarrow & 0 & & \\ & & & & \oplus & & & & \\ & & 0 & \longrightarrow & \underbrace{A}_{\text{hom. deg } 0} & \longrightarrow & 0 & & \end{array}$$

with the first 2 summands contratible complexes of free right A-modules. The lemma follows. $\square$

Rouquier complexes

Now let $A = \Bbbk[x_1, x_2]$. The transposition φ interchanging x_1, x_2 acts on A as an endomorphism. We define A' to be the φ-fixed subring of A, i.e. $A' = \Bbbk[x_1+x_2, x_1x_2]$. Similarly, we define $A^- \triangleq {}_A A_{\varphi(A)}$ ($= A^\varphi$ in the previous notation). Since we assumed that $\mathrm{char}\,\Bbbk \neq 2$, $A = \Bbbk[x_1+x_2, x_1-x_2]$ and $A^- = \Bbbk[x_1+x_2] \otimes_\Bbbk \Bbbk[x_1-x_2]^-$. We set

$$B = A \otimes_{A'} A,$$

and similar as in our toy model, we have:

$$0 \longrightarrow A^- \longrightarrow B \xrightarrow{m} A \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{j} B \longrightarrow A^- \longrightarrow 0$$

where $j(1) \triangleq (x_1 - x_2) \otimes 1 + 1 \otimes (x_1 - x_2)$. (One can think of $x_1 - x_2$ as the x in the toy example, and tensor everything with $\Bbbk[x_1+x_2]$).

More generally, we let $A = \Bbbk[x_1, \cdots, x_n]$ and let the symmetric group S_n act on A by permuting x_i's. We will make A graded by assigning $\deg x_i = 2$. Define for each $s_i = (i, i+1)$,

$$\begin{aligned} A^i &\triangleq s_i\text{-invariants in } A \\ &= \Bbbk[x_1, \cdots, x_{i-1}, x_i + x_{i+1}, x_i x_{i+1}, x_{i+2}, \cdots, x_n] \end{aligned}$$

and

$$B_i = A \otimes_{A^i} A\{-1\}$$

so that $\deg 1 \otimes 1 = -1$. We have s.e.s.

$$0 \longrightarrow A_i^- \longrightarrow B_i\{1\} \xrightarrow{m} A \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{j} B_i\{1\} \longrightarrow A_i^- \longrightarrow 0$$

Lemma 2. (Soergel):

(1). $B_i \otimes_A B_i \cong B_i\{1\} \oplus B_i\{-1\}$

(2). $B_i \otimes_A B_j \cong B_j \otimes_A B_i$ if $|i-j|>1$

(3). $B_i \otimes_A B_{i+1} \otimes_A B_i \cong B_{i,i+1} \oplus B_i$

$B_{i+1} \otimes_A B_i \otimes_A B_{i+1} \cong B_{i,i+1} \oplus B_{i+1}$,

where $B_{i,i+1} = A \otimes_{A^{i,i+1}} A\{-3\}$, and $A^{i,i+1}$ is the ring of invariants in A under permutations s_i, s_{i+1}.

Proof omitted. But note that (3) implies that:

$$B_i \otimes_A B_{i+1} \otimes_A B_i \oplus B_{i+1} \cong B_{i+1} \otimes_A B_i \otimes_A B_{i+1} \oplus B_i$$

□

Similar as in the toy model case, we introduce complexs for each $1 \leq i \leq n-1$:

$$R_i: \ 0 \longrightarrow B_i\{1\} \longrightarrow A \longrightarrow 0$$

$$R_i': \ 0 \longrightarrow A \longrightarrow B_i\{-1\} \longrightarrow 0$$

Prop. We have:

(1). $R_i \otimes_A R_i' \cong A$

(2). $R_i \otimes_A R_j \cong R_j \otimes R_i$ if $|i-j|>1$

(3). $R_i \otimes_A R_{i+1} \otimes_A R_i \cong R_{i+1} \otimes_A R_i \otimes_A R_{i+1}$.

The proof is due to Rouquier but an elementary argument can be found in B. Elias and D. Krasner, Rouquier Complexes Are Functorial over Braid Cobordisms. Part (1) and (2) follows readily from the lemma. For (3), one shows that both sides of the identity are homotopic to the total complex of:

$$\begin{tikzcd}[column sep=small] & & B_i \otimes_A B_{i+2}\{2\} \arrow[r] \arrow[dr] & B_i\{1\} \arrow[dr] & & \\ 0 \arrow[r] & B_{i,i+1}\{3\} \arrow[ur] \arrow[dr] & & & A \arrow[r] & 0 \\ & & B_{i+1} \otimes_A B_i\{2\} \arrow[r] \arrow[ur] & B_{i+1}\{1\} \arrow[ur] & & \end{tikzcd}$$

$\square$

Soergel's theorem

Consider the following monoidal category (Bott - Samuelson category) $\mathcal{B}SC_n$ $(n \geq 1)$, whose:

(1). object: Tensor products of B_i's with grading shifts, and direct sums of these.

(2). morphisms: degree 0 bimodule maps.

For instance, if $n=1$, $\mathcal{B}SC_1$ is generated by

$$A, \quad B, \quad B\otimes_A B$$

with grading shifts, direct sums.

$\mathcal{B}SC_n$ is an additive category, whose Karoubi envelope is what we need.

Recall that, for any additive category $\mathcal{C}$, its Karoubi envelope $Kar(\mathcal{C})$ is the category which has as:

(1). objects: pairs (M, e) where $M \in Ob(\mathcal{C})$ and $e \in Mor_{\mathcal{C}}(M, M)$

s.t. $e^2 = e$ (idempotent)

(2). morphisms : between any two objects (M, e), (M', e'), a morphism is a diagram:

$$\begin{tikzcd} M \arrow[r, "e"] \arrow[d, "f"] \arrow[dr, "f"] & M \arrow[d, "f"] \\ M' \arrow[r, "e'"] & M' \end{tikzcd}$$

(i.e. $f = e'fe \in \mathrm{Mor}_{\mathcal{C}}(M, M')$).

$\mathrm{Kar}(\mathcal{C})$ is also known as the idempotent completion of $\mathcal{C}$. For abelian categories $\mathcal{A}$, $\mathrm{Kar}(\mathcal{A}) \cong \mathcal{A}$. If $\mathcal{C}$ is additive, monoidal, then so is $\mathrm{Kar}(\mathcal{C})$.

Def. The Soergel category SC_n $(n \geq 1)$ is defined to be $\mathrm{Kar}(\mathcal{BSC}_n)$, the Karoubi envelope of $\mathcal{BSC}_n$.

Thm. (Soergel) The split Grothendieck group $K_0(SC_n)$ is a unital, associative ring over $\mathbb{Z}[q, q^{-1}]$, with unit $[A]$, and it's generated by $b_i = [B_i]$ $1 \leq i \leq n-1$, subject to relations:

$$\begin{cases} b_i^2 = (q + q^{-1}) b_i \\ b_i b_j = b_j b_i & |i-j| > 1 \\ b_i b_{i+1} b_i + b_{i+1} = b_{i+1} b_i b_{i+1} + b_i . \end{cases}$$

□

Def. The $\mathbb{Z}[q, q^{-1}]$ algebra $H_n(q)$ is the algebra generated by b_i's $1 \leq i \leq n-1$, subject to the above relations.

Thus Soergel's thm. says that there is an isomorphism of rings

$$\begin{aligned} H_n(q) &\longrightarrow K_0(SC_n) \\ b_i &\longmapsto [B_i] \end{aligned}$$

For the Hecke algebra $H_n(q)$, one usually picks another set of generators $T_i = q b_i - 1$, $1 \leq i \leq n-1$, which satisfies the relations:

$$\begin{cases} T_i^2 = (q^2-1)T_i + q^2 \\ T_i T_j = T_j T_i \quad \text{if } |i-j|>1 \\ T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \end{cases}$$

The first relation, rewritten in another way, says that

$$(T_i - q^2)(T_i+1) = 0.$$

This says that $H_n(q)$ is a deformation of the group algebra $\mathbb{Z}[S_n]$ $(q=1)$.

Via Soergel's thm, we have a dictionary:

$$SC_n \xrightarrow{\quad K_0 \quad} H_n(q)$$

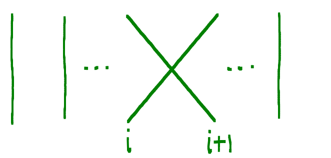
Category of finite dim'l graded vector spaces	$\longmapsto$	The ground ring $\mathbb{Z}[q,q^{-1}]$
A	$\longmapsto$	1
B_i	$\longmapsto$	b_i
$R_i \otimes_A -$	$\longmapsto$	$T_i = q b_i - 1$
$R_i' \otimes_A -$	$\longmapsto$	$T_i^{-1} = q^{-1} b_i - 1$

Remark: $D(A\text{-mod})$ v.s. $\mathrm{Com}(A\text{-mod})$

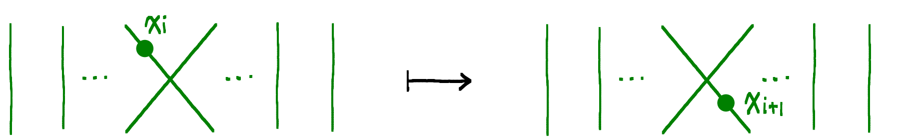
$\mathrm{Com}(SC)$ acts on both $D(A\text{-mod})$ and $\mathrm{Com}(A\text{-mod})$. But recall that on $D(A\text{-mod})$, $R_i \cong A_i^-$ up to a grading shift, so that

$$R_i^2 \cong A$$

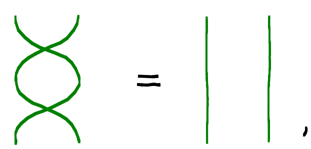
modulo grading shift. What ${}_A(A_i^-)_A$ does is that, when x_i / x_{i+1} passes through it, they get switched, while x_j $(j \neq i, i+1)$ are unaltered. Thus graphically, we can depict the action of $A_i^- \otimes_A -$ as:



which means when x_i passes through the crossing, it becomes x_{i+1}:



Then $R_i^2 \cong A$ just says that, locally, we have:



which says that this is almost the same as the symmetric group action.

On the other hand, the s.e.s.

$$0 \longrightarrow A_i^- \longrightarrow B_i\{1\} \xrightarrow{m} A \longrightarrow 0$$

$$0 \longrightarrow A \xrightarrow{j} B_i\{1\} \longrightarrow A_i^- \longrightarrow 0$$

leads to d.t.'s in $\mathrm{Com}(SC_n)$:

$$A_i^- \longrightarrow B_i\{1\} \longrightarrow A \longrightarrow A_i^-[1]$$

or equivalently,

$$A[-1] \longrightarrow A_i^- \longrightarrow B_i\{1\} \longrightarrow A$$

so that

$$B_i\{1\} = \mathrm{Cone}(A[-1] \longrightarrow A_i^-).$$

Since $A \otimes_A -$ is the identity functor, which can be depicted by the local picture:

$$\Big| \;\; \Big| \, ,$$

this says that $B_i\{1\}$ is the cone:

$$B_i\{1\} = \mathrm{Cone}(\,|\,|\cdots|\,|\cdots|\,|\; \longrightarrow \; |\,|\cdots \times \cdots|\,|\,).$$

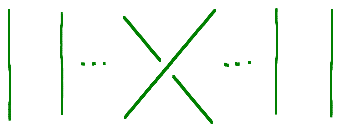
Thus R_i is the cone of the natural quotient map of complexes:

$$R_i = \left[\begin{array}{c} [\,0 \longrightarrow |\,|\cdots|\,|\cdots|\,| \longrightarrow |\,|\cdots \times \cdots|\,| \longrightarrow 0\,] \\ \downarrow \\ [\,0 \longrightarrow |\,|\cdots|\,|\cdots|\,| \longrightarrow 0\,] \end{array} \right].$$

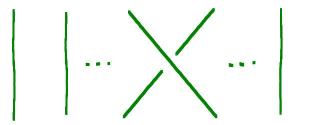
The two copies of identity functors get canceled out in the derived category, but not so in the homotopy category. In this sense, we say that:

> The action of R_i on $\mathrm{Com}(A\text{-mod})$ is a "homological quantization" of the symmetric group action on $D(A\text{-mod})$.

Thus the R_i / R_i' action on $\mathrm{Com}(A\text{-mod})$ represents a genuine braiding:



R_i



R_i'

This phenomenon of "homological quantization" also occurs in matrix factorizations.

Problems:

(1). It's known that the braid group action on $\mathrm{Com}(A\text{-mod})$, but the proof uses very sophisticated methods of geometric representation theory. One can try to find a more elementary topological proof.

(2). The positive crossing is represented by positive complex

$$R_i: \quad 0 \longrightarrow B_i\{1\} \longrightarrow A \longrightarrow 0,$$

where $B_i\{1\}$ sits in homological degree 0. It's an open question how the homological positivity is related to the positivity in knot theory.

More on Hecke algebras

In the previous subsection, we defined the Hecke algebra, and now we recall some basic facts about them, with SC aiding us explaining.

Recall that $H_n(q)$ is the $\mathbb{Z}[q,q^{-1}]$-algebra generated by T_i, $1 \le i \le n-1$, subject to relations:

$$\begin{cases} T_i^2 = (q^2-1)T_i + q^2 \\ T_iT_j = T_jT_i \quad \text{if } |i-j|>1 \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} \end{cases}$$

As for $\mathbb{Z}[S_n]$, the third relation implies that $H_n(q)$ has as an obvious basis:

$$\{T_\omega \triangleq T_{i_1}\cdot T_{i_2}\cdots T_{i_r} \mid \omega\in S_n,\ \omega=s_{i_1}\cdots s_{i_r} \text{ is a reduced expression}\}$$

But it has another more intrinsic basis, defined as follows. For each $\omega\in S_n$, choose a reduced expression $\omega=s_{i_1}\cdots s_{i_r}$ for it. Then there is a unique indecomposable summand B_ω inside $B_{i_1}\otimes_A\cdots\otimes_A B_{i_r}$ such that it doesn't appear as a summand in any $B_{j_1}\otimes_A\cdots\otimes_A B_{j_s}$ for $s<r$.

For instance, when $n=3$, all B_ω for $\omega\in S_3$ are:

$$R,\ B_1,\ B_2,\ B_1\otimes_A B_2,\ B_2\otimes_A B_1,\ B_1\otimes_{A^{1,2}} B_2.$$

Def. The Kazhdan-Lusztig basis $\{C_\omega \mid \omega\in S_n\}$ are the images

$$C_\omega \triangleq [B_\omega] \in K_0(SC_n) = H_n(q).$$

By our def., we have, $\forall\, \omega,\omega'\in S_n$

$$B_\omega\otimes_A B_{\omega'} = \bigoplus_{\omega''\in S_n} B_{\omega''}^{\oplus f_{\omega,\omega'}^{\omega''}}$$

where $f_{\omega,\omega'}^{\omega''}\in\mathbb{Z}_+[q,q^{-1}]$, so that on the Grothendieck group level, we have:

$$C_\omega\cdot C_{\omega'} = \sum_{\omega''} f_{\omega\omega'}^{\omega''} C_{\omega''}.$$

Next we define two operators on $H_n(q)$:

Def. (1). The involution ω: It's the q-antilinear, antihomorphism defined by the properties:

$$\begin{cases} \omega(b_i) = b_i \\ \omega(x \cdot y) = \omega(y)\omega(x), \quad \forall\, x, y \in H_n(q). \end{cases}$$

(2). The trace map $\varepsilon : H_n(q) \longrightarrow \mathbb{Z}[q, q^{-1}]$: It's the q-linear map characterized by, $\forall\, w \in S_n$

$$\varepsilon(T_w) = \begin{cases} 1 & w = 1 \\ 0 & w \neq 1. \end{cases}$$

Using these two maps, we can define a semi-linear on $H_n(q)$:

Def. (Semi-linear form): $\forall\, x, y \in H_n(q)$

$$(x, y) \triangleq \varepsilon(\omega(x) y).$$

It's q-anti-linear in x, and q-linear in y. One easily verifies that,

$$\begin{cases} (1, 1) = 1 \\ (1, b_{i_1} \cdots b_{i_m}) = q^m \quad \text{if } i_1 < \cdots < i_m \end{cases}$$

Rmk: Why do we want a semi-linear form?
When categorifying a ring R acting on a module V. It helps to have a semilinear form on V as above. After categorification, the semi-linear form becomes

$$([X],[Y]) = gr.rk(\oplus_{j\in\mathbb{Z}} \mathrm{Hom}(X, Y\{j\}))$$

where X, Y are 1-morphisms in the categorified category $\mathcal{U}$ of V, and adjoint maps with respect to this bilinear form are lifted to biadjoint 2-morphisms in $\mathcal{R}$ which categorifies R.

Another feature of the bilinear form is that, usually if one specialize $q=1$, it becomes a boring bilinear form, but the graded version:

$$\mathrm{Hom}(X, Y\{1\}) = \mathrm{Hom}(X, Y)\{1\} = \mathrm{Hom}(X\{-1\}, Y)$$

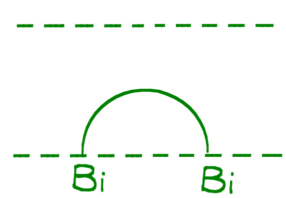
says that the boring bilinear form, after q-deformation, becomes q-semi-linear. A q-anti-linear involution allows us to switch from q-semi-linear to q-bilinear by setting

$$(x, y)' \triangleq (\omega(x), y).$$

Graphical presentation of Soergel category

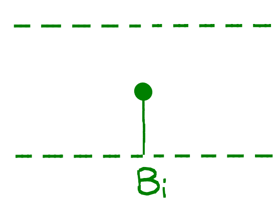
In this subsection, we will use string diagrams of §7 to give a graphical depiction of SC_n. This is the joint work: B. Elias, M. Khovanov - Diagramatics for Soergel Categories.

First off, $B_i \otimes_A -$ is a self-adjoint joint operator on A-mod, which we depict by a cap:

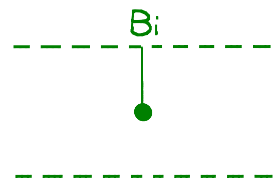


The maps of A-bimodules, or rather, 2-morphisms of functors, will be represented by:

$$B_i \longrightarrow A$$

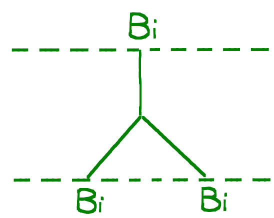


$$A \longrightarrow B_i$$

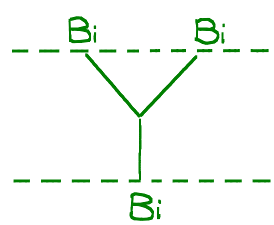


The morphisms:

$$\begin{array}{ccc} B_i \otimes_A B_i \cong A \otimes_{A^i} A \otimes_{A^i} A & \longrightarrow & B_i \\ f \otimes 1 \otimes g & \longmapsto & 0 \\ f \otimes x \otimes g & \longmapsto & f \otimes g \end{array}$$

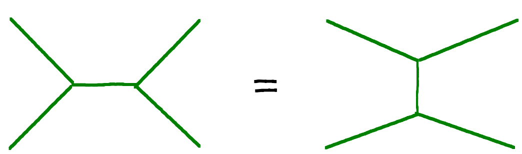


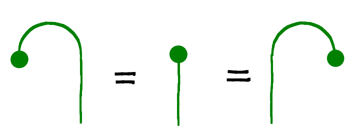
$$\begin{array}{ccc} B_i & \longrightarrow & A \otimes_{A^i} A \otimes_{A^i} A \cong B_i \otimes_A B_i \\ f \otimes g & \longmapsto & f \otimes x \otimes g \end{array}$$

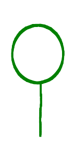


are depicted by trivalent vertices.

The fact that B_i is a Frobenius algebra object over A^i gives us the graphical relations:







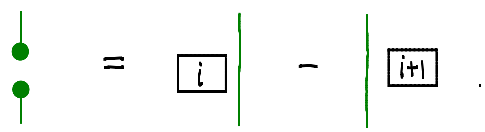
$= \quad 0$

Since A is commutative, multiplication by any elements of A gives endomorphisms on any functor on A-mod. We will depict by drawing a box labeled by elements of A this induced endomorphism:

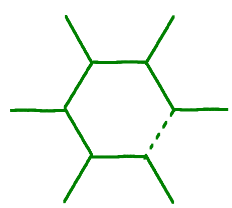
$$\boxed{a} \qquad a \in A$$

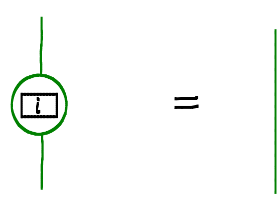
In particular, one checks readily from the def. of B that the closed pictures:

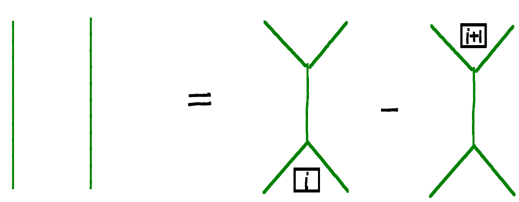
 $= \boxed{i} - \boxed{i+1}$,



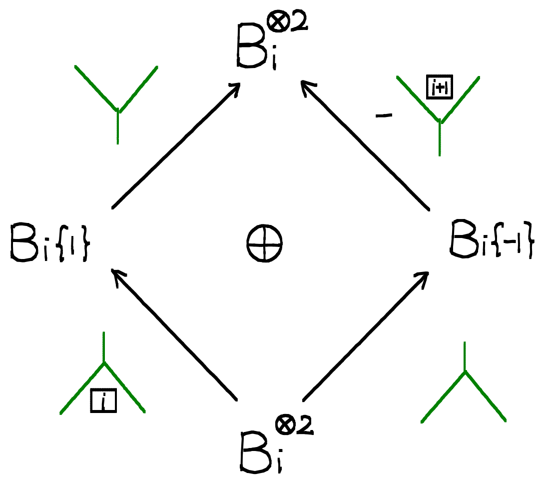
where we denote x_i by a box just labeled i. One can check that (exercise):

 $= 0$





The last two equations imply $B_i \otimes_A B_i \cong B_i\{1\} \oplus B_i\{-1\}$ by setting up maps:



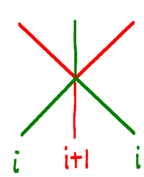
The above relations and their twists by adjunctions give all relations of one "color" i. For adjacent colors i, $i+1$, recall from lemma 2 that we have:

$$\begin{cases} B_i \otimes_A B_{i+1} \otimes_A B_i \cong B_{i,i+1} \oplus B_i \\ B_{i+1} \otimes_A B_i \otimes_A B_{i+1} \cong B_{i,i+1} \oplus B_{i+1}, \end{cases}$$

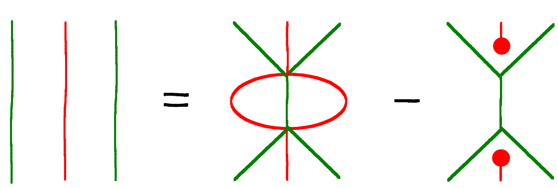
We will depict the composition of the projection $B_i \otimes_A B_{i+1} \otimes_A B_i$ onto $B_{i,i+1}$ and the inclusion of $B_{i,i+1}$ back into $B_{i+1} \otimes_A B_i \otimes_A B_{i+1}$

$$B_i \otimes_A B_{i+1} \otimes_A B_i \longrightarrow B_{i,i+1} \longrightarrow B_{i+1} \otimes_A B_i \otimes_A B_{i+1}$$

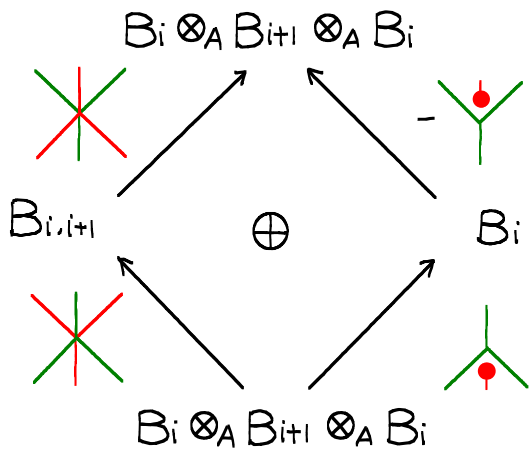
by a 6-valent vertex



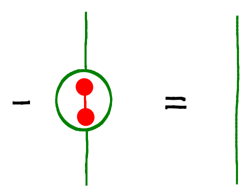
Then it satisfies the relation:



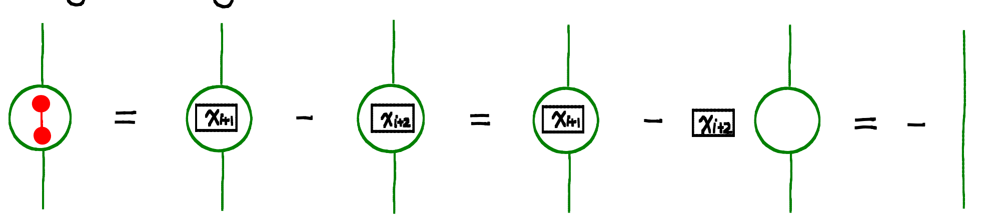
which implies the decomposition $B_i \otimes_A B_{i+1} \otimes_A B_i \cong B_{i,i+1} \oplus B_i$ as for $B_i \otimes_A B_i \cong B_i\{1\} \oplus B_i\{-1\}$.



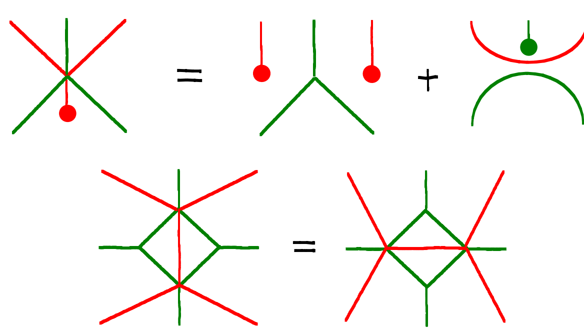
Here we need to use that,



which in turn follows from:



There are more relations about 6 valent vertices such as:

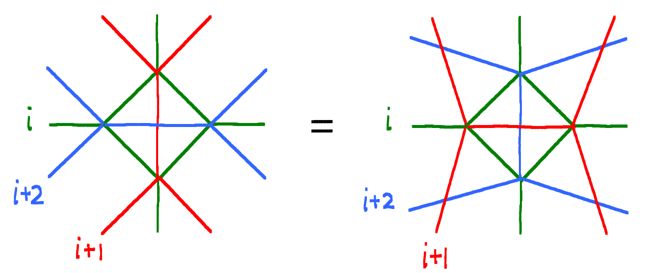


etc.

See the above mentioned paper for all of them.

Next, lines for far away i, j $(|i-j|>1)$ can cross each other at will.

Then the most complicated relation among $i, i+1, i+2$ is the following:



If we denote the graphical monoidal category $\mathcal{G}S_n$ whose objects are sequences of labels $\underline{i} = (i_1 i_2 \cdots i_m)$, $1 \le i_k \le n$, and whose morphisms are $\mathbb{k}$-linear string diagrams modulo the relations above, then we have:

Thm. (Elias-Khovanov). There is an equivalence of $\mathbb{k}$-linear, graded monoidal categories between $\mathcal{G}S_n$ and $\mathcal{B}S_n$.

Furthermore, on the Grothendieck group level, $\forall\, B, C \in \mathrm{Mor}_{\mathcal{G}S_n}$,

$$([B],[C]) = \mathrm{gr.\,rank}_{A\text{-mod}} \mathrm{Hom}(B, \oplus_{j\in\mathbb{Z}} C\{j\})$$

where $(\ ,\ )$ is the semi-linear form on $H_n(q)$ defined above. $\square$

For the proof, see the above mentioned paper. But what one needs to show that any closed string diagram, as endomorphisms of the identity functor $A \longrightarrow A$, reduces to linear combinations of pictures only consisting of boxes labeled by $i \in \{1, \cdots, n\}$, i.e. polynomials in A.

Extension to braid cobordisms

Recall that we have exhibted a braid group action on $\mathrm{Com}(A\text{-mod})$ by assigning to a crossing the complex of bimodules:

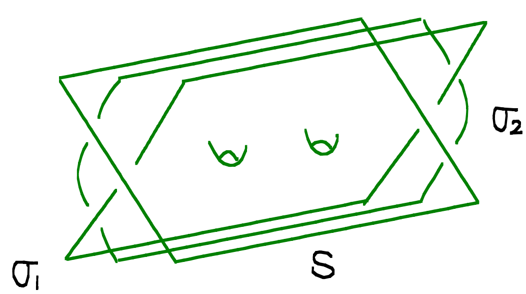
$$R_i : \quad 0 \longrightarrow B_i\{1\} \longrightarrow A \longrightarrow 0 ,$$

In the works:

(1). M. Khovanov, R. Thomas - Braid Cobordisms, Triangulated Categories, and Flag Varieties,

(2). B. Elias and D. Krasner, Rouquier Complexes Are Functorial over Braid Cobordisms,

it's shown that this braid group action extends to braid cobordisms. i.e. to any braid cobordism S between two braids σ_1 and σ_2



we can associate with it a map of chain complexes of A-bimodules

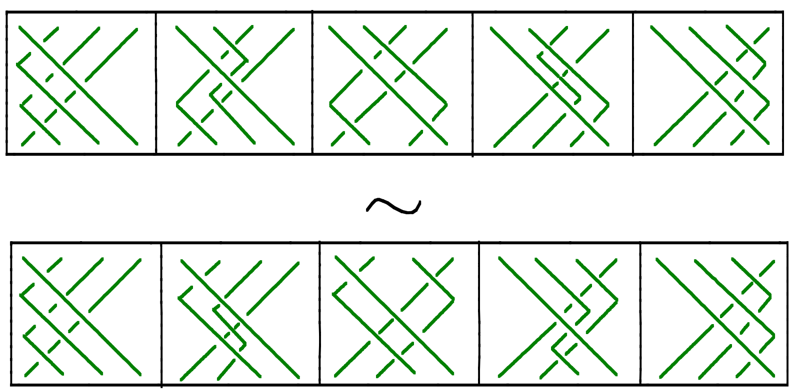
$$F(S) : R_{\sigma_1} \longrightarrow R_{\sigma_2}$$

Then one needs to do a consistency check as we did for the $\mathfrak{sl}_2$ case in §5, i.e. we need to verify that the assignment is invariant under movie moves.

Once again, we have that the ring A can be defined over $\mathbb{Z}$ and the degree 0 part of its center is only $\mathbb{Z}$:

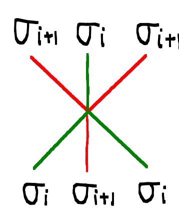
$$Z(A)^0 = A^0 = \mathbb{Z}.$$

The trick in §5 applies to reduce the consistency check to only a sign issue. For instance, the trick says that the two different paths in the most complicated movie move:

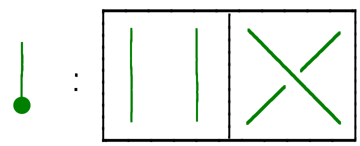


define two maps of chain complexes of A-bimodules up to ± 1. Then the sign issue is easily dealt with if look at only the first few terms of the chain complexes involved. C.f. §5.

Here we also need to mention that in the works of Kamada, Carter-Saito, the same 2D string diagrams are used to represent cobordism of braids in 4D. For instance, they depict by



the cobordism of Reidemeister III move, and they denote the movie move:



which indicates the topology change in the cobordism.

Hoschild homology for Soergel categories

Now given a braid $\sigma = \sigma_{i_1}^{\varepsilon_1} \cdots \sigma_{i_r}^{\varepsilon_r}$ where σ_{i_k} is a simple crossing, we obtain a chain complex of A-bimodules:

$$R_\sigma = R_{\sigma_{i_1}}^{\varepsilon_1} \otimes_A \cdots \otimes_A R_{\sigma_{i_r}}^{\varepsilon_r}$$

($\varepsilon_i \in \{\pm 1\}$, $R_{\sigma_i}^{\varepsilon_i} = R_i$ if $\varepsilon_i = +1$, or R_i' if $\varepsilon_i = -1$). Then one can ask if the Hoschild homology $HH(R_\sigma)$ is an invariant of the braid closure $\hat{\sigma}$.

It turns out $HH(R_\sigma)$ is not that interesting, since it factors through $D(A\text{-bimod})$ so that it's not braided. However, if we write out the chain complex

$$R_\sigma : \ \cdots \longrightarrow R_\sigma^i \xrightarrow{d_i} R_\sigma^{i+1} \longrightarrow \cdots$$

Then each R_σ^i is a graded A-bimodule and the differentials d_i are grading preserving A-bimodule maps. Hence we can take the individual Hoschild homology of R_σ^i and for each $j \in \mathbb{Z}$, define a chain complex:

$$\cdots \longrightarrow HH_j(R_\sigma^i) \xrightarrow{d_i} HH_j(R_\sigma^{i+1}) \longrightarrow \cdots$$

Summing over all j, we get a triply-graded homology theory (Hoschild grading, homological grading, and the interal grading of each R^i_{σ} as a graded A-module), which we denote by $HHH(\sigma)$. We have the following:

Thm. $HHH(\sigma)$ only depends on the braid closure $\tilde{\sigma}$. The Euler characteristic of $HHH(\sigma)$ is the HOMFLY-PT polynomial of the braid closure. $\square$

E.g. The unknot $\bigcirc$ can be regarded as the braid closure of

$$|\,,$$

to which we assign the bimodule $A = \Bbbk[x]$. Using the Koszul resolution of $\Bbbk[x]$:

$$\begin{array}{ccccccccc} 0 \longrightarrow & \Bbbk[x] \otimes \Bbbk[x] & \longrightarrow & \Bbbk[x] \otimes \Bbbk[x] & \xrightarrow{m} & \Bbbk[x] & \longrightarrow 0 \\ & 1 & \longmapsto & 1 \otimes x - x \otimes 1 & & & \end{array}$$

we can see that $HHH(|)$ is just

$$\cdots \longrightarrow 0 \longrightarrow \Bbbk[x] \xrightarrow{0} \Bbbk[x] \longrightarrow 0 \longrightarrow \cdots$$

so that

$$\chi(HHH(|)) = \sum_{i=0}^{\infty} q^{2i} - t \cdot \sum_{i=0}^{\infty} q^{2i} = \frac{1-t}{1-q^2}$$

Rmk: At the moment it's not quite clear how to extend $HHH(\sigma)$ to a fully functorial link homology theory. Recall that as for H^n, to have such an extension, we need $H(\bigcirc)$ to be finite dimensional over $H(\phi)$. However, this is clearly not the case as from the above example, $H(\bigcirc)$ is infinite dim'l but $H(\phi) \cong \mathbb{k}$.

Problems:

1). How can we modify the whole theory to make it functorial?
2). Is there a non-braided description? (See Ozsvath - Szabo - Gilmore using singular braids to give a non-braided description).
3). Seek the connection with the topological vertex. (Talk to Melissa Liu or Andrei Okounkov about this).

Koszul resolutions

Finally, we record the bimodule resolutions of A, B_i so that in principle, this allows us to compute $HHH(\sigma)$ for any braid σ.

For $A = \mathbb{k}[x_1, \cdots, x_n]$, we have

$$A \otimes A \xrightarrow{m} A \longrightarrow 0$$

whose kernel is generated by $\langle x_1 - y_1, \cdots, x_n - y_n \rangle \subseteq \mathbb{k}[x_1, y_1, \cdots, x_n, y_n] \cong A \otimes A$. Since A is regular, we can extend it to the Koszul complex

$$A \otimes A \otimes \Lambda[y_1, \cdots, y_n] \longrightarrow A,$$

which resolves A by a complex of free $A \otimes A$-modules.

For B_i, let $\alpha_1 = (x_1 + x_2) \otimes 1 - 1 \otimes (x_1 + x_2)$, $\alpha_2 = x_1 x_2 \otimes 1 - 1 \otimes x_1 x_2$.

Then the total complex of the following diagram resolves B:

$$\begin{tikzcd}
A \otimes A \arrow[r, "\alpha_1"] & A \otimes A \\
A \otimes A \arrow[u, "\alpha_2"] \arrow[r, "-\alpha_1"] & A \otimes A \arrow[u, "\alpha_2"]
\end{tikzcd}$$