§6. Milnor's Conjecture

In this section, we give some applications of the machinery we developed so far.

Variations of Khovanov Homology

Previously we have seen that to define a flat tangle invariant all we needed was a 2d TQFT over a base ring $\Bbbk$. Equivalently, we needed a commutative Frobenius algebra $A/\Bbbk$. It is only when we wanted RI-invariance that we were forced to require $\dim_{\Bbbk} A = 2$. In this subsection, we give a variation of the tangle invariant by choosing a different 2d TQFT. This construction is due to Bar-Natan.

The idea is to replace the base ring $\Bbbk = \mathbb{Z}$ by $\Bbbk = \mathbb{Z}[t]$, and set our 2d-TQFT to take value:

$$\begin{aligned} F(\bigcirc) &= A \\ &= \Bbbk[x]/(x^2-t) \\ &= \mathbb{Z}[t,x]/(x^2-t) \\ &\cong \mathbb{Z}[t]\cdot 1 \oplus \mathbb{Z}[t]\cdot x \end{aligned}$$

where $\deg t = 2\deg x = 4$. We shall refer to any construction using this 2d TQFT as a "t-theory".

In t-theory, we have

$$F(\underbrace{\bigcirc \cdots \bigcirc}_{m}) = \underbrace{A \otimes_{\Bbbk} \cdots \otimes_{\Bbbk} A}_{m}$$

which is a rank 2^m free $\Bbbk = \mathbb{Z}[t]$-module.

Moreover, in t-theory, we have

$$\begin{cases} \Delta(1) = 1\otimes x + x\otimes 1 \\ \Delta(x) = x\otimes x + t(1\otimes 1) \end{cases}$$

$$\begin{cases} \varepsilon(X) = 1 \\ \varepsilon(1) = 0 \end{cases}$$

One can check algebraically that $A = \mathbb{Z}[x]$ is Frobenius over $\Bbbk = \mathbb{Z}[t]$. A better way to see this is to regard A as the $SU(2)$-equivariant cohomology of $\mathbb{P}^1(\mathbb{C})$:

$$H^*_{SU(2)}(pt) = H^*(BSU(2)) = H^*(\mathbb{HP}^\infty) \cong \mathbb{Z}[t] \quad (\deg t = 4)$$
$$H^*_{SU(2)}(\mathbb{P}^1(\mathbb{C})) \cong H^*_{U(1)}(pt) = H^*(BU(1)) \cong \mathbb{Z}[x] \quad (\deg x = 2)$$

where $H^*_{SU(2)}(\mathbb{P}^1(\mathbb{C})) \cong H^*_{U(1)}(pt)$ since $SU(2)$ acts transitively on $\mathbb{P}^1(\mathbb{C}) \cong S^2$ and the stablizer of any point on S^2 is isomorphic to $U(1)$.

Now one can carry out all the constructions we have done in the previous section for this particular TQFT and obtain Bar-Natan's oriented tangle invariant, which is functorial up to a sign.

Rmk: The most general version of this "relative dim 1" TQFT can be realized by taking

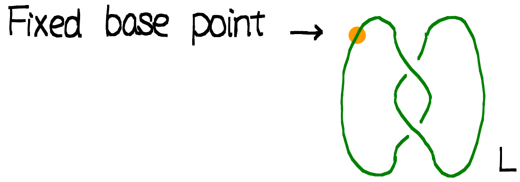
$$\Bbbk = \mathbb{Z}[t, h], \ \deg h = 2, \deg t = 4$$
$$A = \Bbbk[x]/(x^2 - hx - t),$$

which can be regarded as using $U(2)$-equivariant cohomology.

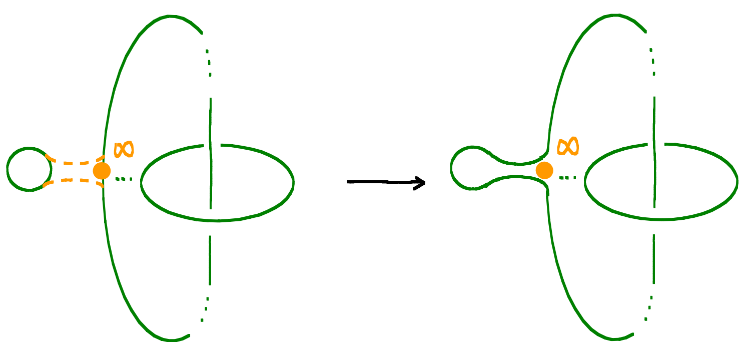
The Rasmussen invariant

We shall denote the t-theory chain complex of an oriented link L by $C_t(L)$ and its cohomology by $H_t(L) \cong \bigoplus_{i,j} H_t^{i,j}(L)$, so that $C_t(L)$, $H_t(L)$ are naturally $\mathbb{Z}[t]$-modules.

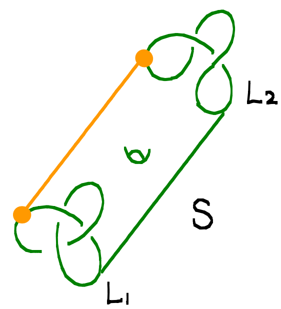
$C_t(L)$ by its definition is a chain complex (up to homotopy) of finitely generated free $\mathbb{Z}[t]$-modules, and each term in the complex is a direct sum of the free module $A \cong \mathbb{Z}[X]$ with some grading shift. We can further extend this $\mathbb{Z}[t]$-module structure on $C_t(L)$ to that of a $\mathbb{Z}[X]$-module by choosing a base point on a fixed component of L:



One can imagine that this point is placed at ∞, and the $\mathbb{Z}[X]$ module structure comes from merging L with circles near ∞, so that it doesn't interfere with any other parts of L.



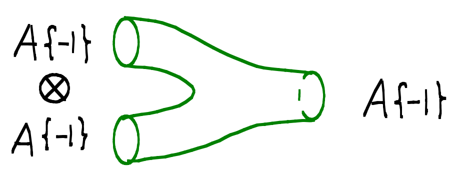
Any cobordism S between L_1 and L_2 are then required to preserve the base points of the links:



Such an S induces an $A \cong \mathbb{Z}[X]$-module map between $C_t(L_1)$ and $C_t(L_2)$, and $H_t(L_1)$ and $H_t(L_2)$ ($X^2 = t$). It is of bidegree $(0, -\chi(S))$ i.e. it doesn't shift homological degrees, by definition, and it maps

$$H^{i,j}(L_1) \longrightarrow H^{i,j-\chi(S)}(L_2)$$

Finally, we have $H(\bigcirc) \cong A\{-1\}$, sitting in homological deg. 0. We shift the q-deg as for the Jones polynomial case to make the multiplication map homogeneous:



In the following we shall pass from $\mathbb{Z}$ to $\mathbb{Q}$ so that for any link L, $C_t(L)$, $H_t(L)$ are finitely generated, bigraded $\mathbb{Q}[t]$-modules. Multiplication by t doesn't change homological degrees. By the classification thm. of finitely generated $\mathbb{Q}[t]$-modules, $H_t(L)$, as a $\mathbb{Q}[t]$-module, must be of the form:

$$\mathbb{Q}[t]^{\ell} \oplus \bigoplus_i \mathbb{Q}[t]/(t^{N_i})$$

((t^{N_i}) are the only homogeneous ideals in $\mathbb{Q}[t]$).

We can compute the value of ℓ for any link L. To do this, we first want to eliminate the torsion part of the homology groups. This can be done by setting $t=1$, at the cost of collapsing the $\mathbb{Z}$-grading from α into a $\mathbb{Z}/4$-grading (recall that $\deg t = 4$).

Def. (E.S. Lee's homology groups). The Lee homology group $H_{Lee}(L)$ of an oriented link L is the homology of the complex $C_t(L)/(t-1)\cdot C_t(L)$:

$$H_{Lee}(L) \triangleq \bigoplus_{i\in\mathbb{Z},\alpha\in\mathbb{Z}/4} H^{i,\alpha}(C_t(L)/(t-1)C_t(L))$$

Thm. 1. (Lee). For any link L, not necessarily oriented,

$$H_{Lee}(L) \cong \mathbb{Q}^{2\cdot\#\{\text{Components of } L\}}$$

as $\mathbb{Q}$-vector spaces.

We will sketch a proof of this thm at the end of this section.

Cor. 2. The rank ℓ of $H_t(L)$ equals $2\cdot\#\{\text{Components of } L\}$.
Pf: Indeed, $C_t(L)$ as complexes of $\mathbb{Q}[t]$-modules, is a direct sum of complexes:

$$(0\longrightarrow \mathbb{Q}[t]\longrightarrow 0)^{\ell}\oplus\bigoplus_i(0\longrightarrow\mathbb{Q}[t]\xrightarrow{t^{N_i}}\mathbb{Q}[t]\longrightarrow 0),$$

since $\mathbb{Q}[t]$ is of homological dimension 1. Evaluating $t=1$ kills all the latter complexes while the remaining $(0\longrightarrow\mathbb{Q}\longrightarrow 0)^{\ell}$ calculates Lee's homology. □

The next observation about H_t over $\mathbb{Q}[t]$ is that, with respect to link cobordisms S, $H_t(S)$ takes torsion subgroups to torsion subgroups, since it commutes with the $\mathbb{Q}[t]$-action.

$$H_t(S): H_t(L_1)^{Tor} \longrightarrow H_t(L_2)^{Tor}$$

This in turn induces a map of free modules:

$$\begin{array}{ccc} H_t^{free}(S): H_t(L_1)/H_t(L_1)^{Tor} & \longrightarrow & H_t(L_2)/H_t(L_2)^{Tor} \\ \text{\rotatebox{90}{$\cong$}} & & \text{\rotatebox{90}{$\cong$}} \\ H_t(L_1)^{free} & & H_t(L_2)^{free} \end{array}$$

which is well-defined up to a sign.

From now on we focus on the case of knots, and we use the base pointed version so that $C_t(L)$, $H_t(L)$ are $A = \mathbb{Q}[X]$-modules. Then

$$H_t(L) \cong X\text{-torsion part} \oplus \mathbb{Q}[X]\{-1+s(L)\}$$

where we have only 1 copy of $\mathbb{Q}[X]$ as the free part by the cor. above ($\mathrm{rank}_{\mathbb{Q}[t]} \mathbb{Q}[X] = 2$).

For knots L, we have the following:

Thm. 3. (1). (Lee) $H_{Lee}(L) \cong A/(t-1)A\{-1+s(L)\}$ for some $s(L) \in 2\mathbb{Z}$.
(2). (Rasmussen). If S is a connected cobordism from two knots L_1 and L_2, then $H_{Lee}(S): H_{Lee}(L_1) \longrightarrow H_{Lee}(L_2)$ is non-trivial.

Sketch of proof

(1). $H_{Lee}(L)$ is rank 1 over $A/(t-1)A \cong \mathbb{Q}[X]/(X^2-1)$ follows from Lee's thm 1 directly. That it's of odd q-deg. also follows from the description of cycles classes in $H_{Lee}(L)$.

(2) basically follows from the construction of Lee's homology theory. See J. Rasmussen, Khovanov Homology and the Slice Genus. □

Thm 3 implies that $s(L) \in 2\mathbb{Z}$ and the free part of $H_t(L)$ sits in homological degree 0.

Def. (Rasmussen invariant). The even number $s(L)$ is called the Rasmussen invariant of a knot L.

E.g. The unknot $\bigcirc$ has

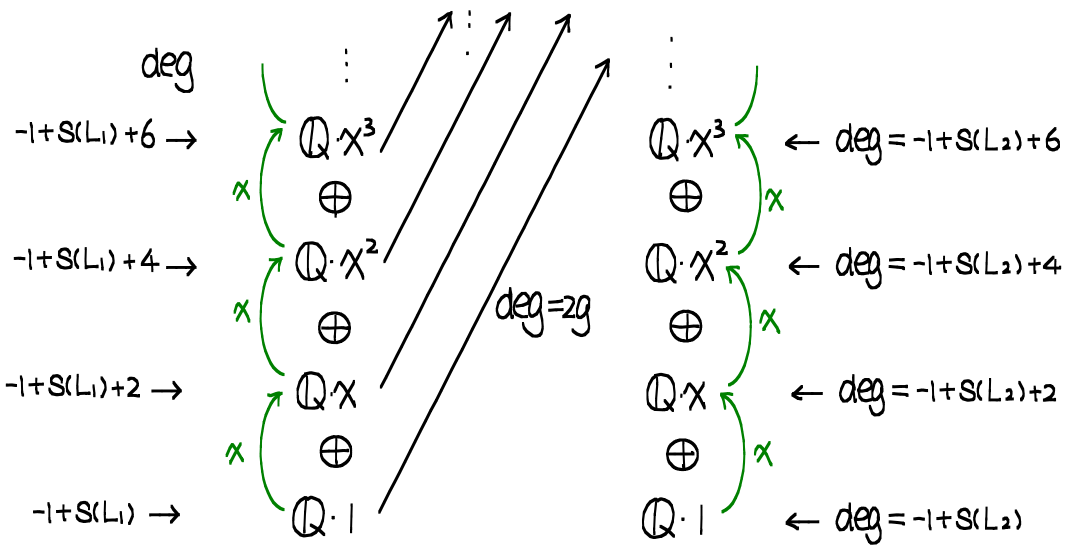
$$s(\bigcirc) = 0$$

This follows since we set $C_t(\bigcirc) = \mathcal{A}\{-1\}$.

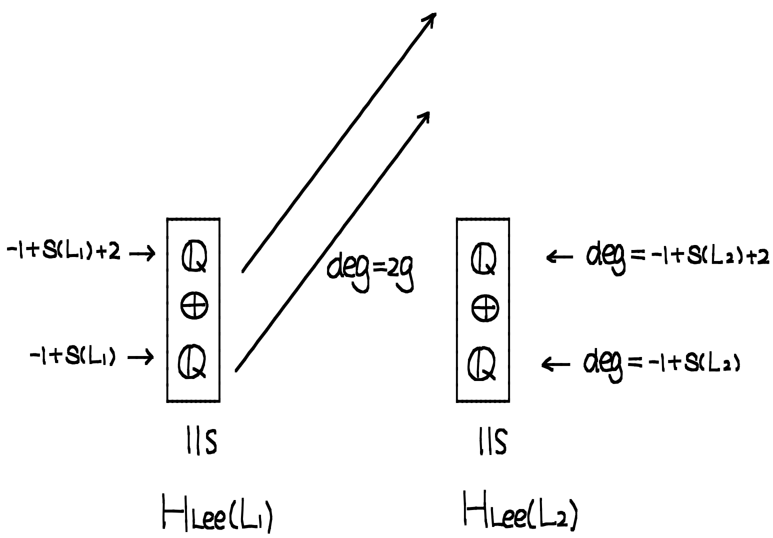
Now if S is a connected genus g cobordism between L_1 and L_2, then $H_t(S)$ is a homogeneous map between $H(L_1)$ and $H(L_2)$ of bidegree $(0, -2g)$. Hence:

$$\begin{array}{ccc} H_t(S): H_t(L_1)^{free} & \longrightarrow & H_t(L_2)^{free} \\ \| \wr & & \| \wr \\ \mathcal{A}\{-1+s(L_1)\} & & \mathcal{A}\{-1+s(L_2)\} \end{array}$$

which, written out componentwise, is of the form:



Setting $t = x^2 = 1$, and taking homology, we should recover the non-trivial homomorphism of Lee's homology groups:



For the arrows to define non-trivial homomorphisms, we must have

$$-1+s(L_2) \leq -1+s(L_1)+2g$$

Viewing S backwards as a cobordism from L_2 to L_1 yields

$$-1+s(L_1) \leq -1+s(L_2)+2g$$

Combined, we obtain:

Thm. 4. (Rasmussen). $|s(L_1)-s(L_2)| \leq 2g$. $\square$

Rmk: The Rasmussen invariant is a homomorphism from the knot concordance group to $2\mathbb{Z}$. Two knots L_1, L_2 are said to be concordant if there is an anulus T $(\chi(T)=0)$ such that

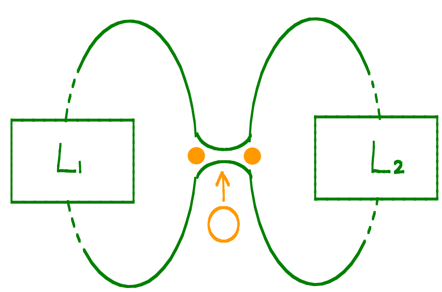
$$\partial T = L_1 \sqcup L_2$$

Thus the thm implies that, concordant knots have the same Rasmussen invariant. To show that it's a homomorphism, one needs to show that

$$s(L_1 \# L_2) = s(L_1) + s(L_2).$$

This follows from the fact that, for the connected sum of two knots L_1, L_2,

$$C_t(L_1 \# L_2) = C_t(L_1) \otimes_A C_t(L_2)$$



A-module structure comes from merging circles at ∞. In the previous section, A corresponds to $H^1 \cong \mathbb{Z}[X]/(X^2)$

so that $H_t^{Free}(L_1 \# L_2)$, and thus $H_{Lee}(L_1 \# L_2)$ inhert the grading shift $\{s(L_1)+s(L_2)\}$.

Rasmussen's proof of Milnor's conjecture

Given any knot $L \subseteq S^3$, it bounds an orientable surface Σ. The Σ with the minimal genus is called the Seifert surface of L. This minimal genus is denoted $g(L)$. Furthermore, if we regard S^3 as the boundary of the 4-ball B^4, and if we allow Σ to be inside B^4 while fixing $\partial\Sigma = L \subseteq S^3$ to be our given knot, we could potentially get a surface with a smaller genus. The smallest such genus is called the 4-genus of L, denoted $g_4(L)$.

Cor 5. $|s(L)| \leq 2 \cdot g_4(L)$

Pf: Indeed, by cutting a small disk out of a minimal bounding surface in B^4, it can then be regarded as a cobordism from the trivial knot L_0 to L. Since $s(L_0) = 0$, thm 5 says that

$$|s(L)| = |s(L_0) - s(L)| \leq 2 \cdot g_4(L)$$

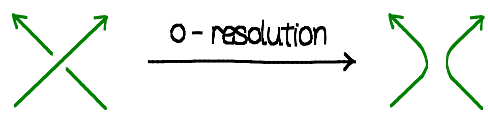
$\square$

In general, it's a hard problem to compute the 4-genus of a knot. Milnor conjectured that, if L is a positive knot (i.e. L admits an oriented projection diagram D with only positive crossings), then $g_4(L) = g(L) = \frac{1}{2}(n-c+1)$. This was first proven by Kronheimer and Mrowka using sophisticated methods of gauge theory. In this subsection we give Rasmussen's combinatorial proof.

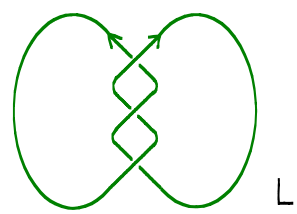
First off, it's easy to see that, for any positive knot L with a positively oriented diagram D,

$$g(L) \leq \frac{n-c+1}{2},$$

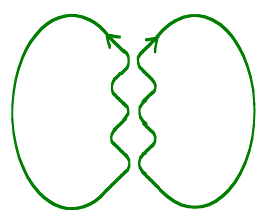
where n is the number of (positive) crossings in L, and c is the number of components when one resolves all crossings of D in the orientation compatible (0-resolution) way.



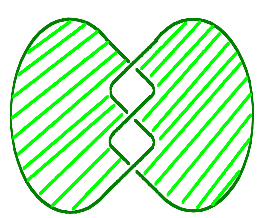
E.g. Consider the positive knot



Its orientation compatible resolution is given by:

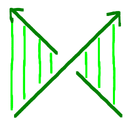


which has 2 components. There is an obvious orientable surface bounding L:

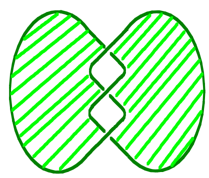


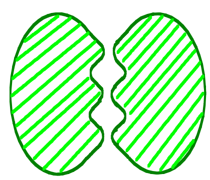
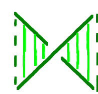
whose genus is 1.

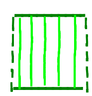
In general, there is an obvious orientable surface Σ bounding L, obtained by shading regions of a positive crossing in the way:



and taking the shaded surface Σ. Its genus $g(\Sigma)$ can be computed from its Euler characteristic $\chi(\Sigma)$:

$\chi(\Sigma) = \chi($  $)$

$= \chi($  $) + c \cdot \chi($  $)$

$= n \cdot 1 + c \cdot \chi($  $)$

$= n - c$

$$\Longrightarrow \quad g(\Sigma) = \frac{n-c+1}{2}$$

(Recall that $2 - 2g - \#(\text{components of } \partial\Sigma) = \chi(\Sigma)$).

Thus by definition, we have a trivial bound:

$$g_4(L) \leq g(L) \leq \frac{n-c+1}{2}$$

Thus it suffices to check that

$$g_4(L) \geq \frac{n-c+1}{2}$$

To do this, we will prove that $s(L) = \frac{n-c+1}{2}$. Then the result will follow from Rasmussen's thm. (thm. 4).

Indeed, in our construction of $C_t(D)$, the chain complex $C_t(D)$ of a positive diagram always sits in non-negative homological degrees, since

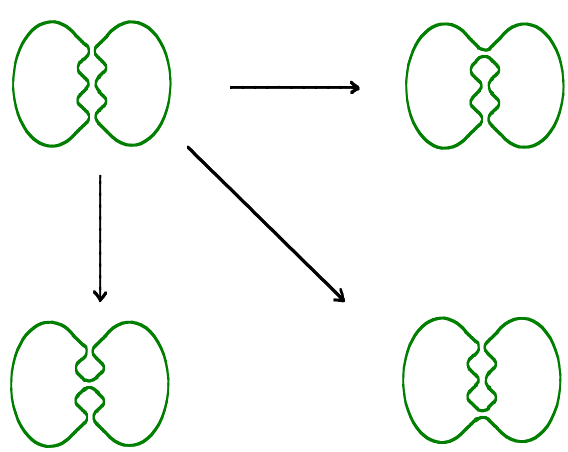
$$C_t(D) = C'_t(D)\,[x(D)]\,\{2x(D) - y(D)\}$$

and $x(D) = \#$ negative crossings in D, which is 0 for D positive.

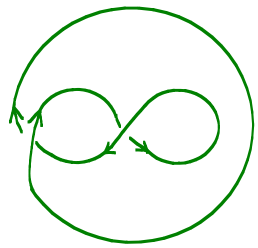
Then, $C_t(D)$ starts with

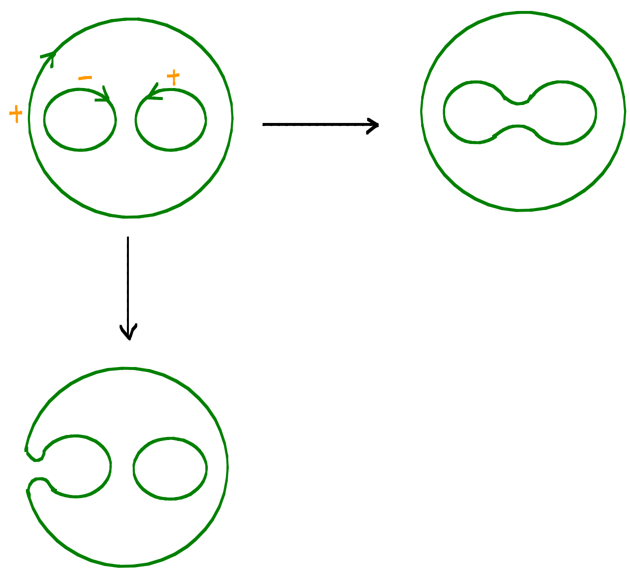
$$0 \longrightarrow A^{\otimes c} \xrightarrow{d} \oplus_n A^{\otimes(c-1)} \longrightarrow \cdots$$

where $A = \mathbb{Q}[x]$ and the tensor product is taken over $\mathbb{Q}[t]$, so that $A^{\otimes c}$ is a free $\mathbb{Q}[t]$-module of rank 2^c. The term $A^{\otimes c}$ sits in homological deg. 0 and c has the same meaning as before. The homological degree 1-term is just n-copies of $A^{\otimes(c-1)}$, since changing any 0-resolution to a 1-resolution merges two circles:



Then, using a standard combinatorial argument, we can assign a sign ± 1 to each of the c circles in the 0-resolution, so that every component of d comes from merging two circles in the 0 resolution with opposite signs. We illustrate this with an example of the unknot diagram:



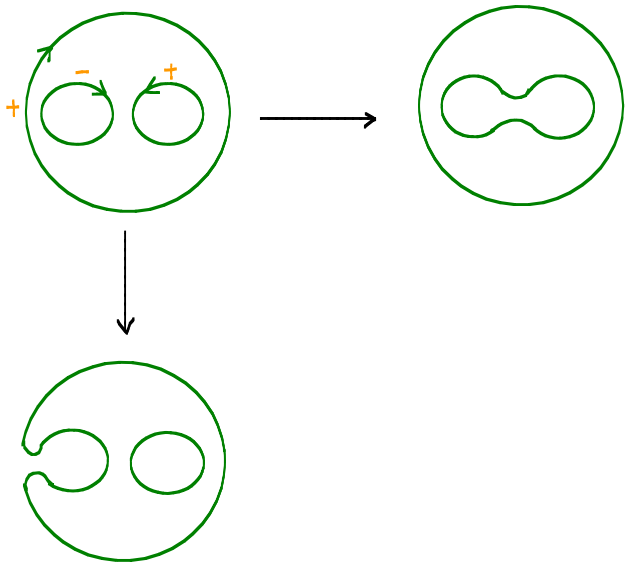


Now we pass back to the ordinary Khovanov homology ($t=1$). Consider the element

$$\alpha = \sum_{\substack{i:\ \text{circles in the} \\ \text{0-resolution}}} \operatorname{sgn}(i) \cdot x \otimes x \otimes \cdots x \otimes \underset{\uparrow \atop i\text{-th position}}{1} \otimes x \cdots \otimes x$$

Then this element maps to 0 under d, since we are always merging circles with opposite signs.

$$1\otimes x\otimes x - x\otimes 1\otimes x + x\otimes x\otimes 1 \longmapsto 0 - x\otimes x + x\otimes x = 0$$



$$\big\downarrow$$

$$x\otimes x - x\otimes x + 0$$

It follows that α represents a non-trivial cohomology class. As a $\mathbb{Q}[x]$-module, $H_t^{Free}(L)$ must be generated by a lifting of this cohomology class, which is of degree:

$$\begin{aligned}\deg\alpha &= ((c-1)\cdot\deg x + \deg(1)) + (2x(D) - y(D))\\ &= (c-1)\cdot 1 - 1 + 2\cdot 0 - n\\ &= c-2-n.\end{aligned}$$

Thus by def. of $s(L)$, we have

$$-1 + s(L) = c-2-n$$

$$\Longrightarrow \quad s(L) = c-1-n.$$

Then cor. 5 implies that

$$g_4(L) \geq \left|\frac{s(L)}{2}\right| \geq \frac{c-1-n}{2}.$$

Combined with the easier half of the inequality, this finishes the proof of Milnor's conjecture.

Rmk: The Rasmussen invariant of a knot is, in general, very hard to compute. Besides positive knots, we know $s(L)$ for alternating knots, which equals the signature $\sigma(L)$ of the knot (Rasmussen).

Lee's homology theory

In this subsection, we give a sketch of proof of Lee's thm. (thm. 1).

Recall that the chain complex $C_t(D)$ for any tangle diagram D is constructed using the 2d-TQFT "t-theory", which is associated with the commutative Frobenius algebra $\mathbb{Z}[x]$ over the base ring $\mathbb{Z}[t]$.

Then Lee's homology theory can be constructed from the quotient Frobenius algebra $\mathbb{Q}[x]/(t=1) = \mathbb{Q}[x]/(x^2=1)$ over $\mathbb{Q}[t]/(t=1) \cong \mathbb{Q}$, which is in fact a semi-simple algebra:

$$\mathbb{Q}[x]/(x^2=1) \cong \mathbb{Q}\cdot\left(\frac{x-1}{2}\right) \oplus \mathbb{Q}\left(\frac{x+1}{2}\right),$$

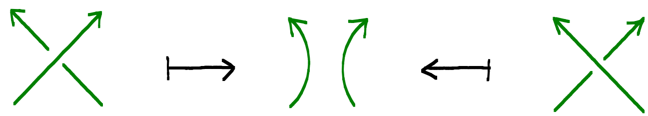
where $\frac{x-1}{2} \triangleq a$, $\frac{x+1}{2} \triangleq b$ are orthogonal idempotents. Using the explicit formula for the multiplication and comultiplication in t-theory, one can see that the above decomposition is in fact a decomposition of Frobenius algebras:

$$\Delta(a) = 2a \otimes a$$

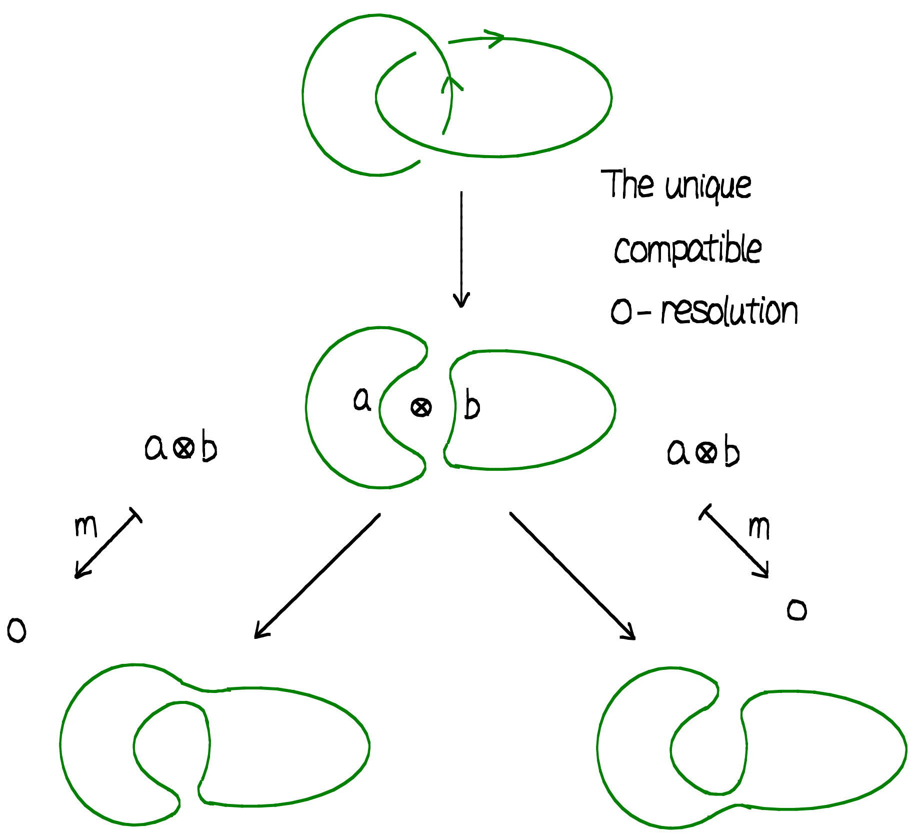
$$\Delta(b) = 2b \otimes b$$

Thus, "t-theory", when mod $(t-1)$, becomes two orthogonal TQFT's.

Using this decomposition, one can construct explicitly the $2^{\#\mathrm{Comp}(L)}$ generators in Lee's homology groups of an oriented link L. One starts with an oriented diagram D, and forgets about its old orientation. Then we orient it in all $2^{\# Comp(L)}$ ways, and for each particular orientation, we resolve it in the unique orientation compatible way (0-resolution):



Then, we assign a and b to circles, so that any circles that are merged in the 1-resolution, are assigned different letters:



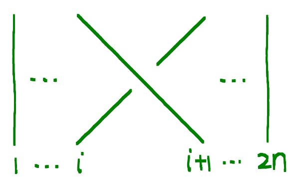
Then, the tensor product of these elements define a cycle in the chain complex. An induction on the total number of crossings in diagram D shows that all these cycles survive in homology and they span $H_{Lee}(L)$.

Some interesting problems

(1). Can one extend the Rasmussen invariant to tangles?

(2). Try to understand Lee's theory for tangles. How do we build analogues of H^n, complexes of bimodules etc.?

(3). There is a natural braid group action Br_{2n} on $Comp(H^n)$, given by sending a braid group generator:

$\sigma_i =$ 

to the *functor* of tensoring with the complex of H^n-bimodules $F(\sigma_i)$ associated with the tangle σ_i.

Is this action *faithful*?

(4). Connect positivity in the category of tangles to the positivity in the category of complexes. A tangle is called positive if it admits an oriented diagram with only positive crossings. A complex is positive if it's homotopic to a complex with terms only in non-negative homological degrees.

We have seen that in the proof of Milnor's conjecture, a knot is positive implies that its chain complex $C(D)$ is positive. Is the converse true?

This question is also related to braid group actions on derived categories of coherent sheaves on certain Springer varieties. See the works of Cautis - Kamnitzer.

(5). *Categorification of 3d-geometry.*

We still don't know how to extend link invariants to 3d geometry invariants. One major step would be to categorify $\mathbb{R}$ as a ring. (it is known that $K_0(\mathrm{II}_1\text{-factor}) \cong \mathbb{R}$, but just as an abelian group), i.e. one needs a monoidal category $\mathcal{C}$ s.t. $K_0(\mathcal{C}) \cong \mathbb{R}$ as a ring.