§ 9. Morita Theory

In this section we give a brief review of Morita equivalences and its derived analogue.

Morita equivalence

Let A, B be rings.

Def. A, B are said to be Morita equivalent if their module categories are equivalent as abelian categories.

In other words, there are functors

$$\begin{tikzcd} A\text{-mod} \arrow[r, bend left, "F"] & B\text{-mod} \arrow[l, bend left, "F'"] \end{tikzcd}$$

s.t. $F' \circ F \cong Id_{A\text{-mod}}$, $F \circ F' \cong Id_{B\text{-mod}}$.

In the case of the def., F takes the left A-module ${}_A A$ to some B-module ${}_B F(A)$. But A is also a right A-module, and right multiplication by elements of A commutes with the left A-action, i.e. right multiplication by elements of A is a left A-module endomorphism. Thus A also acts on ${}_B F(A)$ on the right so that ${}_B F(A)_A$ is a (B, A)-bimodule. Similarly, we obtain the (A, B)-bimodule ${}_A F'(B)_B$.

Lemma 1. F is determined by $F(A)$.

Pf: For any A-module M, choose a presentation:

$$A^{\oplus J} \xrightarrow{\varphi} A^{\oplus I} \longrightarrow M \longrightarrow 0.$$

Since F is an equivalence, it preserves exactness:

$$F(A^{\oplus I}) \longrightarrow F(A^{\oplus J}) \longrightarrow F(M) \longrightarrow 0$$

F also preserves direct sums (colimits) since this is a categorical notion $\Longrightarrow$

$$F(A)^{\oplus I} \longrightarrow F(A)^{\oplus J} \longrightarrow F(M) \longrightarrow 0$$

$\Longrightarrow$ The module $F(M)$ is prescribed. $\square$

Notice that in the above proof we didn't need F to be an isomorphism, but only used that F is additive, right exact, and commutes with taking colimits. We obtain:

Cor. 2. *Any additive, right exact functor*

$$F: A\text{-mod} \longrightarrow B\text{-mod}$$

which commutes with forming colimits is given by tensoring with a (B, A)-bimodule ${}_B N_A$ $(= {}_B F(A)_A)$. $\square$

Hence in the situation of the def., we have

$$F(-) \cong {}_B N_A \otimes_A -$$

$$F'(-) \cong {}_A N'_B \otimes_B -$$

and moreover,

$$N' \otimes_B N \cong A \qquad \text{as } (A, A)\text{-bimodules}$$

$$N \otimes_A N' \cong B \qquad \text{as } (B, B)\text{-bimodules}.$$

Recall the following general fact: If A, B are rings,

$$\mathrm{Hom}_B({}_BN_A \otimes_A M, {}_BL) \cong \mathrm{Hom}_A({}_AM, \mathrm{Hom}_B({}_BN_A, {}_BL))$$

In the situation of the def. of Morita equivalence, we obtain:

$$\begin{aligned}\mathrm{Hom}_A({}_AM, {}_AN'_B \otimes_B L) &\cong \mathrm{Hom}_B({}_BN_A \otimes_A M, {}_BN_A \otimes_A N'_B \otimes_B L) \\ &\cong \mathrm{Hom}_B({}_BN_A \otimes_A M, {}_BL) \\ &\cong \mathrm{Hom}_A({}_AM, \mathrm{Hom}_B({}_BN_A, {}_BL)),\end{aligned}$$

which implies the isomorphism of functors:

$${}_AN'_B \otimes_B - \cong \mathrm{Hom}_B({}_BN_A, -)$$

Cor 3. ${}_BN_A$ is a compact object in B-mod, i.e. $\mathrm{Hom}_B({}_BN_A, -)$ preserves taking direct sums (colimits). $\square$

Ex. Show that an object in B-mod is compact iff it's finitely generated as a B-module.

Rmk: Compact objects are also known as "perfect objects", especially in derived categories.

Since ${}_BN = F(A)$ is also the image of a projective module, it's thus a finitely generated projective B-module. Moreover, by regarding ${}_BN_A$ as an (A^{op}, B^{op})-bimodule, ${}_AN'_B$ as a (B^{op}, A^{op})-bimodule, we obtain the above results for right module categories as well. In particular, ${}_BN_A$ is also a finitely generated, projective right A-module.

Next, observe that:

$$\begin{aligned} B &\cong \mathrm{End}_B(B_B, B_B) \\ &\cong \mathrm{Hom}_A(B_B \otimes_B N_A, B \otimes_B N_A) \\ &\cong \mathrm{End}_A({}_B N_A) \end{aligned}$$

$$\Longrightarrow \quad B \cong \mathrm{End}_A({}_B N_A).$$

We summarize the above discussion in the following:

Prop. 4. Two rings A, B are Morita equivalent iff one of the following equivalent conditions hold:

(1). The left module categories $A\text{-mod} \cong B\text{-mod}$

(2). The right module categories $\text{mod-}A \cong \text{mod-}B$

(3). There is a finitely generated projective right A-module N_A s.t. $N_A^{\oplus r} \supseteq A$ for some $r \in \mathbb{N}$, and

$$B \cong \mathrm{End}_A(N_A) \qquad \square$$

Prop. 5. If A, B are Morita equivalent, then $Z(A) \cong Z(B)$.

Pf: The center of a ring is a categorical notion: it's the endomorphism ring of the identity functor of the module category:

$$Z(A) \cong \mathrm{End}(\mathrm{Id}_{A\text{-mod}}). \qquad \square$$

Morita equivalence for artinian rings

In this subsection, we will construct all rings that are Morita equivalent to an artinian ring A.

By earlier results in §4, we know that an artinian ring A, as a right module over itself, is isomorphic to

$$A_A \cong \bigoplus_{i\in I} P_i^{\oplus r_i},$$

where $\{P_i\}_{i\in I}$ is the isomorphism classes of all indecomposable projective right A-modules. Thus by (3) of Prop 4, we have:

Cor. 6. Any ring B that is Morita equivalent to A is of the form:

$$B \cong \mathrm{End}_A(\bigoplus_{i\in I} P_i^{\oplus n_i})^{op},$$

where n_i are positive integers. □

Thus when people study representation theory of finite dim'l algebras, they try to down-size the rings as much as possible to look for categorical invariants (see Prop 5 for instance). So one takes the Morita equivalent ring:

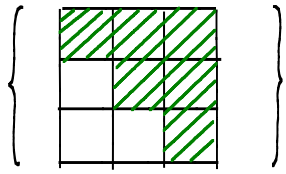
$$B \triangleq \mathrm{End}_A(\bigoplus_{i\in I} P_i)^{op},$$

which is the smallest possible one among such rings.

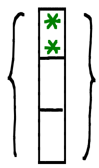
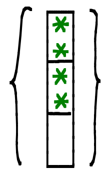
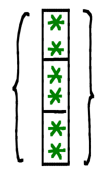
E.g.

(1). Any matrix ring $M_n(A)$ is Morita equivalent to A itself. This works for any ring, not just for artinian rings.

(2). Let A be the block upper triangular matrices with n diagonal blocks of various sizes.

$A =$ 

One can check that a complete set of non-isomorphic indecomposable projective modules is given by

$P_1 =$  $P_2 =$  $\cdots$ $P_n =$ 

and

$$\mathrm{End}_A P_i \cong \Bbbk$$

$$\mathrm{Hom}_A(P_i, P_j) \cong \begin{cases} \Bbbk & \text{if } i \leq j \\ 0 & \text{if } i > j \end{cases}$$

Thus A is Morita equivalent to the $n \times n$ upper triangular matrices, or equivalently, the path algebra of the quiver

$$\begin{tikzcd} \underset{1}{\bullet} \arrow[r] & \underset{2}{\bullet} \arrow[r] & \cdots \arrow[r] & \underset{n}{\bullet} \end{tikzcd}$$

(3). There is a class of finite dimensional algebras in representation theory, that are the smallest possible in their own Morita equivalence classes. These are the algebras all of whose simple modules are 1-dim'l, or equivalently, $A/\mathrm{Jac}(A) \cong \Bbbk \times \cdots \times \Bbbk$. Such algebras are called basic algebras. Examples include path algebras, commutative algebras over algebraically closed fields ($\Bbbk[G]$ when

G is abelian, $H^*(X, \mathbb{k})$ when X is a finite CW complex etc.).

Localization of categories

In this subsection we briefly review how to localize categories and thus obtain the derived categories of abelian categories $\mathcal{A}$ by localizing $\mathrm{Comp}(\mathcal{A})$.

Def. Let $\mathcal{B}$ be a category. A class of morphisms $S \subseteq \mathrm{Mor}(\mathcal{B})$ is called a localizing class if:

1). S is multiplicative, in the sense that $\forall X \in \mathrm{Ob}(\mathcal{B})$, $\mathrm{Id}_X \in S$; and if $s_1, s_2 \in S$, $s_1 s_2 \in \mathrm{Mor}(\mathcal{B})$, then $s_1 s_2 \in S$.

2). Given any diagram:

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

with $s \in S$ (we will use orange arrows to denote morphisms in S), we can complete it into a square, i.e. $\exists\, W, g, t$ s.t. $t \in S$ and:

$$\begin{tikzcd} W \arrow[r, "g"] \arrow[d, "t"] \arrow[dr, phantom, "\circlearrowright"] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

Similarly, we can complete diagrams

$$\begin{tikzcd} & Z \\ X & Y \arrow[l] \arrow[u] \end{tikzcd} \Longrightarrow \begin{tikzcd} W & Z \arrow[l] \\ X \arrow[u] \arrow[ur, phantom, "\circlearrowright"] & Y \arrow[l] \arrow[u] \end{tikzcd}$$

3). Given $X \underset{g}{\overset{f}{\rightrightarrows}} Y$. There exists $s: Z \longrightarrow X$ s.t. $fs = gs$ iff there exists $t: Y \longrightarrow W$ s.t. $tf = tg$.

Rmk: We will formally invert all morphisms in S, so that a diagram:

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \text{(resp.} \quad \begin{tikzcd} & Z \\ X & Y \arrow[l, "g"'] \arrow[u, "t"'] \end{tikzcd} \quad \text{)}$$

formally represents a morphism $s^{-1}f: X \longrightarrow Z$ (resp. $gt^{-1}: Z \longrightarrow X$). Condition (2) above is to guanrantee that when we are composing such morphisms $s_1^{-1}f_1$, $s_2^{-1}f_2$, we can find a "common denominator" for s_1 and s_2.

Def. (Localization of $\mathcal{B}$ with respect to S).
We define the localization of $\mathcal{B}$ with respect to the localizing class S to be the category $\mathcal{B}[S^{-1}]$ with:

$\mathrm{Ob}\,\mathcal{B}[S^{-1}] = \mathrm{Ob}\,\mathcal{B}$

$$\mathrm{Mor}(X, Y) \triangleq \left\{ \begin{tikzcd} & X' \arrow[dl, "s"'] \arrow[dr, "f"] & \\ X & & Y \end{tikzcd} \ (\triangleq fs^{-1}),\ s \in S,\ f \in \mathrm{Mor}_{\mathcal{B}}(X', Y) \right\} \Big/ \sim$$

where $fs^{-1} \sim gt^{-1}$ iff $\exists\, X'''$ and arrows making the diagram below commute:

$$\begin{tikzcd} & X''' \arrow[dl] \arrow[dr] & \\ X' \arrow[d, "s"'] \arrow[drr, "f" near end] & & X'' \arrow[d, "g"] \arrow[dll, "t"' near end] \\ X & & Y \end{tikzcd}$$

Composition of morphisms is defined as follows. Given $fs^{-1}: X \longrightarrow Y$, $gt^{-1}: Y \longrightarrow Z$,

$$\begin{tikzcd}
& X' \arrow[dl, "s"'] \arrow[dr, "f"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

we can find X'' and arrows by property 2) of S:

$$\begin{tikzcd}
& & X'' \arrow[dl, "r"'] \arrow[dr, "h"] & & \\
& X' \arrow[dl, "s"'] \arrow[dr, "f"] \arrow[rr, phantom, "\circlearrowright"] & & Y' \arrow[dl, "t"'] \arrow[dr, "g"] & \\
X & & Y & & Z
\end{tikzcd}$$

By property 1) of S, $s \circ r \in S$, and the composition is defined to be:

$$\begin{tikzcd}
& X' \arrow[dl, "s \circ r"'] \arrow[dr, "g \circ h"] & \\
X & & Y
\end{tikzcd}$$

One can check that "$\sim$" is an equivalence relation, and that the composition is well-defined and associative. We will check the composition, and leave the rest as exercise.

Suppose we have r_1, r_2, g_1, g_2 s.t. the diagram commutes:

$$\begin{tikzcd}
& X'' \arrow[d, "r_1"'] \arrow[drr, "g_1", near start] & & X''' \arrow[dll, "r_2"', near start] \arrow[d, "g_2"] & \\
& X' \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & & Y' \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\
X & & Y & & Z
\end{tikzcd}$$

We would like $g_1 r_1^{-1} \sim g_2 r_2^{-1}$ so that $f_2 g_1 r_1^{-1} s_1^{-1} \sim f_2 g_2 r_2^{-1} s_1^{-1}$. We can complete $X'' \xrightarrow{r_1} X' \xleftarrow{r_2} X'''$ into a square, using property 2) of S, so that $r_1 \circ t_1 = r_2 \circ h$.

$$\begin{tikzcd}
 & & X^4 \arrow[dl, "t_1"'] \arrow[dr, "h"] & & \\
 & X'' \arrow[d, "r_1"'] \arrow[drr, "g_1"] & & X''' \arrow[dll, "r_2"'] \arrow[d, "g_2"] & \\
 & X' \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & & Y' \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\
X & & Y & & Z
\end{tikzcd}$$

Then $s_2 g_2 h = f_1 r_2 h = f_1 r_1 t_1 = s_2 g_1 t_1$. Property 3) of S implies that we can find $t_2 : X^5 \longrightarrow X^4$ s.t. $g_1 t_1 t_2 = g_2 h t_2$:

$$\begin{tikzcd}
 & & X^5 \arrow[d, "t_2"'] & & \\
 & & X^4 \arrow[dl, "t_1"'] \arrow[dr, "h"] & & \\
 & X'' \arrow[d, "r_1"'] \arrow[drr, "g_1"] & & X''' \arrow[dll, "r_2"'] \arrow[d, "g_2"] & \\
 & X' \arrow[dl, "s_1"'] \arrow[dr, "f_1"] & & Y' \arrow[dl, "s_2"'] \arrow[dr, "f_2"] & \\
X & & Y & & Z
\end{tikzcd},$$

showing that $g_1 r_1^{-1} \sim g_2 r_2^{-1}$. $\square$

Rmk: The proof that $\mathcal{B}[S^{-1}]$ is well-defined consists of "filling up diagrams with directed cubes". An interesting problem would then be whether there is some sort of "directed homotopy theory" that is hidden behind.

Now, let $\mathcal{A}$ be an abelian category. Recall that we can form the abelian category of chain complexes in $\mathcal{A}$, denoted $\mathcal{K}om(\mathcal{A})$. Then by modding out all null-homotopic chain maps we obtain the category $Com(\mathcal{A})$ of chain complexes up to homotopy. It is no longer abelian, but additive and triangulated. The cohomology functor $H^*: \mathcal{K}om(\mathcal{A}) \longrightarrow \mathcal{A}$ descends to $Com(\mathcal{A})$ and a morphism $f: M \longrightarrow N$ is called a quasi-isomorphism (qis) in both $\mathcal{K}omp(\mathcal{A})$ and $Com(\mathcal{A})$ if:

$$H^*(f): H^*(M) \xrightarrow{\cong} H^*(N).$$

Prop. 7. (a). $S = \{qis\}$ is a localizing class in $Com(\mathcal{A})$ (but not in $\mathcal{K}om(\mathcal{A})$).

(b). $D(\mathcal{A}) \triangleq Com(\mathcal{A})[S^{-1}]$, the derived category of $\mathcal{A}$ is additive and triangulated.

Sketch of pf.

One just checks the def. of a localizing class. A useful tool is the cone construction in $\mathcal{K}om(\mathcal{A})$ / $Com(\mathcal{A})$. For instance, to check 2), we are then given maps of chain complexes:

$$\begin{tikzcd} & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

The way to produce a square would be to regard

$$X \oplus Z \xrightarrow{(f,s)} Y$$

as maps of chain complexes and shift the degree by $[-1]$. In other words, we just complete the diagram by:

$$\begin{tikzcd}
\left(\begin{tikzcd}[sep=small] & Z \arrow[d, "s"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \right) \arrow[r] \arrow[d] & Z \arrow[d, "s"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

and the maps are the obvious projections. The diagram is only commutative in $\mathrm{Com}(\mathcal{A})$.

To check 3) we note that since $\mathrm{Com}(\mathcal{A})$ is additive, it suffices to show that

$$\left\| \begin{array}{l} Z \xrightarrow{s} X \xrightarrow{f} Y \ : \ fs = 0 \ \text{in } \mathrm{Com}(\mathcal{A}) \quad \text{iff} \quad \exists\, W,\ t : Y \longrightarrow W \\ X \xrightarrow{f} Y \xrightarrow{t} W \ : \ tf = 0 \ \text{in } \mathrm{Com}(\mathcal{A}) \end{array} \right.$$

W is then constructed as the "cone" of $Z \xrightarrow{s} X \xrightarrow{f} Y$, namely the total complex of W is just $Z[2] \oplus X[1] \oplus Y$ with differentials d_Z, d_X, d_Y and s, f, $f \circ s$ with appropiate signs. In other words, W can be regarded as:

$$Z \xrightarrow{s} X \xrightarrow{f} Y \xrightarrow{id_Y} \underbrace{(Z \xrightarrow{s} X \xrightarrow{f} Y)}_{W}$$

Again $tf = 0$ is only true in $\mathrm{Com}(\mathcal{A})$ but not $\mathcal{K}om(\mathcal{A})$. $\square$

Rmk: People know how to localize commutative rings and categories, but not much is done about the intermediate case, namely localizing non-commutative rings. For instance, it's known that for the first Weyl algebra

$$A = \mathbb{C}\langle x, \partial_x \rangle / \langle \partial_x \cdot x - x \cdot \partial_x - 1 \rangle$$

the set $S = A \setminus \{0\}$ is localizing. But not much is know about the localized algebra $A[S^{-1}]$. It's a risky but rewarding area of new math to be explored.

In summary, throughout the process of localization, we pass from any abelian category $\mathcal{A}$ to its derived category $D(\mathcal{A})$, which is additive, triangulated (see the next subsection).

$$\mathcal{A} \longrightarrow \mathcal{K}om(\mathcal{A}) \longrightarrow Com(\mathcal{A}) \longrightarrow D(\mathcal{A})$$

Furthermore, $\mathcal{A}$ fully and faithfully embeds in $D(\mathcal{A})$