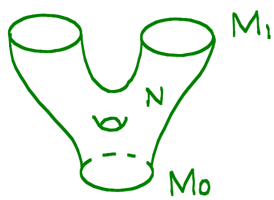
§2. Topological Quantum Field Theories

Cob_n and n-d TQFT

First, we describe the category Cob_n. It has as:

(1). objects : closed oriented $(n-1)$-dim'l manifolds

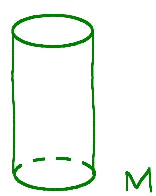
(2). morphisms: oriented n-manifolds with boundaries (up to diffeomorphism relative to boundaries), which are regarded as morphisms from the incoming bounday to the outgoing boundary:



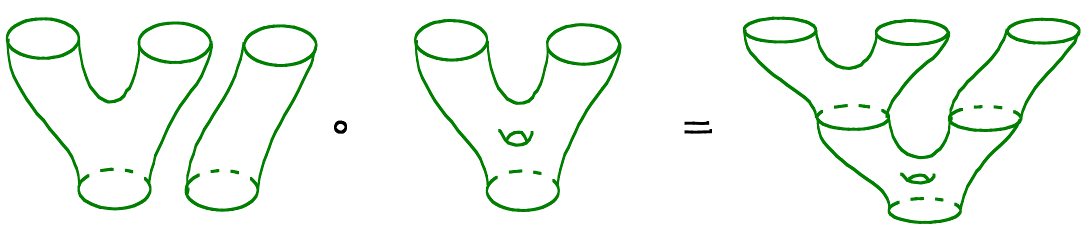
$$\partial N = -M_0 \sqcup M_1$$

where $-M_0$ means M_0 with the opposite orientation.

The identity morphism id_M is given by the product manifold $M\times[0,1]$:



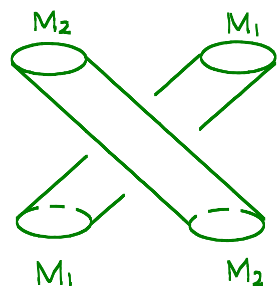
Composition of morphisms are given by gluing manifolds:



(3). symmetric monoidal structure, given by disjoint union of objects and morphisms, with the empty $(n-1)$ manifold as the monoidal unit:

(a). $\bigcirc_M \sqcup \phi = \bigcirc_M$

(b). $\bigcirc_{M_1} \sqcup \bigcirc_{M_2} \cong \bigcirc_{M_2} \sqcup \bigcirc_{M_1}$:



(the braiding)

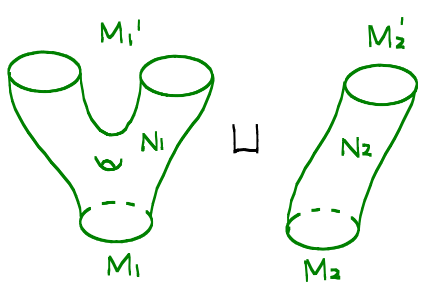
(c). The associativity of $\sqcup$:

$$(M_1 \sqcup M_2) \sqcup M_3 \cong M_1 \sqcup (M_2 \sqcup M_3)$$

together with various coherence conditions on these structures.

Def. (n-d TQFT) An n-d TQFT is a tensor (monoidal) functor from Cob_n to some additive symmetric monoidal category (i.e. a functor preserving (a). (b). (c) above, e.g. the category of $\mathbb{k}$-vector spaces with the usual tensor structure).

Thus, an n-d TQFT F sends:



$$\longmapsto \quad \begin{tikzcd} F(M_1' \sqcup M_2') = F(M_1') \otimes F(M_2') \\ F(M_1 \sqcup M_2) = F(M_1) \otimes F(M_2) \arrow[u, "F(N_1 \sqcup N_2) = F(N_1) \otimes F(N_2)"'] \end{tikzcd}$$

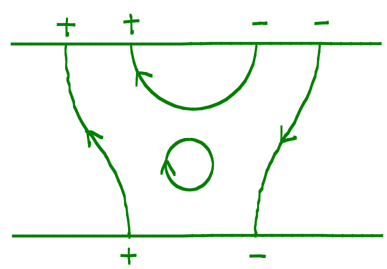
Notice that it's easy to construct / classify 1 and 2d TQFT's. But 3d TQFT is much harder. We will mostly be dealing with these cases in this course. For 4d TQFT's, only some toy models are known and beyond 4d, no interesting example is known.

1d TQFT's (over a field $\mathbb{k}$)

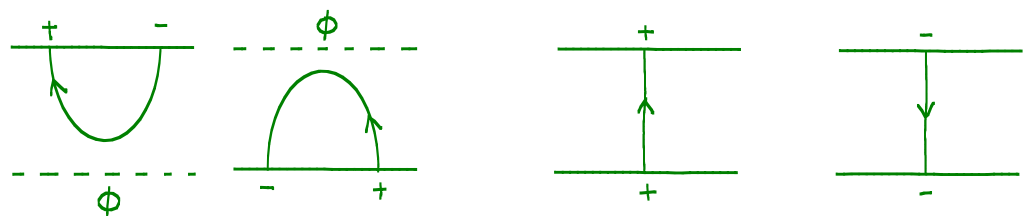
There are only two connected oriented 0-dim'l manifolds:

$$\{\bullet^+, \bullet^-\}$$

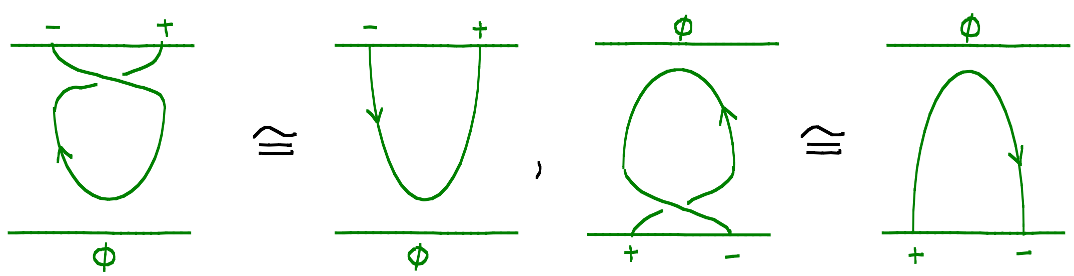
Cobordisms between 0 dim'l manifolds are just oriented 1 manifolds:



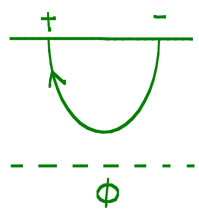
and they are built up from the following basic building blocks:



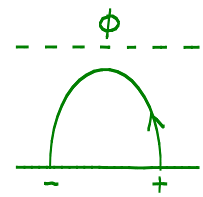
The "cup" and "cap" in the opposite orientation can be constructed by twisting:



Now let F be any 1d TQFT, and set $V = F(\bullet^{+})$, $W = F(\bullet^{-})$, and it follows from the monoidal unit axiom (a) that $F(\phi) = \Bbbk$, the ground field. Then for the cup and cap, we get:

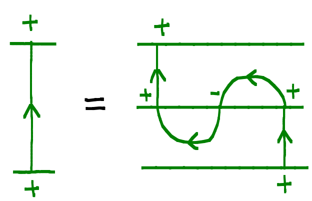


$\xrightarrow{F}$ $\Bbbk \xrightarrow{\alpha} V \otimes W$;



$\xrightarrow{F}$ $W \otimes V \xrightarrow{\beta} \Bbbk$

The following isotopy gives:



$$\xrightarrow{F} \quad V \xrightarrow{Id_V} V \quad = \quad V \xrightarrow{\alpha \otimes Id_V} V \otimes W \otimes V \xrightarrow{Id_V \otimes \beta} V$$

i.e. $Id_V = (Id_V \otimes \beta)(\alpha \otimes Id_V)$. If we set $\alpha(1) = \sum_{i=1}^{n} v_i \otimes w_i$, we get that $\forall\, v \in V$,

$$\begin{aligned} v &= (Id_V \otimes \beta)(\alpha \otimes Id_V)(v) \\ &= (Id_V \otimes \beta)\Big(\sum_{i=1}^{n} v_i \otimes w_i \otimes v\Big) \\ &= \sum_{i=1}^{n} v_i \cdot \beta(w_i \otimes v) \end{aligned}$$

In particular, V is spanned by $\{v_i,\ i = 1, \cdots, n\}$. Similarly, using the other pair of cup and cap, we obtain the finite dimensionality of W.

Lemma 1. Both V and W are finite dimensional. $\square$

Using $\alpha: \Bbbk \longrightarrow V \otimes W$ and $\beta: W \otimes V \longrightarrow \Bbbk$, we get transposed maps $\alpha': V^* \longrightarrow W$, $\beta': W \longrightarrow V^*$. Then

$$Id_V = (Id_V \otimes \beta)(\alpha \otimes Id_V) \iff \beta' \circ \alpha' = Id_{V^*} \Longrightarrow \dim V = \dim V^* \le \dim W.$$

Similarly, reversing all the arrows and signs in the pictures, one obtains that:

$$\dim W \leq \dim V^* = \dim V.$$

Combined, these give:

Lemma 2. $\alpha' : V^* \xrightarrow{\cong} W$, $\beta' : W \xrightarrow{\cong} V^*$ are isomorphisms of finite-dimensional vector spaces. $\square$

Using the identification $\beta' : W \xrightarrow{\cong} V^*$, we obtain the canonical forms of α and β:

$$\alpha : \Bbbk \longrightarrow V \otimes V^*, \quad 1 \longmapsto \sum_{i=1}^{\dim V} e_i \otimes e_i^*$$

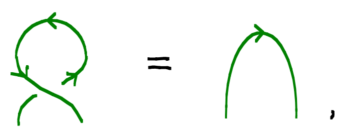
$$\beta : V \otimes V^* \longrightarrow \Bbbk, \quad v \otimes w^* \longmapsto w^*(v),$$

where $\{e_i\}$ is a basis of V and $\{e_i^*\}$ its dual.

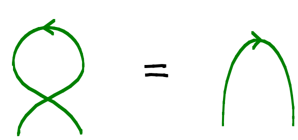
E.g. (1). Look at $F(\bigcirc) : \Bbbk \longrightarrow \Bbbk$ which is just a number. Then

$$\begin{aligned} F(\bigcirc) &= F(\frown) \circ F(\times) \circ F(\smile) \\ &= \sum_{i=1}^{\dim V} e_i^*(e_i) \\ &= \dim V \end{aligned}$$

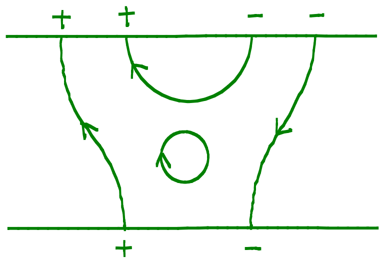
Here we used the fact that



which, without regard to ambient spaces, could also be depicted as



(2). Let's apply F to the cobordism at the beginning of the section:

F() $= \dim V \cdot \begin{array}{c} V\otimes V\otimes V^*\otimes V^* \\ \uparrow \scriptstyle{Id_V\otimes\alpha'\otimes Id_{V^*}} \\ V\otimes V^* \end{array}$

We summarize the above discussions into:

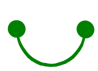
Thm 3. We have a bijection of sets:

$$\left\{\begin{array}{c}\text{1d TQFT's over}\\ \text{a field } \Bbbk\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Finite dim'l } \Bbbk\text{-}\\ \text{vector spaces}\end{array}\right\} \qquad \square$$

Unoriented 1d TQFT

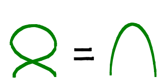
Without orientation, there is only one connected 0-manifold, namely a single point. Going over similar arguments as before, if F is an unoriented 1d TQFT / $\Bbbk$, then

(i). $F(\bullet) = V$, a finite dimensional $\Bbbk$-space;

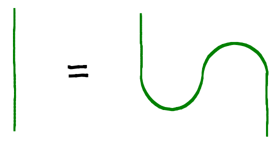
(ii). F() $= \begin{array}{c} V\otimes V \\ \uparrow \\ \Bbbk \end{array}$: a distinguished symmetric element;

F() $= \begin{array}{c} \Bbbk \\ \uparrow \\ V\otimes V \end{array}$: a symmetric non-degenerate bilinear form.

The symmetries come from:

 ,

and the bilinear form is non-degenerate since



It also follows from this that, if we denote the distinguished element by $\sum a_{ij} v_i \otimes v_j$, and the bilinear form by $\sum b_{ij} v_i^* \otimes v_j^*$ with respect to some basis $\{v_i\}$ of V, then

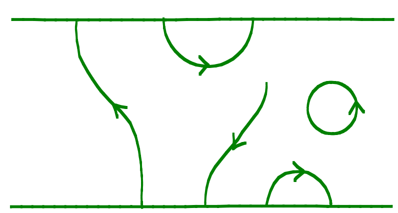
$$[a_{ij}] = [b_{ij}]^{-1}$$

and the matrices $[a_{ij}]$, $[b_{ij}]$ are symmetric. Hence we have show that:

Thm. 4. We have a bijection of sets:

$$\left\{\begin{array}{l}\text{Unoriented 1d-}\\ \text{TQFT's over}\\ \text{a field } \Bbbk\end{array}\right\} \longleftrightarrow \left\{\begin{array}{c}\text{Finite dim'l } \Bbbk\text{-spaces}\\ \text{with a non-degenerate}\\ \text{symmetric bilinear form}\end{array}\right\} \qquad \square$$

Ex. Classify generalized 1d-TQFT's where cobordisms are allowed to end in the middle:



2d TQFT's (over a field $\Bbbk$)

Before specializing to 2d, we first prove a general result:

Lemma 5. Let F be any n-d TQFT and M a closed, oriented

$(n-1)$ dim'l manifold. Then $\dim_{\mathbb{k}} F(M) < \infty$.

Pf: For any $M \in Ob(Cob_n)$ we can construct a functor

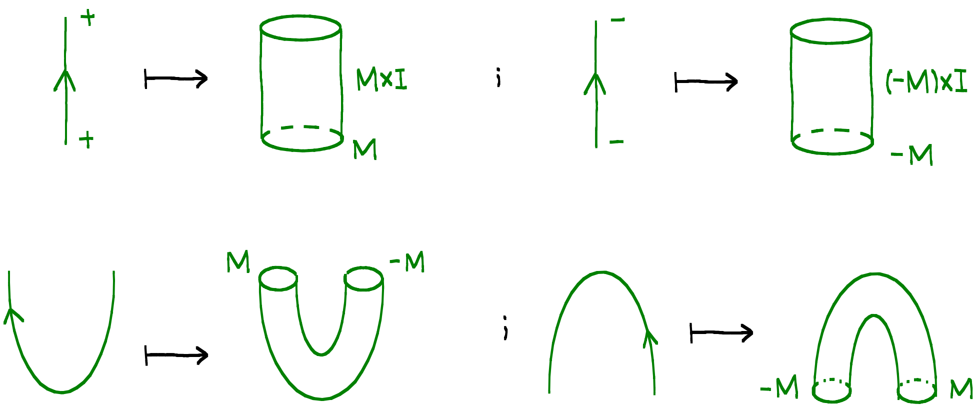
$$C_M : Cob_1 \longrightarrow Cob_n,$$

defined as follows:

on objects:

$$\bullet^{+} \longmapsto M \quad ; \quad \bullet^{-} \longmapsto -M$$

on morphisms:



Now composing F with C_M we get a 1d TQFT, which assigns

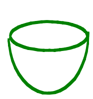
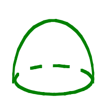
$$F \circ C_M(\bullet^{+}) = F(M).$$

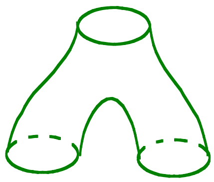
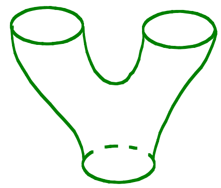
Then we are done by lemma 1. $\square$

Now we look at Cob_2, whose objects are multiple copies of circles:

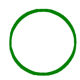
$$Ob(Cob_2) = \left\{ \underbrace{\bigcirc\bigcirc \cdots \bigcirc}_{m} \;\middle|\; m \in \mathbb{Z}_{\geq 0} \right\},$$

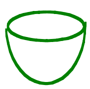
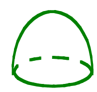
and whose morphisms are glued from the following basic pieces:

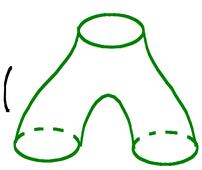
 cup ;  cap

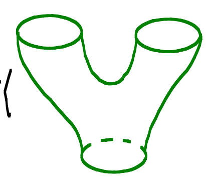
 pants ;  copants

Suppose F is a 2d TQFT/$\mathbb{k}$. We define the following:

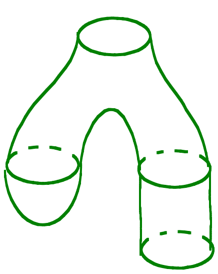
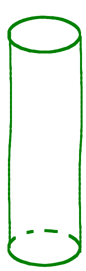
$F($  $) = A$

$F($  $) = \mathbb{k} \xrightarrow{i} A$: unit , $F($  $) = A \xrightarrow{\varepsilon} \mathbb{k}$: trace

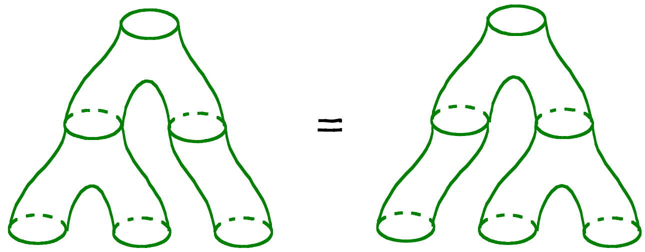
$F($  $) = A \otimes A \xrightarrow{m} A$: multiplication,

$F($  $) = A \xrightarrow{\Delta} A \otimes A$: comultiplication.

Then the following relations imply:

 $=$  $\overset{F}{\Longrightarrow}$ $m \circ (i \otimes id_A) = id_A : A \longrightarrow A$

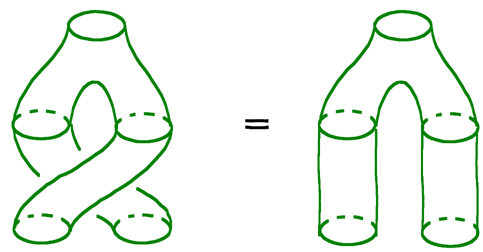
Next.



$\overset{F}{\Longrightarrow}$

$$\begin{tikzcd}
& A & \\
A^{\otimes 2} \arrow[ur, "m"] & \circlearrowright & A^{\otimes 2} \arrow[ul, "m"'] \\
& A^{\otimes 3} \arrow[ul, "m\otimes id_A"] \arrow[ur, "id_A\otimes m"'] &
\end{tikzcd}$$

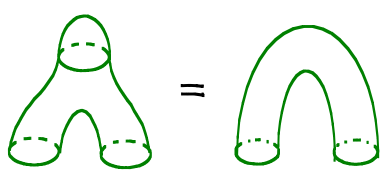
i.e. A is an associative algebra. It's also commutative by considering:



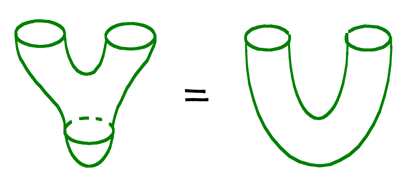
(They are diffeomorphic by taking a twist!)

Similarly by using the copants, we can show that A is also a cocommutative coalgebra.

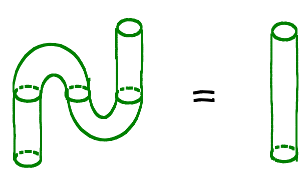
Furthermore, by gluing the cap and the pants, the cup and copants, we obtain the U-turns



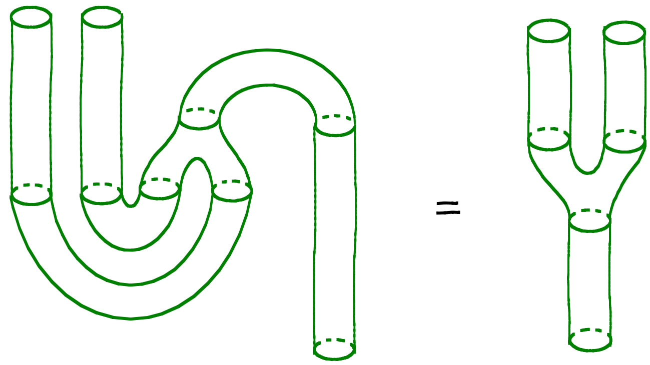
;



which give rise to the relation:



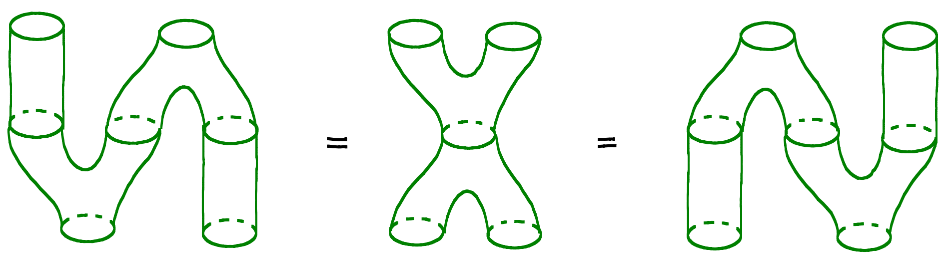
It follows just as in the 1d TQFT case that $\varepsilon\circ m$ is a symmetric, non-degenerate bilinear form on A (non-degenerate meaning that $\forall a\in A,\ a\neq 0,\ \exists\, b\in A$ s.t. $\varepsilon(a\cdot b)\neq 0$). In particular, $A\cong A^*$ via $\varepsilon\circ m$. Under this identification, we see from:



that the comultiplication Δ is just the dual of multiplication:

$$\Delta: A \cong A^* \xrightarrow{m^*} (A\otimes A)^* \cong A^*\otimes A^* \cong A\otimes A.$$

It follows that Δ is a map of (A,A)-bimodules since m is. But this also admits the following pictorial proof:



The above discussion reminds us of some basic notions of representation theory:

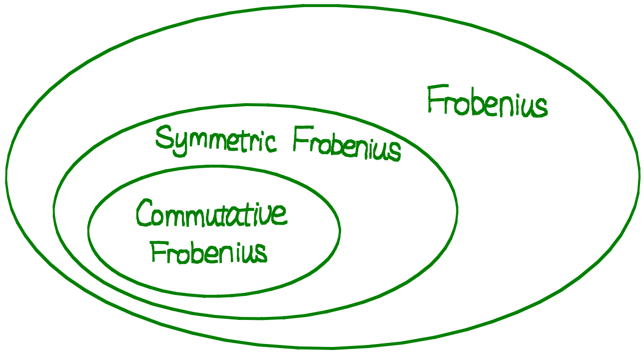
Def. Let A be a unital, associative, finite dim'l $\Bbbk$-algebra.

1). A is called Frobenius if $\exists\ \varepsilon: A \longrightarrow \Bbbk$ a non-degenerate form called the trace. Here non-degenerate means that $\forall a\in A,\ a\neq 0,\ \exists b\in A$ s.t. $\varepsilon(ab)\neq 0$

2). A is called symmetric Frobenius if $\varepsilon(ab)=\varepsilon(ba),\ \forall a,b\in A$.

3). A is called commutative Frobenius if it's commutative and Frobenius.

It follows from def. that:



All the discussions above lead to the following:

Thm. 6. We have a bijection of sets:

$$\left\{ \begin{array}{c} \text{2D TQFT over} \\ \text{a field } \mathbb{k} \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Commutative Frobenius} \\ \text{algebras over } \mathbb{k} \end{array} \right\}$$

$\square$

E.g. Frobenius algebras.

1). If G is a finite group, then $\mathbb{k}[G]$ is symmetric Frobenius, with trace form:

$$\varepsilon(g) = \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise} \end{cases}$$

2). $\mathrm{Mat}(n, \mathbb{k})$ is symmetric Frobenius with the usual trace form. E.g. 1) in case when $\mathbb{k}[G]$ is semisimple is just a product of this case.

3). If M is a closed compact $\mathbb{k}$-oriented manifold, then Poincaré duality tells us that

$$\varepsilon = \int_M : H^{top}(M) \longrightarrow \mathbb{k}$$

gives a non-degenerate trace on the cohomology ring $H^*(M, \mathbb{k})$. It's super-commutative and thus it's a commutative Frobenius algebra in the category of super-vector spaces.

In particular, the even part subalgebra $H^{2*}(M, \mathbb{k})$ is a commutative Frobenius algebra if M is real even dimensional. For instance, consider $\mathbb{CP}^{n-1}$:

$$H^*(\mathbb{CP}^{n-1}, \Bbbk) = H^{2*}(\mathbb{CP}^{n-1}, \Bbbk)$$
$$\cong \Bbbk[x]/(x^n),$$

with the trace form on the latter given by:

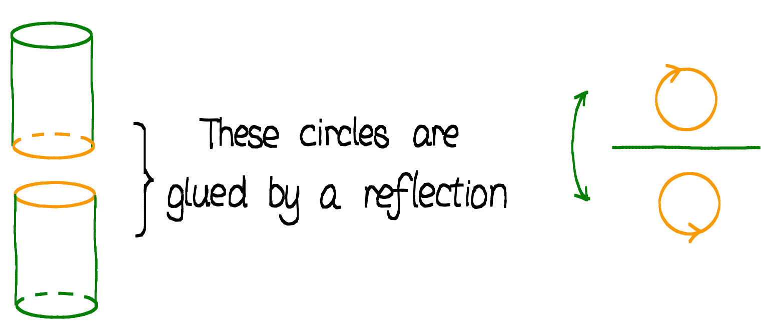
$$\varepsilon(x^i) = \begin{cases} 1 & i = n-1 \\ 0 & \text{otherwise} \end{cases}$$

Unoriented 2d TQFT's

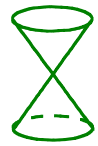
Let's now enlarge the category of Cob_2 by allowing unoriented 2-manifolds as morphisms in the category. Again let F be any 2d TQFT from this unoriented Cob_2 to the category of $\Bbbk$-vector spaces, and set $A = F(\bigcirc)$.

As before, all the oriented cobordism relations force A to be a commutative Frobenius algebra / $\Bbbk$. But now, we have some extra cobordism generators and relations that put more conditions on A.

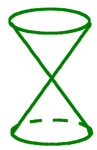
We have an extra generator that exchanges the orientation of S^1, constructed from:



and for convenience, we depict it by:

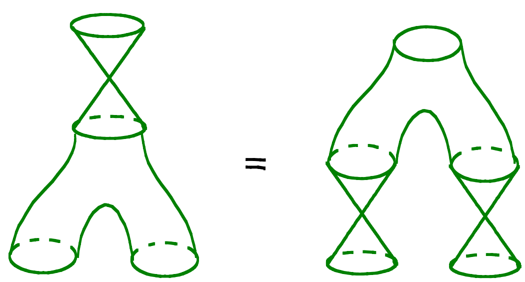


Then:

F() $\triangleq$ $\varphi : A \longrightarrow A$

is an order 2 $\Bbbk$-linear map. We have:

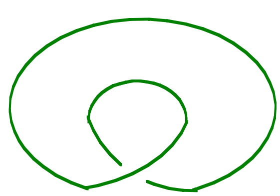
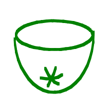
says that φ preserves the unit and trace. And



The top "reverser" slides down to change the orientation on the bottom

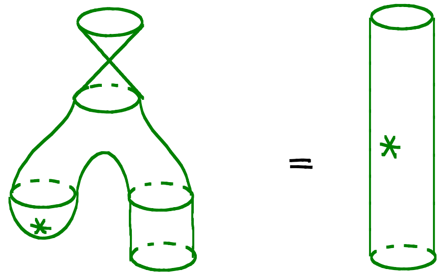
implies φ is an algebra homomorphism. Hence, φ is an involution of A as a commutative Frobenius algebra.

Next, the Mobius band, since it has only one boundary component, can be regarded as an unoriented cobordism from ϕ to S^1:

 denoted by 

Then it picks out a special element $\Bbbk \longrightarrow A$, $1 \longmapsto \theta$.

The topological relation:

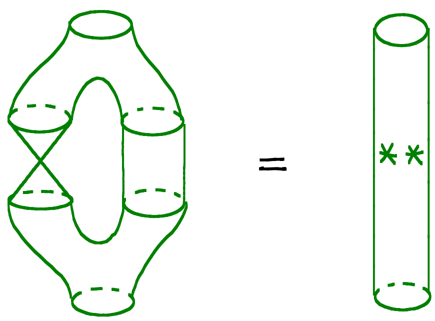


This is true since if we close both up by a cup and cap, they all become $\mathbb{RP}^2$'s.

implies that, $\forall a \in A$, we have:

$$\varphi(\theta \cdot a) = \theta a.$$

Similarly, the topological relation:



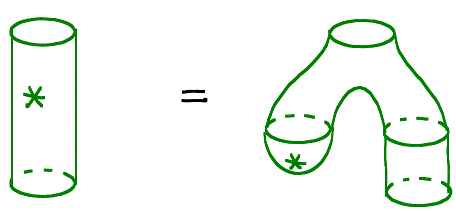
They are equal since both are Klein bottles with 2 punctures.

gives rise to the algebraic:

$$m(\varphi \otimes Id_A)\Delta(1) = \theta^2.$$

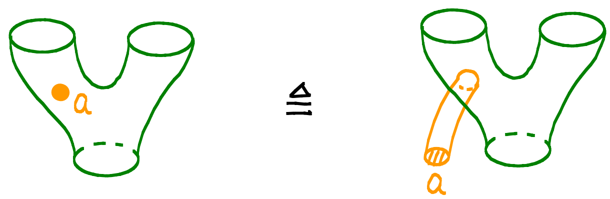
Rmk: An extension of diagramatics:

In the above discussion, we used the diagramatics of representing the map $A \longrightarrow A$, $a \mapsto \theta a$ by the picture:

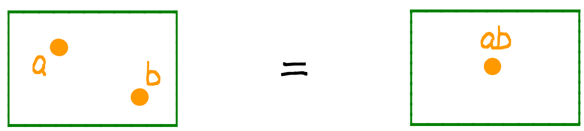


In general, we can extend this "multiplication by an element in A"

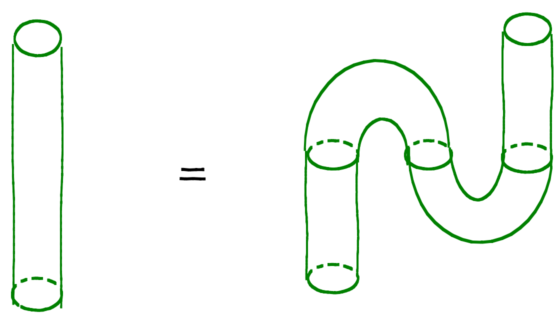
operation by locally "sewing" a patch labeled by the element. For instance,

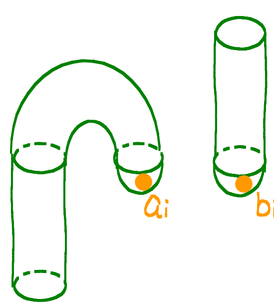


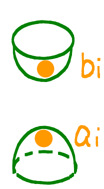
These local pictures can merge by associativity:



In this notation, if we write $\Delta(1) = \sum_{i=1}^{\dim A} a_i \otimes b_i$, where $\{a_i\}$, $\{b_i\}$ are dual basis of A under ε, then we have:

$Id_A =$ 

$= \sum_{i=1}^{\dim A}$ 

$= \sum_{i=1}^{\dim A}$ 

which, algebraically, just says that, $\forall x \in A$,

$$x = \sum_{i=1}^{\dim A} \varepsilon(a_i x) \cdot b_i$$

We summarize this subsection in the following:

Thm. 7. Unoriented 2d TQFT's over a field $\Bbbk$ are in bijection with commutative Frobenius algebras A with an involution φ, a distinguished element θ s.t. $\forall a \in A$, the following holds:

(i). $\varphi(\theta a) = \theta a$

(ii). $m(\varphi \otimes Id_A)\Delta(1) = \theta^2$.

Ex. 1). Prove thm. 7 by checking that there are no other constraints.
2). Find such A's in representation theory of finite dimensional algebras.
Notice that if char$\Bbbk \neq 2$, φ decomposes A into two parts:

$$A \cong A^+ \oplus A^-$$

s.t. $\varphi|_{A^+} = Id_{A^+}$, $\varphi|_{A^-} = -Id_{A^-}$. $\theta \in A^+$.

Module theoretical characterization of Frobenius algebras

We collect some basic module theoretical properties of finite dim'l Frobenius algebras, not necessarily commutative.

- A is Frobenius $\iff A \cong A^*$ as left A-modules
  $\iff A \cong A^*$ as right A-modules

Pf: First of all A^* is an (A,A)-bimodule, $\forall a,b,c \in A$, $f \in A^*$,

$$(a \cdot f \cdot b)(c) \triangleq f(acb).$$

Now $\varepsilon \in A^*$. Then $\forall a \in A$, the map

$$A \longrightarrow A^* : a \mapsto a \cdot \varepsilon$$

is an isomorphism of left A-modules: $\forall a, b \in A$

$$ba \longmapsto \varepsilon(- \cdot ba) = b \cdot \varepsilon(- \cdot a) = b \cdot (a \cdot \varepsilon)$$

To prove for right A modules take $a \mapsto \varepsilon \cdot a$ instead. Conversely, to recover ε from the module isomorphisms, just take the image of $1 \in A$ in A^*. $\square$

- A is symmetric Frobenius $\Longleftrightarrow$ $A \cong A^*$ as (A, A)-bimodules.

Indeed, fix $a, b \in A$, $\forall c \in A$,

$$\begin{aligned} a \cdot c \cdot b \mapsto \varepsilon(a \cdot c \cdot b \cdot -) &= a \cdot \varepsilon(c \cdot b \cdot -) \\ &= a \cdot \varepsilon(- \cdot c \cdot b) \\ &= a \cdot \varepsilon(- \cdot c) \cdot b \\ &= a \cdot \varepsilon(c \cdot -) \cdot b \end{aligned}$$ $\square$

- Projective modules over A coincide with injective A-modules.

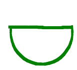
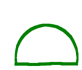
Indeed, taking the vector space dual maps projective modules to injective modules. $\square$

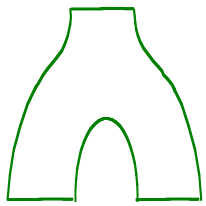
Compare with the case of $\mathbb{Z}$, $\Bbbk[x]$, or even Coh(projective varieties), whose collection of projective objects and injective objects are vastly different!

Thin surface TQFT's

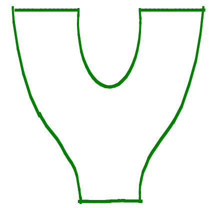
How can one obtain non-commutative Frobenius algebras as TQFT's? One possible cobordism category is the following cobordism of thin surfaces. The objects are closed intervals, and morphisms thin surfaces with boundaries and corners:

Objects:

Morphisms:  cup (unit)  cap (counit)

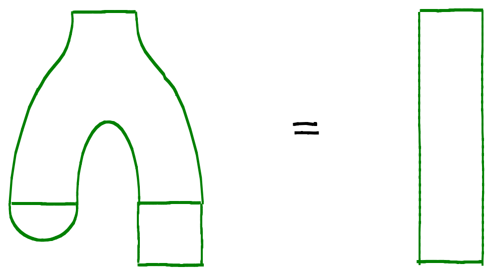


pants
(multiplication)



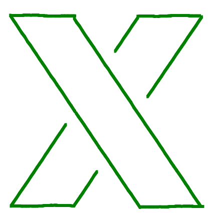
copants
(comultiplication)

subject to cobordism relations like:

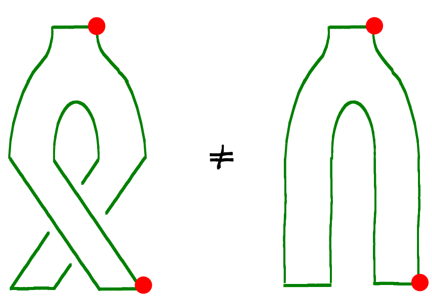


etc.

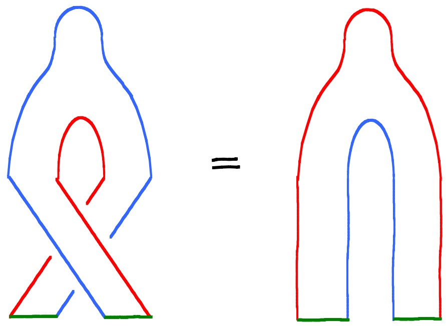
The braiding is given by:



Then we have:

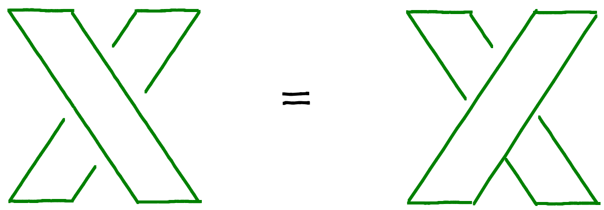


since the reds points on the right are connected by a cobordism boundary arc while the ones on the left are not. However, if we consider any TQFT from this category to $\Bbbk$ vector spaces, we always get symmetric Frobenius algebras. This is because:



There is a diffeomorphism of strips sending blue boundaries to blue boundaries, red to red, and fixing the bottoms.

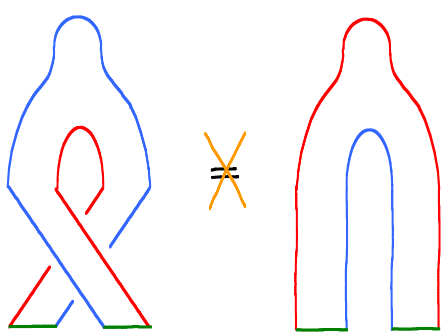
Note also that the braiding satisfies:

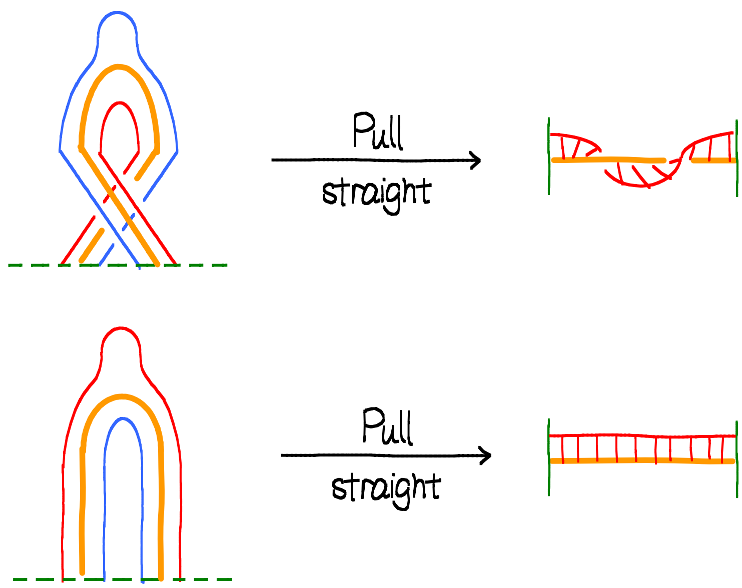


for the same reason as above.

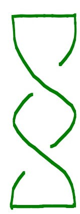
Ex. An attempt for non-symmetric cases.

If we want to obtain non-symmetric ones, we should differentiate the above pictures. Notice that inside $\mathbb{R}^3$, the pictures are different if we take into account of the winding number of the boundary around the central curve:





Hence we may try to "unwind" the above situation, by introducing the following "untwistor":



,

which might play the role of the Nakayama automorphism when applying a TQFT:

Recall the notion of the "Nakayama automorphism" of a Frobenius algebra. Let A be a Frobenius algebra, and fix $a \in A$. Since ε is non-degenerate, although $\varepsilon(a \cdot -) \in A^*$ need not be the same as $\varepsilon(- \cdot a) \in A^*$, it must be equal to $\varepsilon(- \cdot a')$ for some unique $a' \in A$. The Nakayama automorphism of A is then the linear map:

$$\tau : A \longrightarrow A$$

$$a \mapsto a'.$$

i.e. it's defined by the property that, $\forall\, b \in A$,

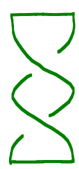
$$\varepsilon(a \cdot b) = \varepsilon(b \cdot \tau(a)).$$

It's easy to see that τ is an algebra automorphism: $\forall a, b, c \in A$,

$$\begin{aligned} \varepsilon(c \cdot \tau(ab)) &= \varepsilon(a \cdot b \cdot c) \\ &= \varepsilon(b \cdot c \cdot \tau(a)) \\ &= \varepsilon(c \cdot \tau(a) \cdot \tau(b)), \end{aligned}$$

and ε is non-degenerate. A is symmetric iff $\tau = id_A$.

From this discussion, we see that the image of the "untwistor ribbon" under a TQFT should be the Nakayama automorphism.

$F($  $) = \tau : A \longrightarrow A.$