§13. Hopf Algebras

We will be using G. Kuperberg's graphical notation. Two basic references are his papers:

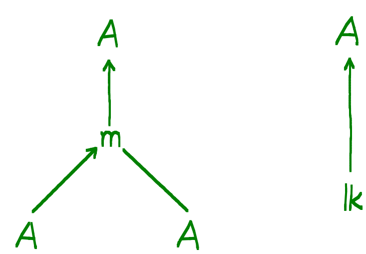
[1]. G. Kuperberg, Involutory Hopf Algebras and 3-Manifold Invariants

[2]. G. Kuperberg, Non-involutory Hopf Algebras and 3-Manifold Invariants

Algebras and coalgebras

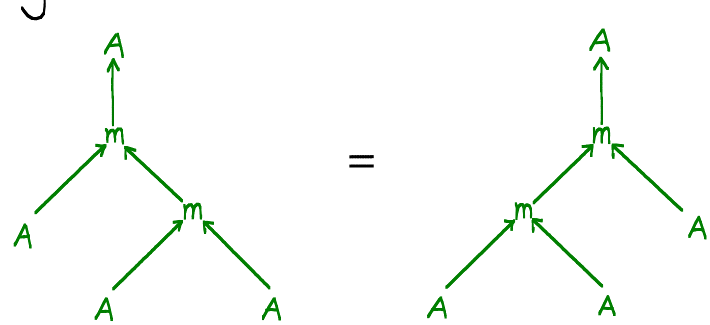
We will use a graphical notation to recall their definitions.

Def. A $\Bbbk$-algebra over a ground field $\Bbbk$ is a $\Bbbk$-vector space A equipped with linear maps m, i:



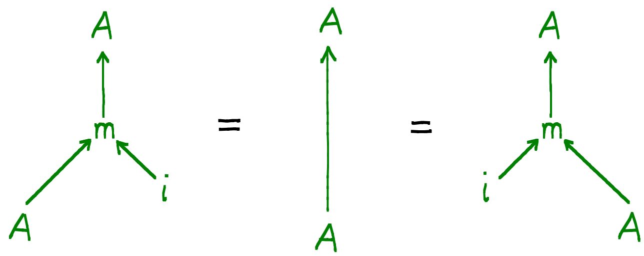
satisfying the axioms:

(1). Associativity:

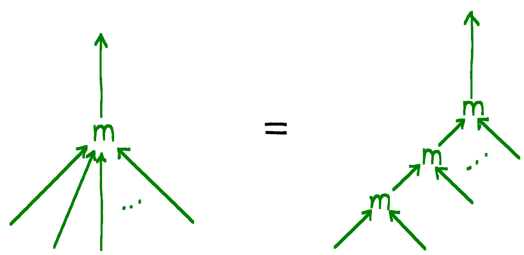


,

(2). Unit:



We will stop writing A on the ends in what followes. Note that associativity allows us to define unambigously:

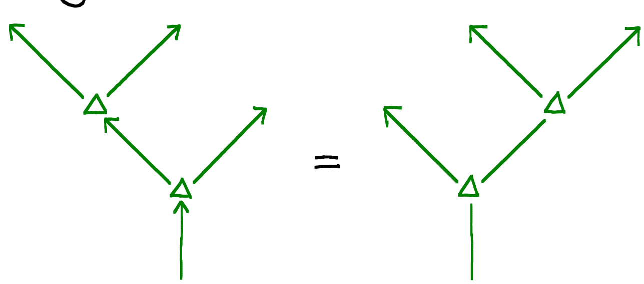


Reversing all the arrows, we obtain the definition of a coalgebra.

Def. A $\mathbb{k}$-linear coalgebra C is a $\mathbb{k}$-vector space C equipped with linear maps

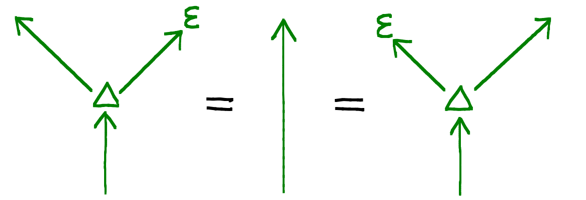
satisfying the axioms:

(1). Co-associativity

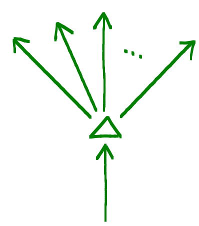


,

(2). Co-unit:



We will also define the unambigous notation:



There is an obvious duality between finite dimensional algebras and coalgebras by taking the vector space duals. This is, however, not the case when $\dim_{\Bbbk} A = \infty$. We only have:

$$\begin{array}{ccc}
\text{Co-algebras} & \longrightarrow & \text{Algebras} \\
C & \longmapsto & C^* \\
(C \xrightarrow{\Delta} C \otimes C) & \longmapsto & ((C \otimes C)^* \xrightarrow{\Delta^*} C^*) \\
(C \xrightarrow{\varepsilon} \Bbbk) & \longmapsto & (\Bbbk \xrightarrow{\varepsilon^*} C^*)
\end{array}$$

$$(C\otimes C)^* \supseteq C^* \otimes C^* \quad \text{(by restriction)}$$

Problem occurs when we dualize $m : A \otimes A \to A$:

$$m^* : A^* \longrightarrow (A \otimes A)^* \not\supseteq A^* \otimes A^*$$

People usually bypass the problem by restricting to the subspace $A^\circ \subseteq A^*$ (restricted dual), which is defined by the condition that

$$A^\circ = \{ \varphi \in A^* \mid m^*(\varphi) \in A^* \otimes A^* \}$$

One can show that A° consists of φ s.t. φ vanishes on a finite codim'l ideal of I, and $A^\circ \otimes A^\circ = (A \otimes A)^\circ$. Hence $m^*(\varphi) \in A^\circ \otimes A^\circ$ if $\varphi \in A^\circ$.

E.g. $n \times n$-matrix coalgebra.
This is the vector space dual of the usual $n \times n$-matrix algebra. Thus $C \cong \Bbbk[C_{ij}]_{i,j=1}^{n}$,

$$\Delta(C_{ij}) = C_{ik} \otimes C_{kj}$$
$$\varepsilon(C_{ij}) = \delta_{ij}$$

E.g. Let $\mathfrak{g}$ be a Lie algebra, and $A = \mathcal{U}(\mathfrak{g})$ its universal enveloping algebra. Its restricted dual

$$A^\circ = \bigoplus_{i \in I} (V_i \otimes_{\Bbbk} V_i^*)$$

where I is a complete set of isomorphism classes of finite dim'l $\mathfrak{g}$ representation. A° is equipped with the direct sum coalgebra structure of each matrix coalgebra $V_i \otimes V_i^*$.

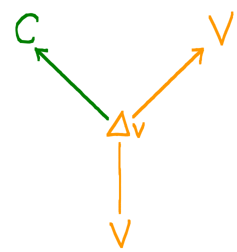
Rmk: Any coalgebra is a union of its finite dimensional sub-coalgebras: Take any $c \in C$,

$$\Delta(c) = \sum c_{(1)} \otimes c_{(2)}$$

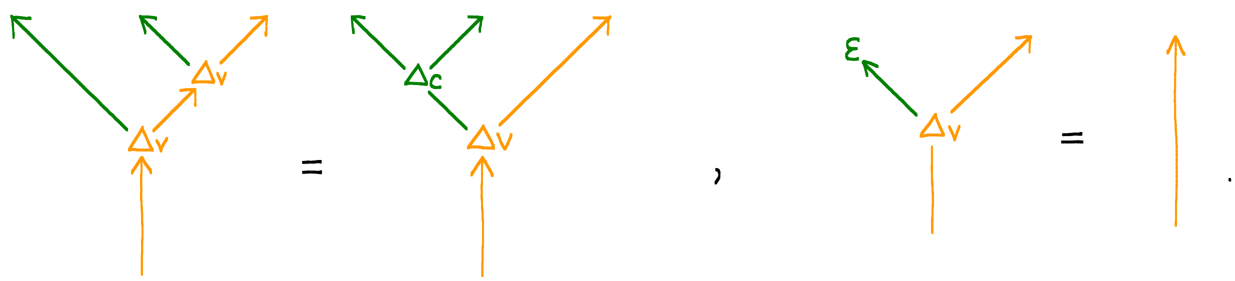
(Sweedler's notation). One can check that the subspace spanned by $\langle c_{(1)}, c_{(2)} \rangle$ is a sub-coalgebra.

Next, we introduce a dual categorical notion of a module (rep.) over an algebra.

Def. Let C be a coalgebra. A left comodule V over C is a $\mathbb{k}$-vector space V equipped with a linear map:



such that:



We list some standard facts about the category of comodules over a coalgebra C : C-Comod:

(1). C-Comod is an abelian category which admits infinite direct sums (not nessarily infinite direct products).

(2). C itself is an injective C-comodule.

(3). Any comodule embeds into an injective comodule (enough injectives). But there is not always enough projectives.

(4). Similar as that any finite dimensional algebra A has a smallest Morita equivalent algebra A_{basic}, any coalgebra C has a smallest "co-Morita equivalent" coalgebra C_{basic}, whose simple subcoalgebras are duals of division algebras.

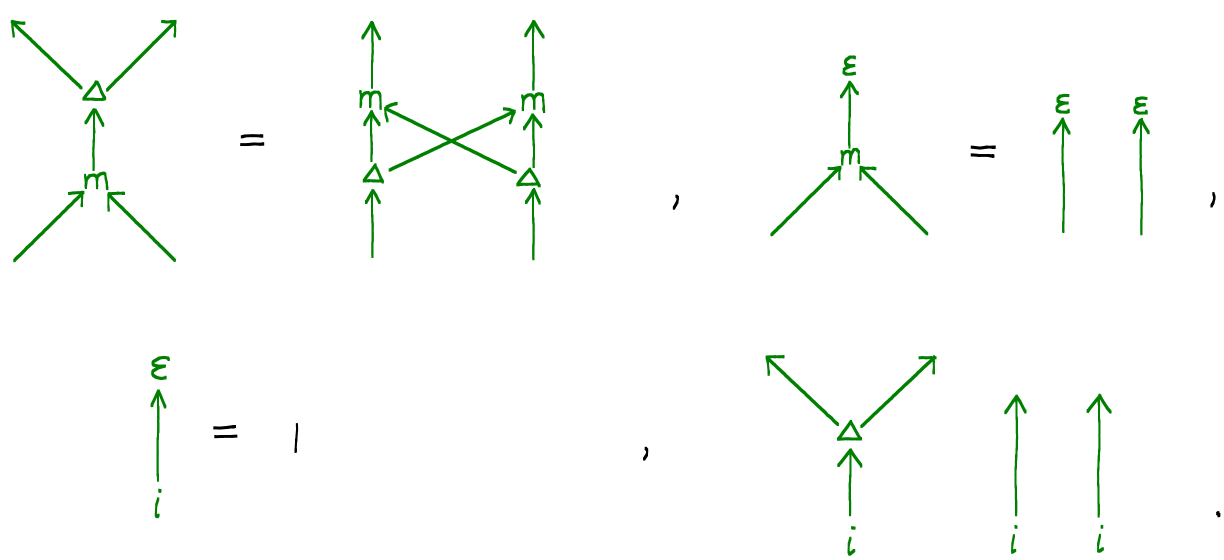
E.g. Another important family of coalgebras are provided by

$$H_*(X, \Bbbk)$$

homology groups of topological spaces, which are also graded.

If we combine the notions of algebra and coalgebra, we obtain that of a bialgebra.

Def. A bialgebra B is a $\Bbbk$-vector space equipped with $\Bbbk$-linear compatible algebra and coalgebra structures. In other words, (Δ, ε) are algebra homomorphisms (this turns out to be equivalent to requiring (M, ι) to coalgebra homomorphisms).



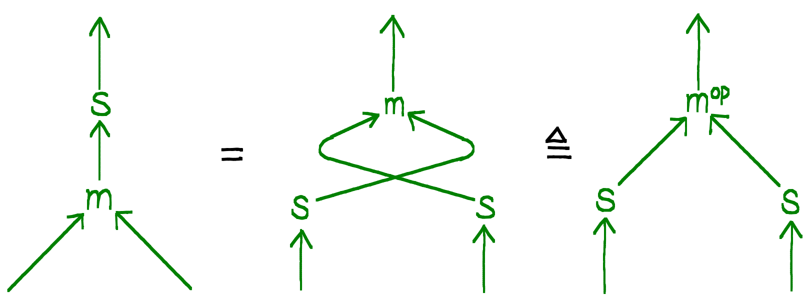
E.g. If G is a semigroup, the semigroup algebra $\Bbbk[G]$ becomes a bialgebra by setting, $\forall g \in G$,

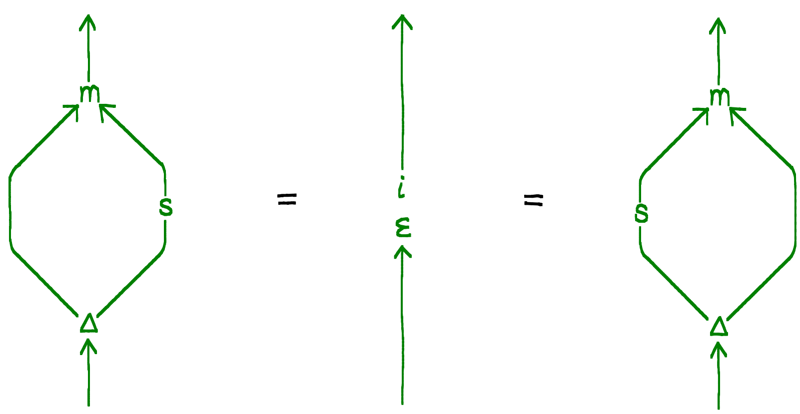
$$\Delta(g) = g \otimes g,$$

$$\varepsilon(g) = 1.$$

The def. of a Hopf algebra is modeled on the above example when G is a group so that $g \in G$ can be inverted.

Def. A Hopf algebra H is a $\Bbbk$-bialgebra equipped with an anti-homomorphism $S : H \longrightarrow H^{op}$:





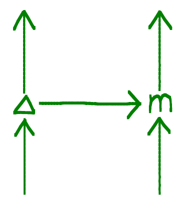
It turns out that these two axioms also imply that S is a coalgebra anti-homomorphism.

From now on, we will focus on Hopf algebras.

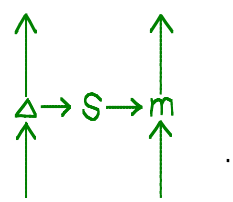
Hopf algebras

Let $(H, m, i, \Delta, \varepsilon, S)$ be a Hopf algebra over $\Bbbk$.

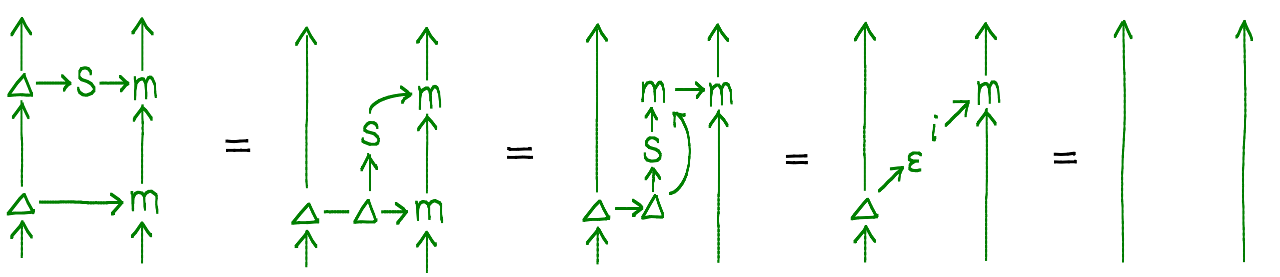
Lemma.1. The following endomorphism of $H^{\otimes 2}$ (called the ladder):



admits a 2-sided inverse:



Pf: We prove one side, and the other side is similar.



$\square$

Now we will start talking about the representation category of H.

Let. V, W be two H-modules. We have:

(1). $V \otimes W$ becomes an H-module via:

$$H \xrightarrow{\Delta} H \otimes H \curvearrowright V \otimes W$$

(2). The ground field $\Bbbk$ becomes an H-module via ε, which we denote by $\underline{\Bbbk}$. $V \otimes \underline{\Bbbk} \cong \underline{\Bbbk} \otimes V \cong V$.

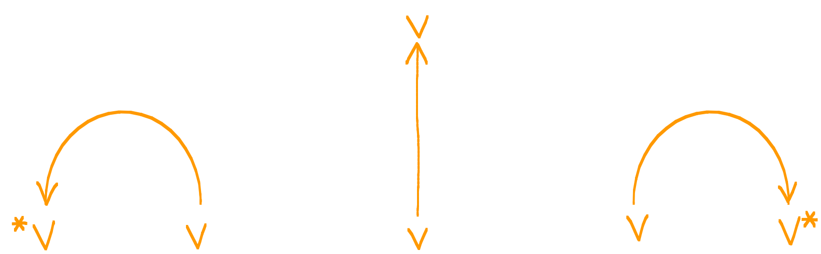
(3). We have internal Hom of H-modules, i.e. $\mathrm{Hom}_{\mathbb{k}}(V,W)$ is an H-module as follows, $\forall h \in H$, $f \in \mathrm{Hom}_{\mathbb{k}}(V,W)$, $x \in V$,

$$(h\cdot f)(v) \triangleq \sum h_{(1)} \cdot (f(S(h_{(2)})\cdot v)).$$

In particular, V^* becomes an H-module by $\forall f \in V^*$, $h \in H$, $v \in V$,

$$(h\cdot f)(v) = f(S(h)\cdot v).$$

In summary, the category H-mod is a $\mathbb{k}$-linear monoidal category with internal Hom's. Furthermore, notice that if S is invertible, we can define another dual ${}^*V$ for H using S^{-1} instead of S in (3). Then it's easy to see that ${}^*V \cong V^*$ iff $\exists\, u \in H$ s.t. $S^2(h) = uhu^{-1}$, $\forall h \in H$. We say in this case that H is reflexive (or rigid) since we are allowed to bend $\mathrm{id}_V : V \longrightarrow V$ in two ways:



Rmk: Similarly, one can define the category of comodules over H. The tensor product of comodules is defined using multiplication of H instead:

$$\left.\begin{array}{l} V \longrightarrow H\otimes V \\ W \longrightarrow H\otimes W \end{array}\right\} \Longrightarrow V\otimes W \longrightarrow V\otimes H\otimes W\otimes H \xrightarrow{m_H} V\otimes W\otimes H$$

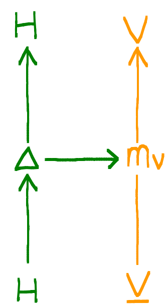
If H is finite dimensional, the comodule category is isomorphic to H^*-mod.

From now on, we will be working with finite dimensional Hopf algebras, so that we can bend arrows freely.

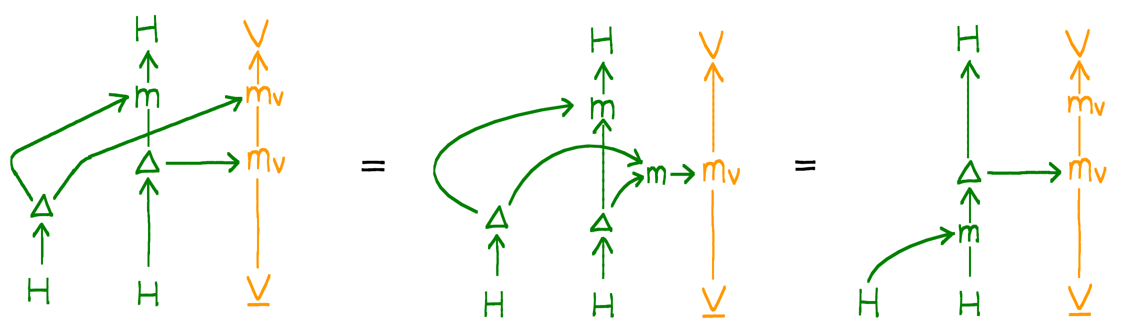
Prop. 2. Let V be any H-module. Then

$$H \otimes V \cong H^{\dim V} \cong H \otimes \underline{V}$$

where $\underline{V}$ denotes the vector space V with trivial H-module structure.
Pf: We will show that, the following map is an H-module map:



Indeed, we have,

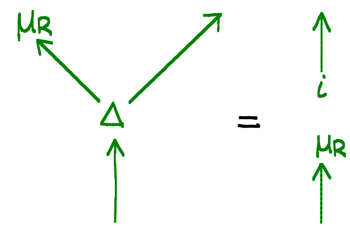


Moreover, this is invertible for the same reason as in lemma 1. □

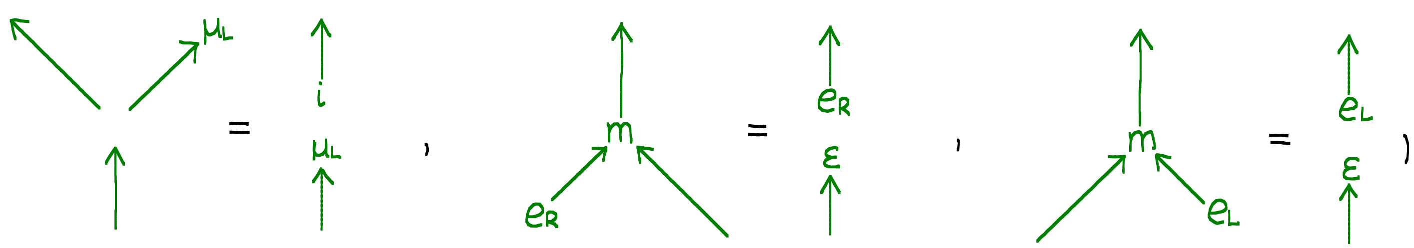
Cor. 3. $P \otimes V$ is a projective H-module if P is. □

Ex. Let C be a coalgebra. Then $I \otimes V$ is an injective comodule whenever I is.

Def. (Integrals for Hopf algebras).
We define a right integral $\mu_R \in H^*$ (resp. left integral $\mu_L \in H^*$, right cointegral $e_R \in H$, left cointegral $e_L \in H$) by the property:



(resp.



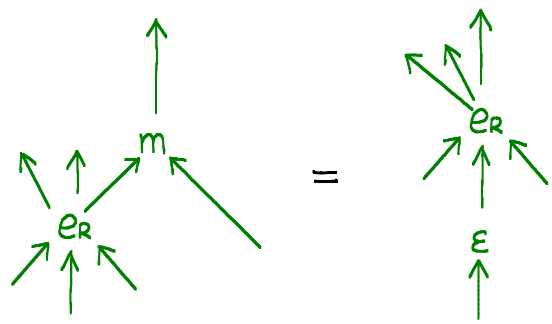
E.g. If $H = \Bbbk[G]$ with G a finite group, we can take

$$e_R = e_L = \sum_{g \in G} g$$

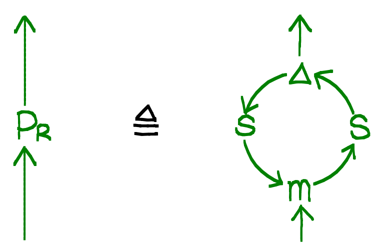
$$\mu_L = \mu_R = \begin{cases} 1 & g = 1 \\ 0 & g \neq 1 \end{cases}$$

Rmk: The def. of a right cointegral says that $\Bbbk e_R$ spans a 1-dim'l right submodule of H. Furthermore, any non-zero element in $\Bbbk e_R$ is also a right cointegral. Later we will see that the space of right cointegrals in H is 1-dim'l. Similar comments apply to left cointegrals, left integrals, left right integrals.

Def. (Generalized integrals) A (generalized) right cointegral is a tensor $e_R \in H^{\otimes r} \otimes (H^*)^{\otimes s}$ satisfying

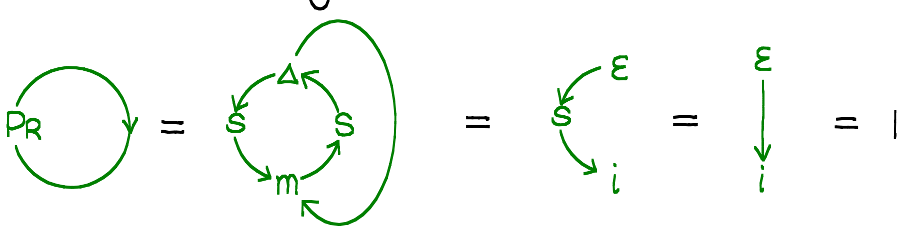


Prop. 4. Define a tensor $P_R \in H \otimes H^*$ by

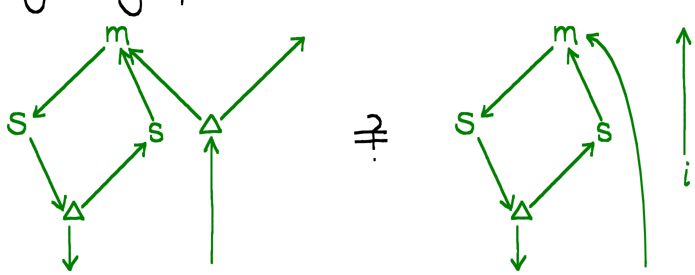


Then P_R is both a right integral and a right cointegral and it has trace 1.

Pf: That it has trace 1 follows as

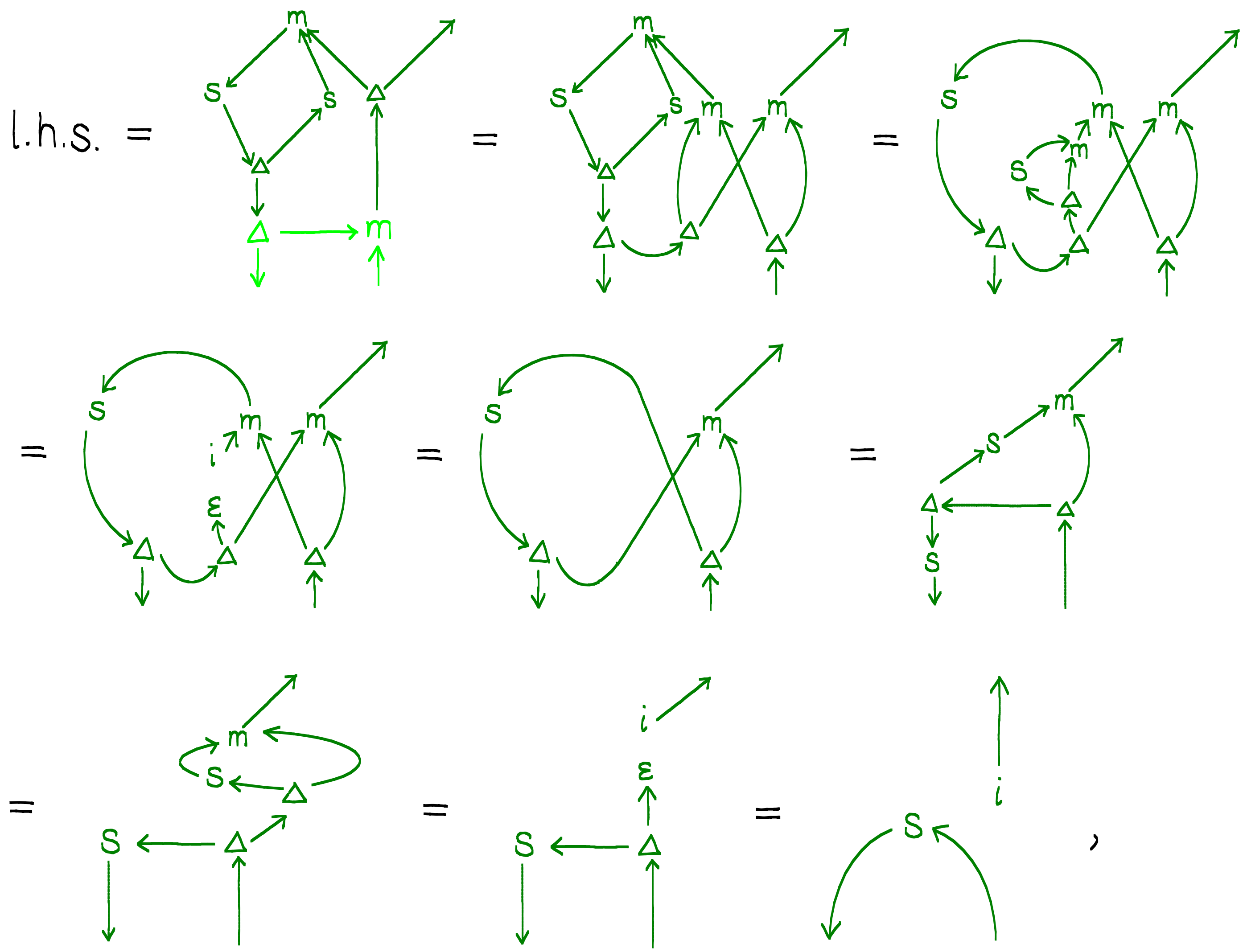


To show that it's a right integral, we use that H is finite dim'l to bend the defining picture:



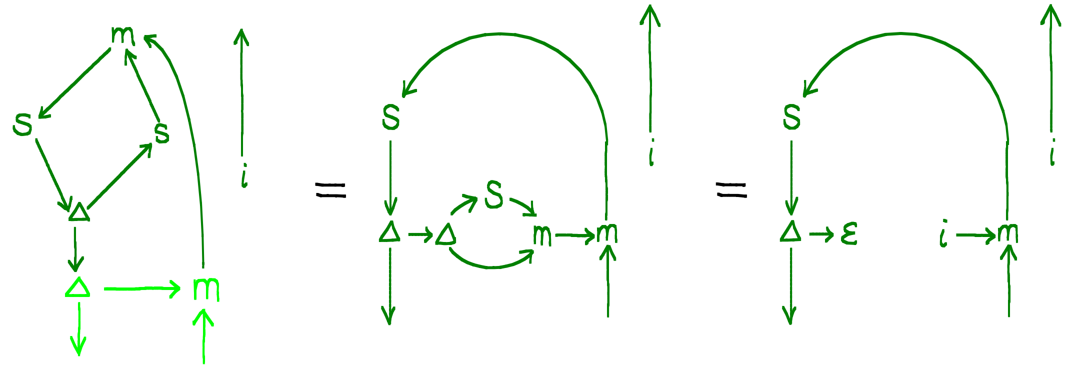
Apply a ladder to both sides, we obtain:

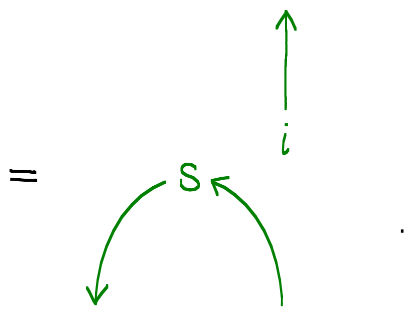
l.h.s. =



while the

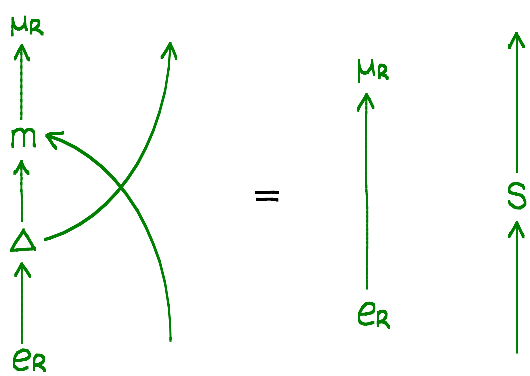
l.h.s. =



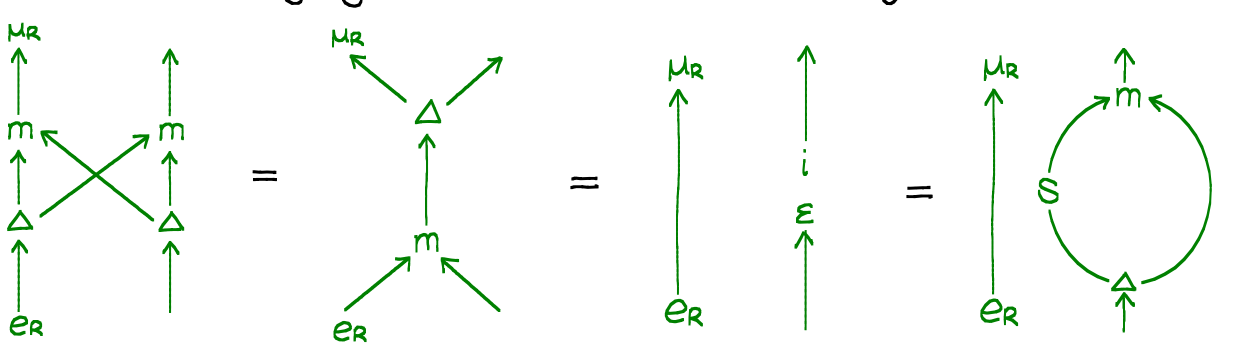


Thus both sides are equal, and that P_R is a right integral follows from lemma 1. The proof that P_R is a right cointegral is similar and left as exercise. □

Lemma 5. Given a right integral μ_R and a right cointegral e_R, we have

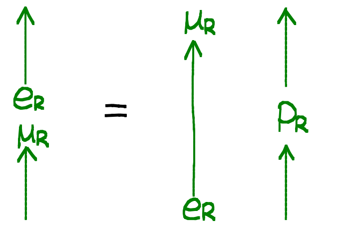


Pf: Indeed, applying a ladder to the l.h.s. gives us



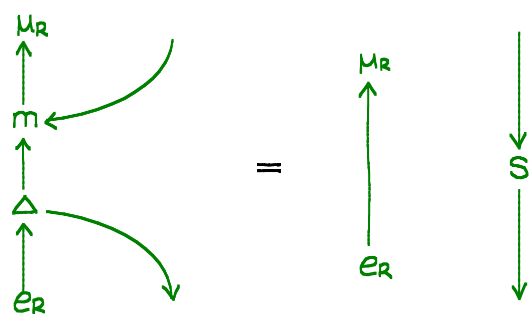
The lemma follows by inverting the ladder (lemma 1). □

Cor 6. (Uniqueness of integrals). Given a right integral μ_R and a right cointegral e_R, we have

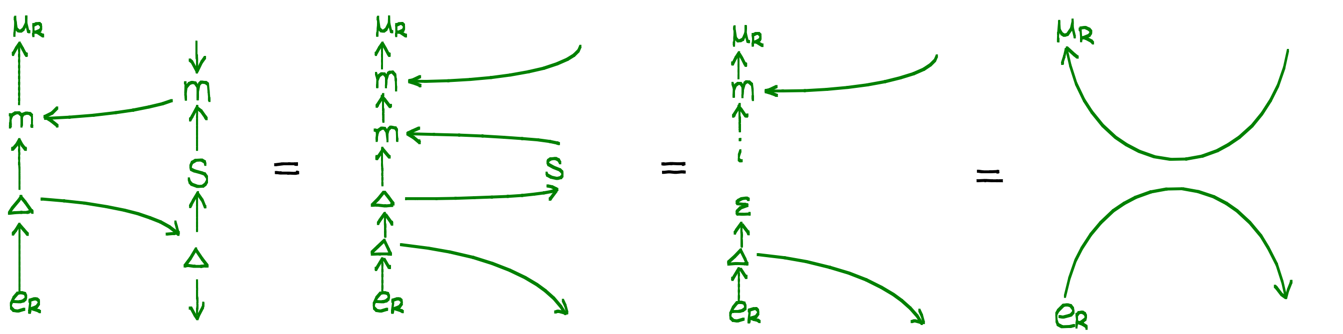


In particular, since the l.h.s. is a non-zero tensor in $H \otimes H^*$, $\mu_R(e_R) \neq 0$.

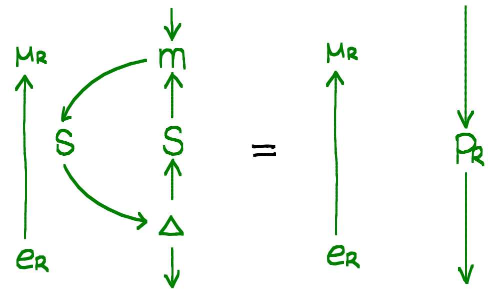
Pf: Undo the crossing on the l.h.s. of the lemma, we have



Adding an inverse ladder to the equality, we obtain:



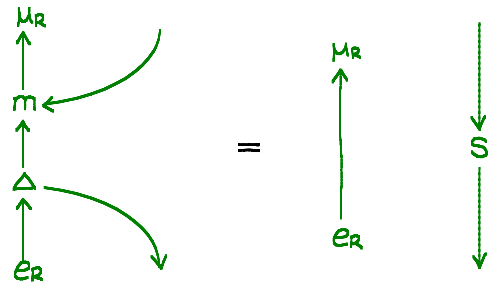
while the r.h.s. becomes



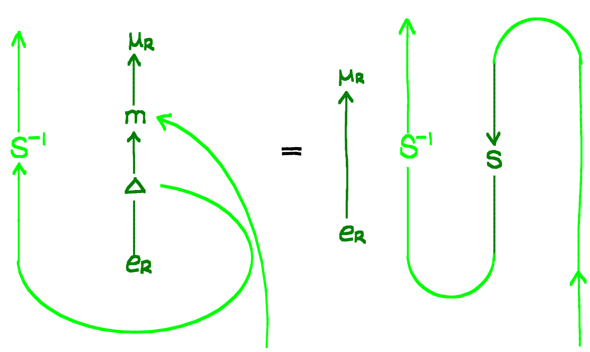
Rotate the pictures 180° gives us the desired result. □

Cor. 7. μ_R defines a non-degenerate trace form on H, so that H is Frobenius.

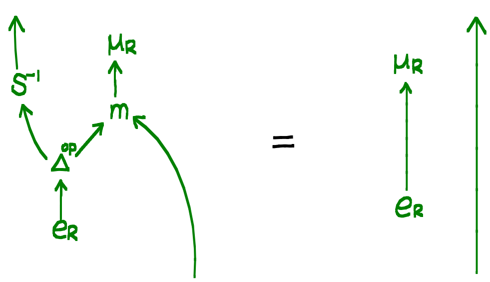
Pf. The relation



gives rise to:



so that



showing that μ_R is non-degenerate form on H. $\square$

We summarize our discussions into the following:

Thm. 8. 1). For any finite dimensional Hopf algebra H, there exist left and right integrals μ_L, μ_R (as well as cointegrals e_L, e_R), which are unique up to rescaling. μ_L, μ_R are non-degenerate bilinear forms on H, so that H is Frobenius.
2). $\varepsilon(e_L) \neq 0$ iff H is semisimple as an algebra.
3). If H is not finite dimensional, then μ_R exists iff projective H-comodules are exactly injective H-comodules.

Rmk: The thm. allows us to consider the stable category of H: H-$\underline{\text{mod}}$ (or H-$\underline{\text{comod}}$), which are monoidal triangulated, so that $K_0(H\text{-}\underline{\text{mod}})$ is a ring. This will be discussed more carefully in what follows.

Sketch of proof of thm. 8.
Part 1) of the thm. is proven in previous lemmas. For 3), see S. Dascalescu, C. Nastasescu and S. Rainu, Hopf Algebras, An Introduction.
We now show that part 2) of the thm. is true.
If $\varepsilon(e_L) \neq 0$ for H, then the left H-module map:

$$H \xrightarrow{\varepsilon} \Bbbk \longrightarrow 0$$

admits a splitting as H-modules via:

$$\begin{tikzcd} H \arrow[r, "\varepsilon"] & \mathbb{k} \arrow[r] \arrow[l, bend left] & 0 \\ \frac{e_L}{\varepsilon(e_L)} & 1 \arrow[l, mapsto] \end{tikzcd}$$

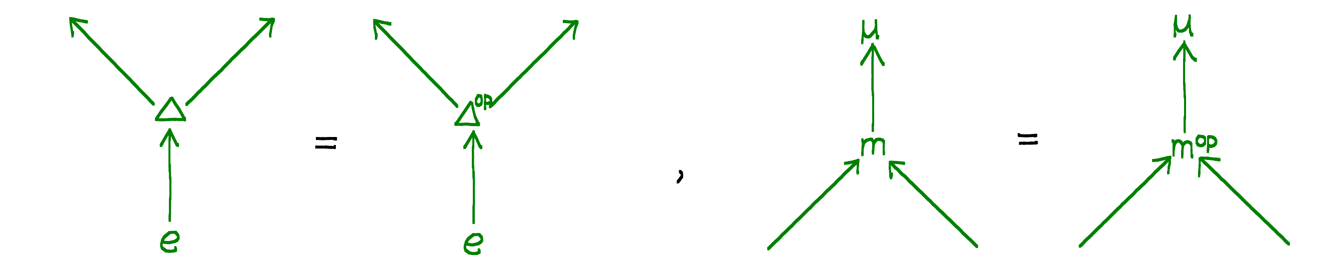
Hence $H \cong \mathbb{k} \oplus \ker\varepsilon$ as H-modules and $\mathbb{k}$ is projective as an H-module. It follows from cor. 3. that any H-module $V \cong \mathbb{k} \otimes_{\mathbb{k}} V$ is projective, so that H is semisimple.

On the other hand, if $\varepsilon(e_L) = 0$, we have $e_L^2 = \varepsilon(e_L) e_L = 0$. If H were semisimple, $H \cong \bigoplus_i \mathrm{Mat}(n_i, D_i)$ as an algebra, where D_i is a division ring $/\mathbb{k}$. Consider the submodule $H \cdot e_L = \mathbb{k} e_L \subseteq H$, which is simple, and thus must be of the form $D_i^{\oplus n_i}$ for some i. By counting dimensions we have $D_i = \mathbb{k}$ and $n_i = 1$. Then we would have e_L is a multiple of the idempotent of projection onto $\mathbb{k}$, so that $e_L^2 \neq 0$. Contradiction. $\square$

Kuperberg's invariant of 3-manifolds

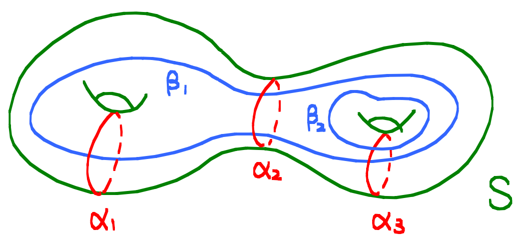
In this subsection, we mostly follow Kuperberg's paper [1]. The non-involutory case is slightly more involved. We will make the assumption that $S^2 = \mathrm{Id}$, $e_L = e_R = e$, $\mu_L = \mu_R = \mu$, $\mu(e) = 1$. This holds, for instance, when $H = \mathbb{k}[G]$ for a finite group G, or $H = \mathbb{k}[x]/(x^p)$, where $\mathrm{char}\,\mathbb{k} = p$.

Lemma. Under this assumption, we have

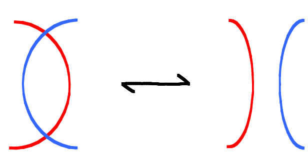


The lemma is a corollary of lemma 3.9 of [2], we omit it here

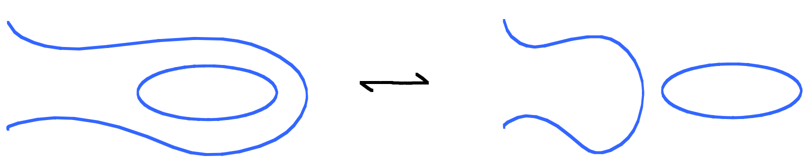
Recall that to a 3-manifold M we can take a Heegaard splitting of M and associate with it a Heegaard diagram D which is a Riemann surface S together with some α and β cycles on it such that they span $H_1(S)$ and the intersection matrix of α and β curves on S has maximal rank:



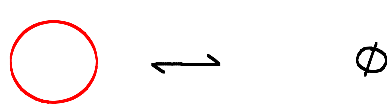
Two different Heegaard diagrams give rise to diffeomorphic M iff they are related by the following moves:



(the two point move)

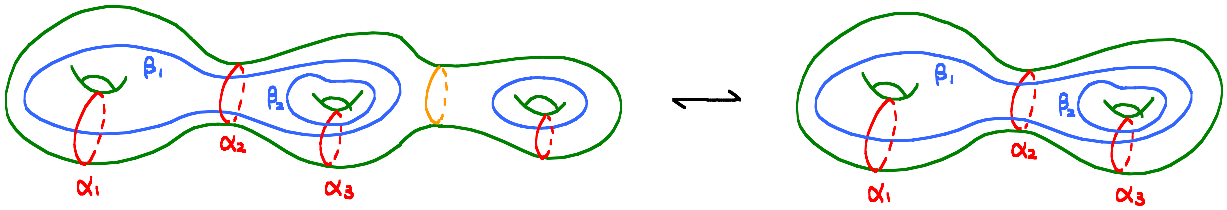


(the slide move of any same color)

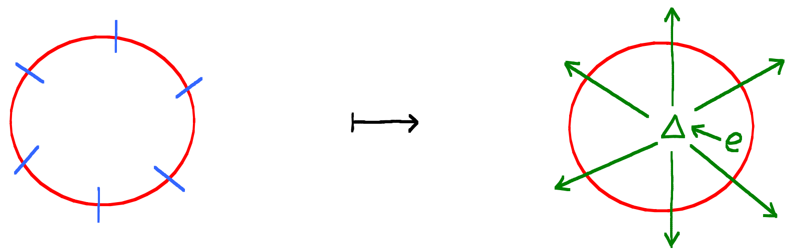


(removing or creating a trivial circle)

and finally stablization, i.e. adding or removing a handle as follows:

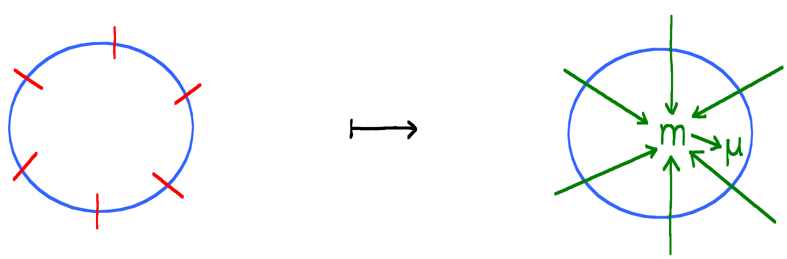


Now we associate with a Heegaard diagram an invariant as follows: to an α-curve with blue markings of the intersection points, we assign the tensor:



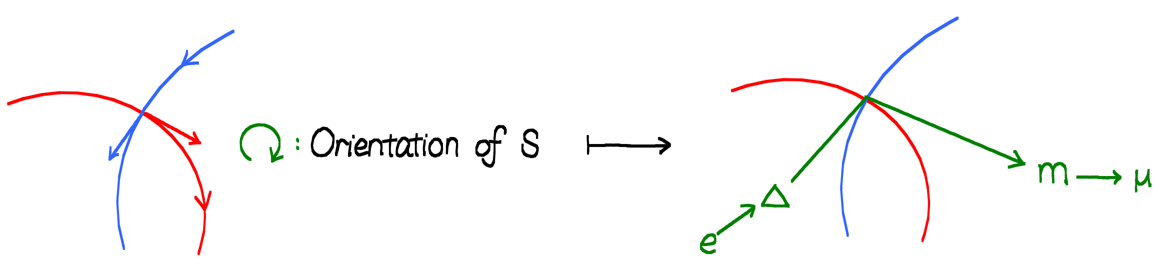
,

while to a β-curve with red markings, we assign the tensor



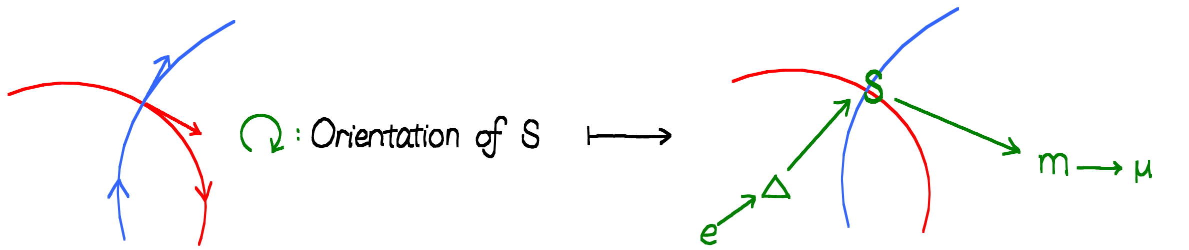
.

Then we connect the arrows corresponding to the same marking in the following way: First we orient the α, β curves arbitrarily, (we need to show then that the invariant is independent of the orientations). Then if at a given intersection point, the orientation $\langle \alpha, \beta \rangle$ agrees with the underlying orientation of S, we merge the arrows:



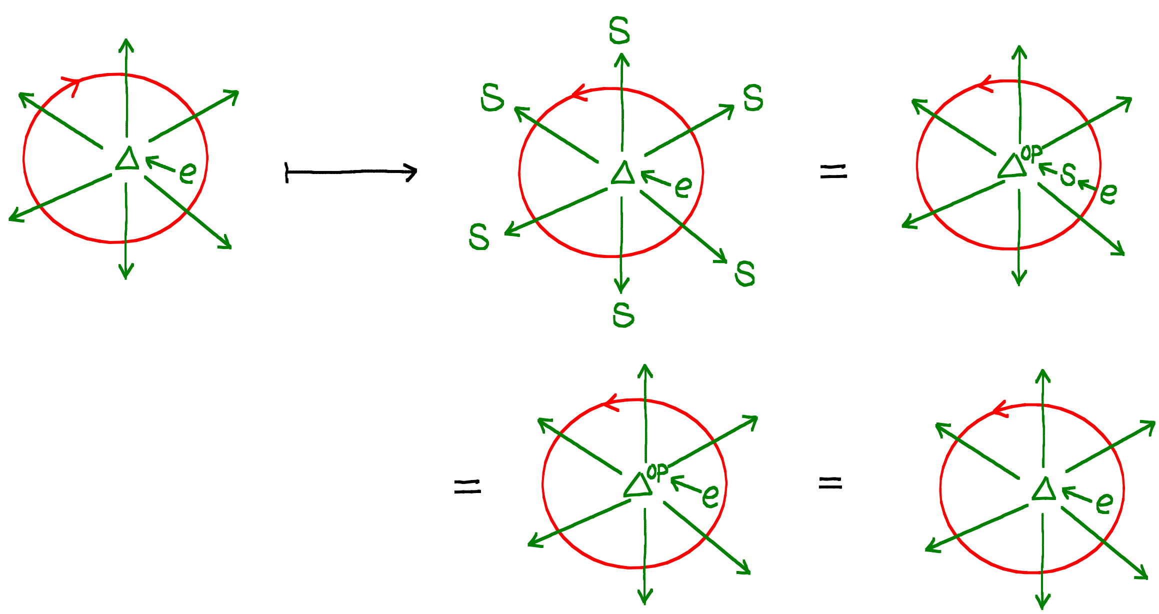
,

and if the orientation $\langle \alpha, \beta \rangle = -$ orientation of S, we insert an anti-pode between the arrows and then join them:



This process associates with any Heegaard diagram a closed web of arrows and S, Δ, m, and thus a number in the base field $\mathbb{k}$. Before checking that this number is invariant under the moves, we need to see that this assignment is independent of the orientation we chose for the curves.

Indeed, reversing the orientation of any circle corresponds to adding an extra S to each arrow coming out of / pointing into the circle ($S^2 = id$) :



where we used the lemma at the beginning of this subsection.

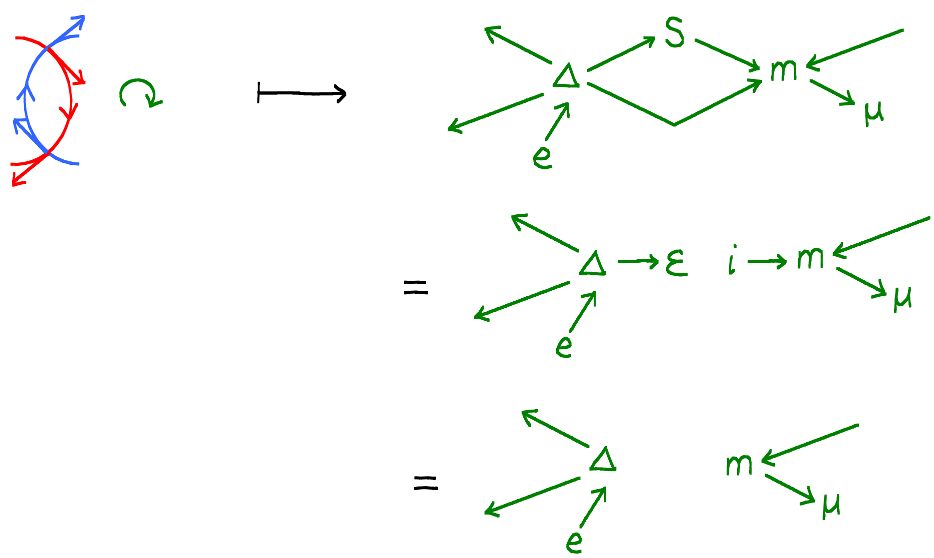
Now we check that this number is invariant under the moves above.

(1). Stablization. This corresponds to adding or removing:

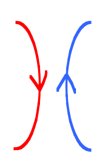
$$e \longrightarrow \mu$$

which we assumed to be 1.

(2). The two point move. When oriented, the two intersection points are of opposite orientations, to which we associate



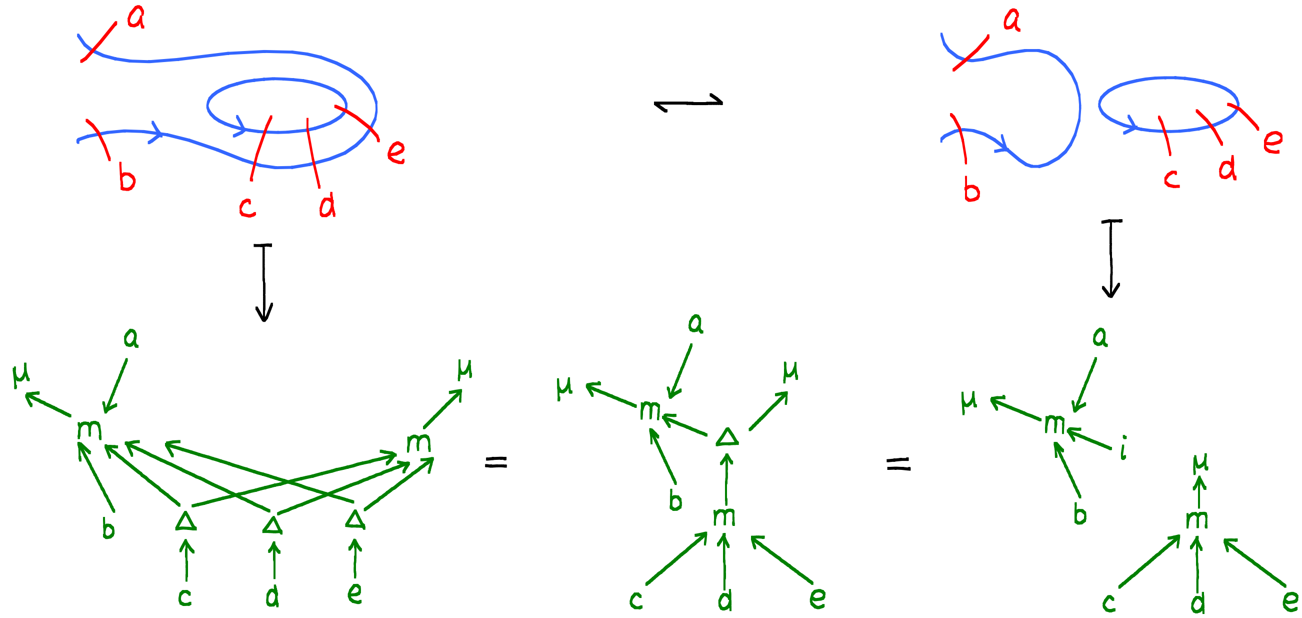
which is the diagram associated with:



(3). Adding or removing a trivial circle. The invariance follows from:

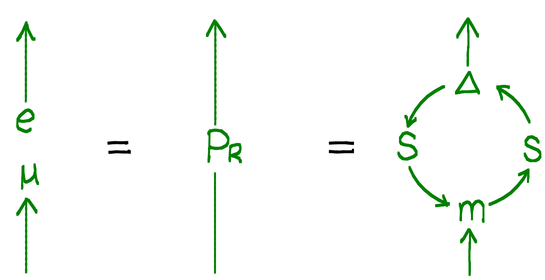
$$i \longrightarrow e = 1 = \mu \longrightarrow \varepsilon$$

(4). Finally, the slide move invariance follows from the compatability of Δ and m:

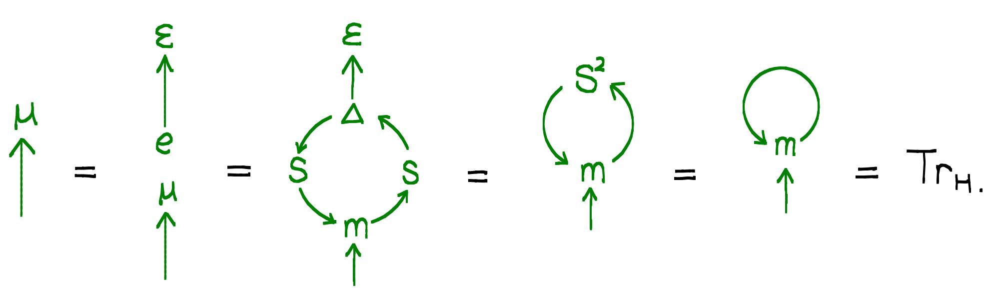


This finishes the proof.

Rmk: In the semisimple case, we can identify μ with the trace on H as follows. Recall that:



Thus when $\varepsilon(e) \neq 0$ (assume it's 1 then), we have:



Rmk: Kuperberg's invariant also extends to a super version. In fact, the facts we obtained so far extends with no effort to super Hopf algebras, and thus the same procedure produces for any 3-manifold an invariant in the ground field.

For instance, the exterior algebra Λ on 1 generator x becomes a super Hopf algebra if we assign to x an odd degree, so that

$$(1 \otimes x) \cdot (x \otimes 1) = -x \otimes x$$

in the product of two copies $\Lambda \otimes \Lambda$. The corresponding Kuperberg invariant is just $\det(M)$ for any 3-manifold M.

In this sense, the original def. of Heegaard-Floer homology of Ozsvath-Szabo for 3-manifolds can be viewed as a categorification of Kuperberg's invariant for the super Hopf algebra Λ, since the latter is the Euler characteristic of the Heegaard-Floer homology groups.

Problems:

1). What's the topological meaning of Kuperberg's invariant for the Hopf algebra $H = \Bbbk[x]/(x^p)$, where $\Bbbk$ is a field of char p?

2). How do we categorify this invariant?

3). Maybe a good starting point is first to extend the grid diagram description of knot Floer homology to be a homology theory of p-complexes over $\Bbbk$ ($\mathrm{char}\,\Bbbk = p$) (p-complexes means that $d^p = 0$ instead of $d^2 = 0$).

Categorification of $\mathbb{F}_p$

Before tackling the problems above, we will need a categorification of the ground field $\Bbbk$ where our 3-manifold invariants lie inside. Since any field of finite characteristic contains $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$, we will categorify $\mathbb{F}_p$ as a first step.

Recall that if $\operatorname{char} \Bbbk = p$, $H = \Bbbk[x]/(x^p)$ is a finite dimensional Hopf algebra, and thus its module category is Frobenius by our earlier results, i.e. the projective modules are the same as injective modules. For such Frobenius module categories, we can define its associated stable category $H\text{-}\underline{\mathrm{mod}}$ as follows:

The objects of $H\text{-}\underline{\mathrm{mod}}$ are the same as that of $H\text{-mod}$, and for any two obects $M, N \in H\text{-}\underline{\mathrm{mod}}$, we define

$$\mathrm{Hom}_{H\text{-}\underline{\mathrm{mod}}}(M,N) \triangleq \mathrm{Hom}_{H\text{-mod}}(M,N) / (\text{Morphisms that factors through an injective module}),$$

where we say that an H-module map $f: M \longrightarrow N$ factors through an injective ($=$ projective) if $\exists$ I an injective H-module s.t. the diagram below commutes:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr] & & N \\ & I \arrow[ur] & \end{tikzcd}$$

On $H\text{-}\underline{\mathrm{mod}}$, we define the shift functor $[1]$ as follows. For any H-mod M, choose an injection $M \longrightarrow I$ where I is an injective H-module (say $I = H \otimes M$ by Prop. 2), and $M[1]$ is defined to be I/M. It's easily checked that different choices of I result in isomorphic quotient modules in H-$\underline{\text{mod}}$, so that $[1]$ is well-defined.

Furthermore, given any morphism $f: M \longrightarrow N$ in H-mod, we can define the cone of f to be the push-out H-module C_f so that we have the commutative diagram:

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
M \arrow[r, "f"] \arrow[d, "i"'] & N \arrow[d, "g"] \\
I \arrow[r] \arrow[d] & C_f \arrow[d, "h"] \\
M[1] \arrow[r, equal] \arrow[d] & M[1] \arrow[d] \\
0 & 0
\end{tikzcd}$$

The sextuple $M \xrightarrow{f} N \xrightarrow{g} C_f \xrightarrow{h} M[1]$ viewed in H-$\underline{\text{mod}}$ is called a standard distinguished triangle, and any sextuple which is isomorphic to a standard distinguished triangle in H-$\underline{\text{mod}}$ is called a distinguished triangle.

Equipped with these structures, we have the following:

Thm. 1). H-$\underline{\text{mod}}$ is a triangulated category.
2). The tensor product on H-mod descends to H-$\underline{\text{mod}}$ (see Cor. 3), so that H-$\underline{\text{mod}}$ is triangulated monoidal.

For the proof, see D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras. A working def. of triangulated monoidal categories is given by V. Voevodsky in his PhD thesis: Homology of Schemes and Covariant Motives.

E.g. Consider the graded super Hopf algebra $H = \Bbbk[d]/(d^2)$ where $\deg d = 1$. The graded version H-<u>gr.mod</u> is no other than the homotopy category (= derived category) of complexes of $\Bbbk$-vector spaces.

Cor. $G_0(H\text{-}\underline{\text{mod}})$ is a ring, with unit given by the class of the trivial H-module: $[\underline{\Bbbk}]$. □

Now, let's look at the case $H = \Bbbk[x]/(x^p)$ where $\Bbbk$ is a field of char p. By Jordan's thm., it's easy to see that any indecomposable module over H is of the form $\Bbbk[x]/(x^{k+1}) \triangleq V_k$, $1 \le k \le p$, with V_0 being the trivial module $\underline{\Bbbk}$. The category H-mod is Krull-Schmit so that any H-module V is isomorphic to:

$$V \cong \bigoplus_i V_i^{a_i}.$$

Since there is only 1 simple H-module V_0 and only 1 indecomposable projective module, we have:

$$G_0(H\text{-mod}) \cong \mathbb{Z}.$$

Furthermore, in $G_0(H\text{-mod})$, from the s.e.s.

$$0 \longrightarrow V_0 \longrightarrow V_n \longrightarrow V_{n-1} \longrightarrow 0,$$

we deduce that $[V_n] = (n+1)[V_0]$, so that

$$G_0(H\text{-mod}) = \mathbb{Z} \cdot [V_0]$$

However, it's more interesting when we pass to H-$\underline{\text{mod}}$. Since in H-$\underline{\text{mod}}$, we have

$$V_{p-1} \cong H \cong 0$$

the same argument as above says that inside $G_0(H\text{-}\underline{\text{mod}})$,

$$[V_{p-1}] = p \cdot [V_0] = 0,$$

which implies that:

Cor. As a ring, $G_0(H\text{-}\underline{\text{mod}}) \cong \mathbb{F}_p$. $\square$

For more results along these lines, and proofs of some of these results, see M. Khovanov, *Hopfological Algebra and Categorification at a Root of Unity: the First Steps*.