§8. $\mathfrak{sl}_3$ Link Homology

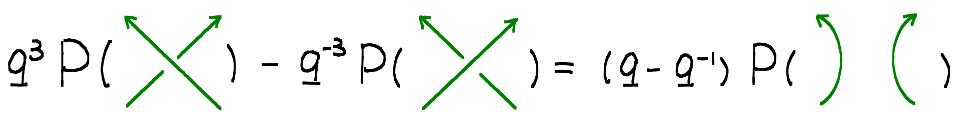
In this section we define a new link homology theory by categorifying the $U_q(\mathfrak{sl}_3)$-link invariant.

Recall that the $U_q(\mathfrak{sl}_3)$-link invariant is the assignement

$$P: \text{Oriented links} \longrightarrow \mathbb{Z}[q, q^{-1}]$$
$$L \longmapsto P(L)$$

determined by:

(1). the Skein relation



(2). The normalization condition:

$$P(\bigcirc) = q^2 + 1 + q^{-2} = [3]$$
$$P(\emptyset) = 1.$$

The Skein relation implies that, for any oriented link L, we have:

$$P(L \sqcup \bigcirc) = [3] \cdot P(L)$$

In particular,

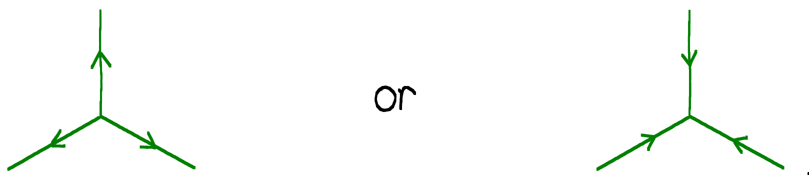
$$P(\underbrace{\bigcirc \bigcirc \cdots \bigcirc}_{k}) = [3]^k$$

Kuperberg's $\mathfrak{sl}_3$ spiders.

G. Kuperberg developed a planar graphical calculus for $U_q(\mathfrak{sl}_3)$

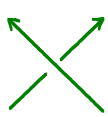
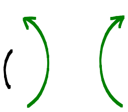
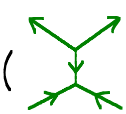
representation theory. (We will talk more about representation theory of quantum groups later. For the moment, one can just regard the quantum group $U_q(\mathfrak{sl}_3)$ as $U(\mathfrak{sl}_3)$, the universal enveloping algebra of the Lie algebra $\mathfrak{sl}_3$.)

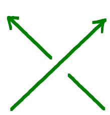
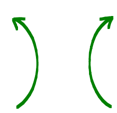
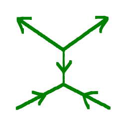
The planar graphs of Kuperberg, called webs, are planar oriented trivalent graphs whose vertices look like either



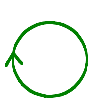
For instance, the following are webs:

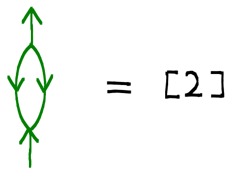
Note that webs are naturally bipartite. Kuperberg uses these webs to define the $U_q(\mathfrak{sl}_3)$ link invariant, by resolving crossings as:

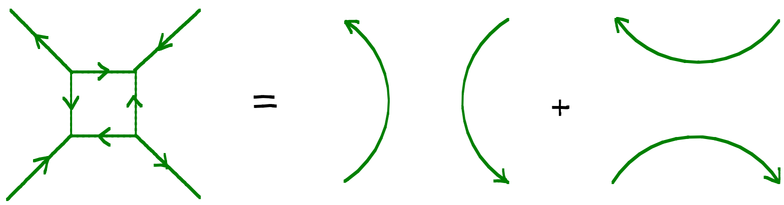
$P($  $) = q^{-2} P($  $) - q^{-3} P($  $)$

$P($  $) = q^{2} P($  $) - q^{3} P($  $)$.

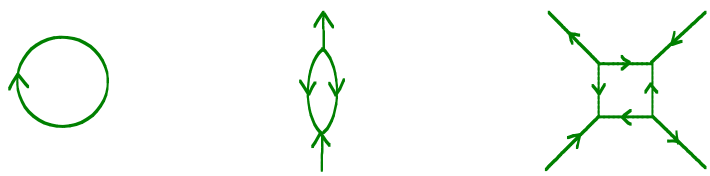
Webs are subject to the simplifications:

(1).  $= [3]$

(2). 

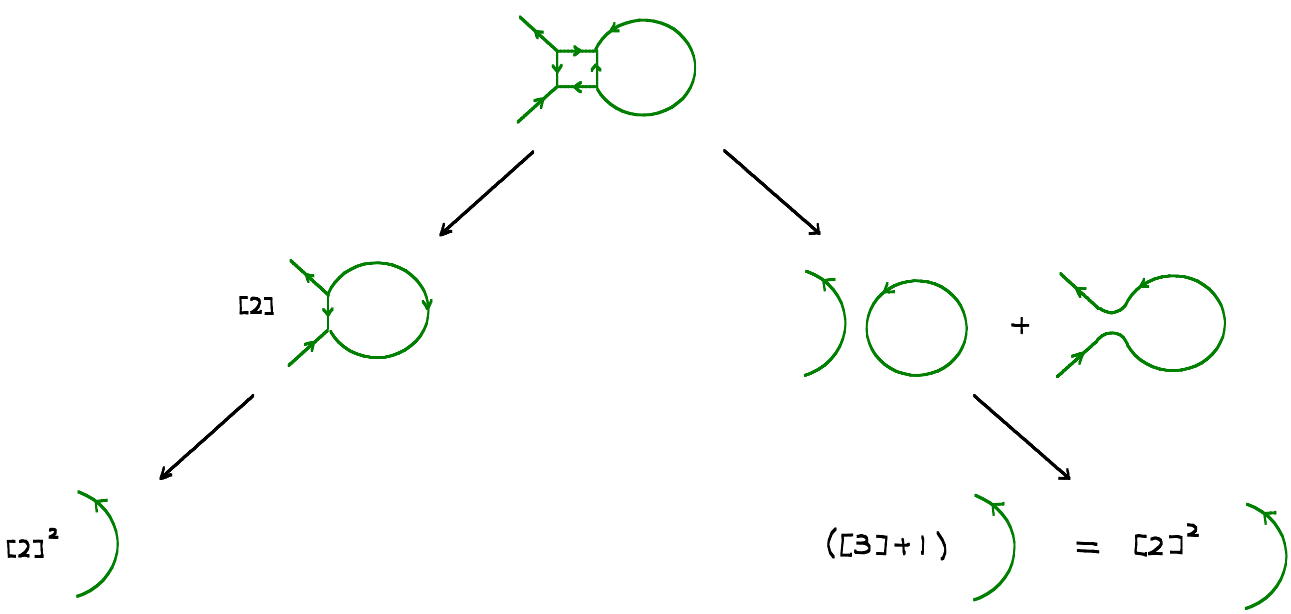
(3). 

Since any web contains either



,

using these simplification rules, we associate with any web a Laurent polynomial in $\mathbb{Z}[q, q^{-1}]$. Note that in fact these Laurent polynomials all have non-negative coefficients ($\in \mathbb{Z}_{\geq 0}[q, q^{-1}]$). One needs to check the consistency of this procedure,

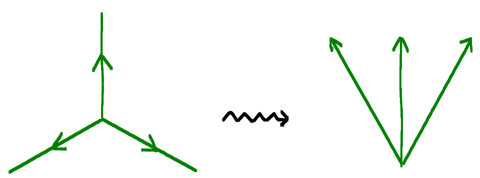


For this, see Kuperberg: Spiders for Rank 2 Lie Algebras.

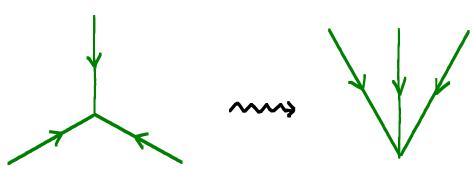
From a representation theory point of view, trivalent vertices can be regarded as intertwiners of $U_q(\mathfrak{sl}_3)$-modules. Namely, to points with +/- signs, we assign the standard $U_q(\mathfrak{sl}_3)$-module V / its dual V^*:

$$\overset{+}{\bullet} \longmapsto V \qquad\qquad \overset{-}{\bullet} \longmapsto V^*$$

A trivalent vertex, after bending a little bit, is assigned to the intertwiner of $U_q(\mathfrak{sl}_3)$-modules:



$$\longmapsto \quad \mathbb{C} \hookrightarrow V^{\otimes 3} \cong \underbrace{\Lambda^3 V}_{\cong \mathbb{C}} \oplus S^3 V \oplus \cdots$$



$$\longmapsto \quad \mathbb{C} \hookrightarrow V^{*\otimes 3} \cong \underbrace{\Lambda^3 V^*}_{\cong \mathbb{C}} \oplus S^3 V^* \oplus \cdots$$

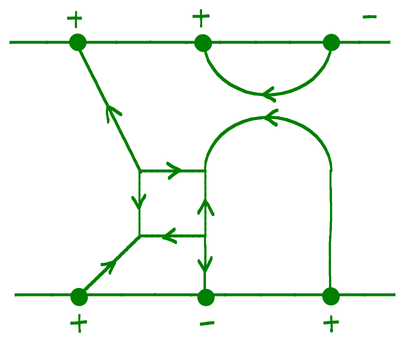
One readily sees that this assignment is invariant under rotations of the trivalent vertices.

The following thm. of Kuperberg states that, the braided monoidal category of $U_q(\mathfrak{sl}_3)$-modules is equivalent to the category of webs with boundaries modulo the above simplifications.

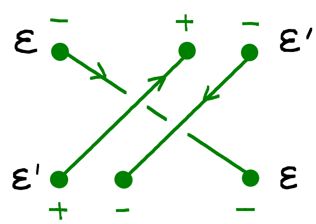
Thm. 1. (Kuperberg). (1). We have a braided monoidal category $\mathcal{G}$, with

Objects: sequences of signed points:

Morphisms: $\mathbb{C}(q)$-linear combinations of webs with boundaries:



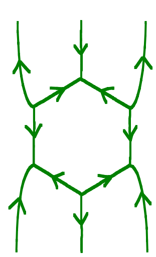
modulo the simplification relations. The braiding is given by resolutions of the usual braidings:



(2). More over, for any sequence of signs $\varepsilon, \varepsilon'$:

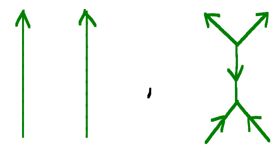
$$\mathrm{Hom}_{\mathcal{G}}(\varepsilon, \varepsilon') = \mathrm{Hom}_{U_q(\mathfrak{sl}_3)}(V^{\otimes \varepsilon}, V^{\otimes \varepsilon'}).$$

The $\mathbb{C}(q)$-space $\mathrm{Hom}_{\mathcal{G}}(\varepsilon, \varepsilon')$ has as basis the elliptic webs, which are webs like below (more complex than hexagons):



For the proof of thm 1, see the paper of Kuperberg.

E.g. The $\mathbb{C}(q)$-vector space $\mathrm{Hom}(++,++)$ is spanned by



Now, any oriented tangle T, after resolution, becomes a linear combination of webs with boundaries, and can thus be regarded as an element in $\mathrm{Hom}_{\mathcal{G}}(\varepsilon,\varepsilon')$ for some $\varepsilon,\varepsilon' \in \mathrm{Ob}(\mathcal{G})$. Thm 1 then tells us that this is a tangle invariant.

Rmk: Kuperberg in fact defined spiders for all rank 2 Lie algebras: $\mathfrak{sl}_3$, $\mathfrak{so}(5) \cong \mathfrak{sp}(2)$, G_2. The latter ones have similar diagrammatics as $\mathfrak{sl}_3$. But there are rational coefficients for web diagrams that occur in the intermediate steps that make them hard to categorify.

Cobordisms of webs

As for the Jones polynomial, we want to categorify the $\mathfrak{sl}_3$-link invariant $P(L)$ to a functorial (up to ± 1) bigraded homology theory $H(L) = \bigoplus_{i,j} H^{i,j}(L)$, so that

$$\chi(H(L)) = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{rank} H^{i,j}(L)\, q^j = P(L).$$

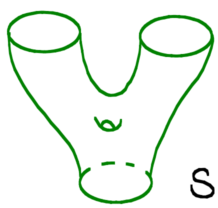
To achieve this, we again must assign a commutative Frobenius algebra A to a circle, whose graded rank is

$$\mathrm{gr.\,rk}\, A = [3].$$

By analogy, we will just take $A \cong H^*(\mathbb{C}P^2) \cong \mathbb{Z}[x]/(x^3)$, and degrees need to be shifted down accordingly:

$$A \triangleq H^*(\mathbb{C}P^2)\{-2\} \cong \underbrace{\mathbb{Z}\cdot 1}_{\deg -2} \oplus \underbrace{\mathbb{Z}\cdot x}_{\deg 0} \oplus \underbrace{\mathbb{Z}\cdot x^2}_{\deg 2}$$

Then this determines a 2d TQFT F, which when applied to an oriented surface S with boundary gives a deg. $-2\chi(S)$ map:

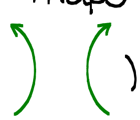
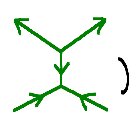

S

$$\longmapsto \qquad A^{\otimes k} \xrightarrow{F(S)} A^{\otimes \ell}$$

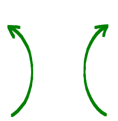
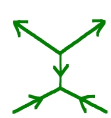
However, now we have more than just circles after resolving oriented link diagrams: we have webs Γ. But the fact that $P(\Gamma) \in \mathbb{Z}_+[q, q^{-1}]$ gives us a hint that $H(\Gamma)$, just as for circles, should only live in homological degree 0:

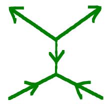
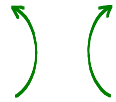
$$H(\Gamma) \cong \bigoplus_{j \in \mathbb{Z}} H^{0,j}(\Gamma)$$

Furthermore, we want maps

H() $\rightleftarrows$ H()

in both directions to take care of the resolution of crossings. In fact, we want chain complexes:

$0 \longrightarrow$ $\underbrace{\text{H(} }$$\underbrace{\text{)}\{-2\}}_{\text{Homological deg } 0}$ $\longrightarrow$ H()$\{-3\} \longrightarrow 0$,

$0 \longrightarrow H($  $)\{3\} \longrightarrow H($  $)\{2\} \longrightarrow 0$.

Homological deg 0

We also need a multiplicative categorification to take care of:

$$P(\Gamma_1 \sqcup \Gamma_2) = P(\Gamma_1) \cdot P(\Gamma_2)$$

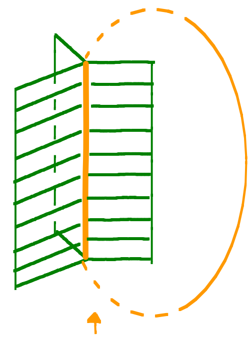
The most straight forward way to categorify this is to require:

$$H(\Gamma_1 \sqcup \Gamma_2) = H(\Gamma_1) \otimes_{\mathbb{Z}} H(\Gamma_2) .$$

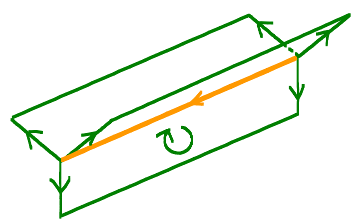
By the expected functoriality, we need to look for cobordisms between the webs with boundaries:

Such cobordisms are called *foams*, which are 2-dim'l CW complexes with singularities along circles / arcs:

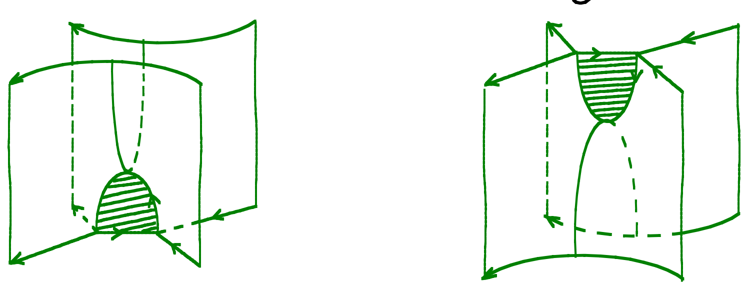
Near the singularities, foams look like "$Y \times I$" :



Finally, we require each 2-dim'l smooth cell of foams to be oriented so that the induced orientations on the boundaries of the cells agree with the prescribed orientations on the webs.



Together with the usual cobordisms in Cob_2 (birth of a circle, death of a circle, saddle moves), the foams:

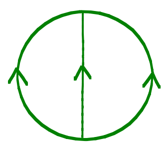


generate the cobordisms of webs (with boundaries).

Cohomology of flag varieties

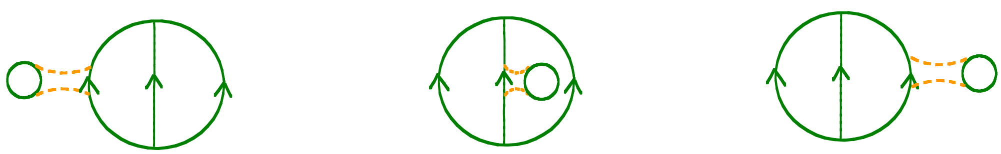
Before moving on to the TQFT of foams, we will first look at a special example of a web. A trick later will enable us to reduce closed foams to this case.

We now try to determine what graded module $H(\Theta)$ we should assign to the "Θ-web":



The "Θ-web"

Note that the three edges of Θ allow us to merge circles into it in three ways:



Thus we expect $H(\Theta)$ to be an A-module in 3 different ways. In general, if a web Γ has k edges, we would expect $H(\Gamma)$ to be an $A^{\otimes k}$-module.

Next, notice that, by the rule of removing a digon, we have:

$$P(\Theta) = [3]\cdot[2]$$

Since $\mathrm{gr.rk}\, H^*(\mathbb{C}P^1) = [2]$, $\mathrm{gr.rk}\, H^*(\mathbb{C}P^2) = [3]$, the most natural way to lift $P(\Theta)$ is to realize it as the cohomology ring of a manifold, which can be regarded as a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$ in 3 different ways. There is a unique such example, namely the full flag variety Fl_3 of the Lie algebra $\mathfrak{sl}_3$.
Recall that Fl_3 consists of full flags in $\mathbb{C}^3$:

$$Fl_3 = \{ (V_i)_{i=1}^{2} \mid \dim V_i = i,\ 0 \subseteq V_1 \subseteq V_2 \subseteq \mathbb{C}^3 \}$$

Note that we have $\mathbb{P}(\mathbb{C}^3) \cong \mathbb{CP}^2$ choices for V_1, and for each fixed V_1, there is $\mathbb{P}(\mathbb{C}^3/V_1) \cong \mathbb{CP}^1$ choices for V_2. This gives us a description of Fl_3 as a $\mathbb{P}^1$-bundle over $\mathbb{P}^2$.

The 3 different $\mathbb{P}^1$-bundle realizations of Fl_3 are given by regarding Fl_3 as a real analytic manifold. Namely, if we fix a hermitian inner product on $\mathbb{C}^3$, then any full flag (V_1, V_2) is completely determined by first choosing $L_1 = V_1$, and taking the orthogonal complement L_2 of V_1 in V_2, and finally letting L_3 be the direction perpendicular to V_2 in $\mathbb{C}^3$. In this way, we have set up a bijection of full flags Fl_3 with the collection:

$$\{ (L_i)_{i=1}^{3} \mid L_i \perp L_j,\ \forall i,j \}$$

We can permute L_i's and start by choosing any of them first ($\mathbb{CP}^2$ worth of choices). Thus we obtain the three different $\mathbb{P}^1$-bundle descriptions of Fl_3.

In general, if G is a compact Lie group and T is a maximal torus of G, the flag variety Fl of G can be regarded as the homogeneous manifold G/T, or holomorphically as $G^{\mathbb{C}}/B$, where $G^{\mathbb{C}}$ is the complexification of G, and B is a Borel subgroup of $G^{\mathbb{C}}$. The permuation of L_i's above is realized by the Weyl group $W = N_G(T)/T$ action on Fl, which doesn't preserve the holomorphic structure of Fl.

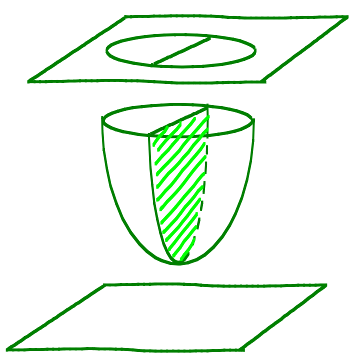
Algebraically,

$$H^*(Fl_3) \cong \mathbb{Z}[x_1, x_2, x_3]/(x_1+x_2+x_3,\ x_1x_2+x_1x_3+x_2x_3,\ x_1x_2x_3)$$

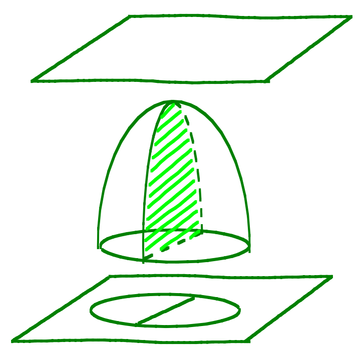
where x_i is the first Chern class of the line bundle on Fl_3, which assigns to the point (L_1, L_2, L_3) the line L_i. This is a Frobenius algebra of dim 6, with basis $\{x_1^r x_2^s \mid r \leq 2, s \leq 1\}$. The trace map is:

$$\varepsilon(x_1^r x_2^s) = \begin{cases} 1 & \text{if } r=2,\ s=1 \\ 0 & \text{otherwise} \end{cases}$$

We take $H(\Theta) \triangleq H^*(Fl_3, \mathbb{Z})$. The Frobenius algebra structure gives us a unit map ι (resp. trace map ε), which must come from a foam from ϕ to Θ (resp. Θ to ϕ):

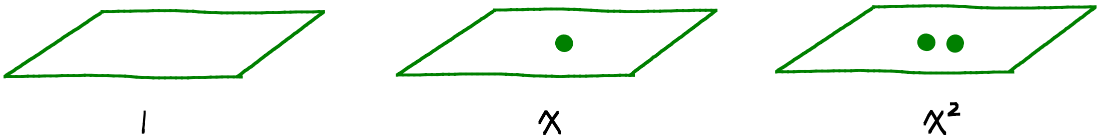


(resp.

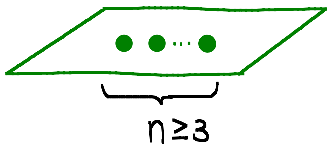


)

Recall that in §2, we introduced the handy graphical notation of depicting "multiplication by elements of A" by sewing a patch labeled by that element. Now we have patches from $A = \mathbb{Z}[x]/(x^3)$:

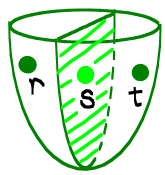


and

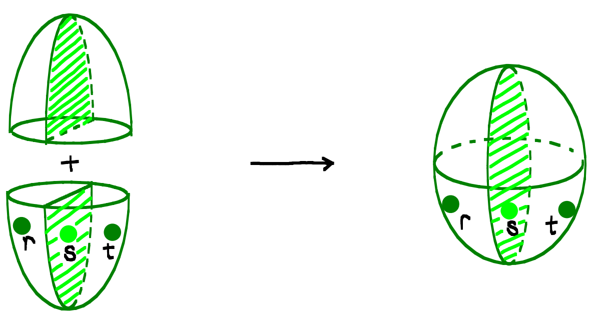


$= 0$

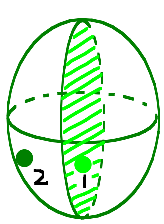
Thus the unit map with patches:



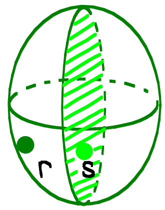
denotes the inclusion of the element $x_1^r x_2^s x_3^t$ into $H(\Theta)$. Closing it up with the cap:



represents the integer $\varepsilon(x_1^r x_2^s x_3^t)$. In particular, we have:

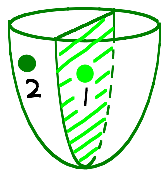


$= 1$

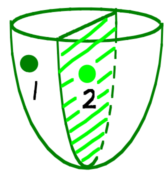


$= 0 \quad (r \neq 2 \text{ or } s \neq 1)$

Also note that by modding out symmetric functions, the inclusion of elements is an anti-symmetrizer :

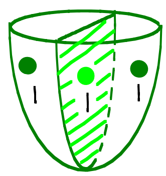


$= -$



$$\begin{aligned}(\because\ x_1^2x_2 + x_1x_2^2 &= (x_1+x_2)x_1x_2 \\ &= -x_1x_2x_3 \\ &= 0 \quad \text{in } H(\Theta)\ )\end{aligned}$$

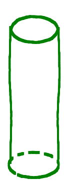
In particular, any symmetric distribution of dots around a singular circle is always 0 :



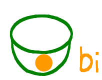
$= 0$

Evaluating closed foams

Now, we recall a trick from §2 that will always allow us to reduce closed foams to closed surfaces and "Θ bubbles," possibly with dots on them. This is done by locally decomposing any cylinder into patches :



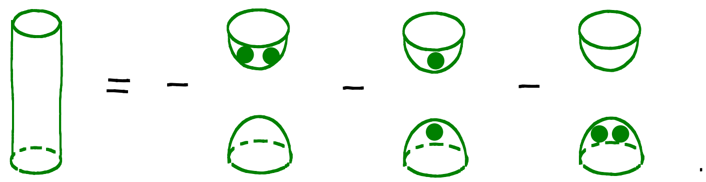
$= \sum_{i=1}^{\dim A}$



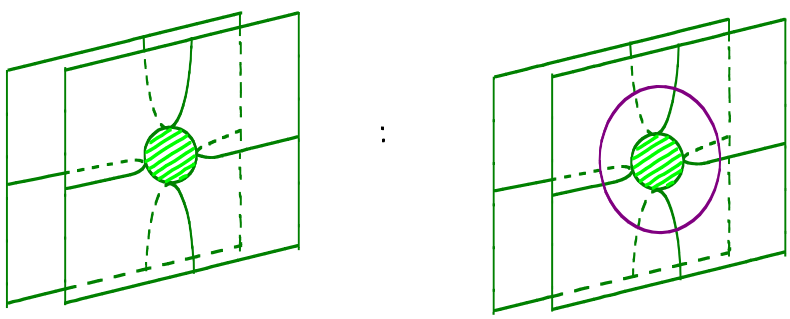
,

where $\{a_i\}$, $\{b_i\}$ are dual bases under ε. Here for later convenience,

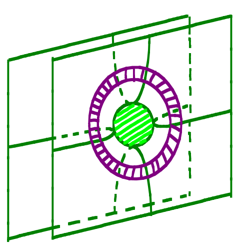
we will take $-\varepsilon$ as our trace pairing for A (or $A = H^*(\mathbb{CP}^2, \mathbb{Z})$).
so that:

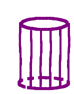


Then, inside any (closed) foam, near a singular circle, we can do a "surgery" around a smooth loop winding around the singular circle, and sewing back the above patches:

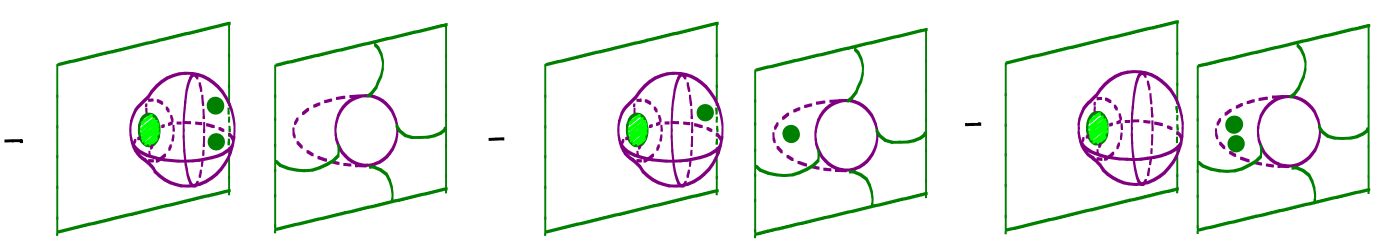


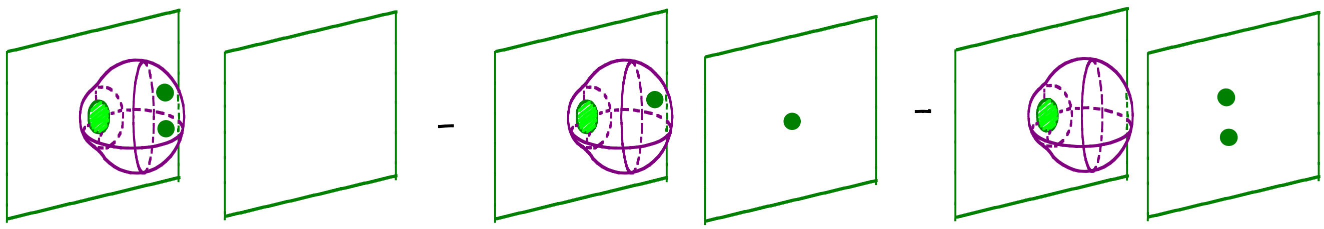
We will perform a surgery around the purple circle.

(1) 

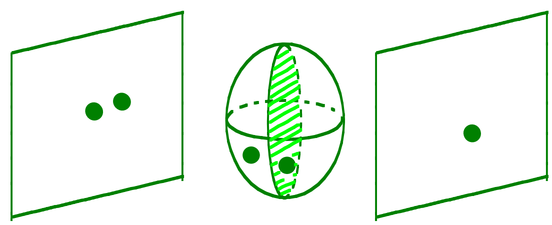
Cut out the purple band, which is diffeomorphic to $S^1 \times I$ 

(2). Paste back the patches:



$= -$ 

(3). Repeat this process to the left wall as well. We will get linear combinations of pictures where a "Θ-bubble" lies in between two walls, all carrying some dots:

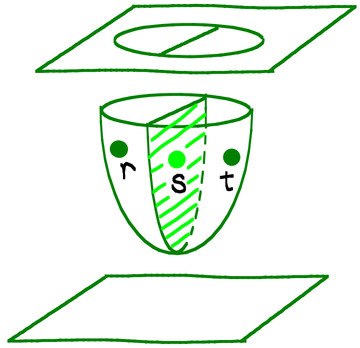
$\sum (\pm 1)$ 

In this way we will always reduce closed foams to the case of closed surfaces and Θ-bubbles carrying dots, to both of which we already know how to assign values. Although it may be impractical, it will help to determine $H(\Gamma)$ for webs Γ, as described below

Graded modules for webs

Being able to evaluate closed foams that carry dots will also enable us to determine the graded module $H(\Gamma)$ for any web Γ, at least in theory.

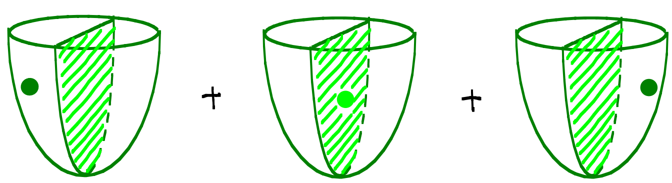
Indeed, we have a set of generators for $H(\Gamma)$, namely, we can regard any element of $H(\Gamma)$ as a cobordism from ϕ to



This picture gives the element $x_1^r x_2^s x_3^t \in H(\Theta)$

Γ, possibly carrying dots. Now we reverse this process by taking the free graded abelian group $\widetilde{H}(\Gamma)$ generated by all such cobordisms carrying dots.

Then we need to mod out all the relations among generators. A linear combination of the above cobordism pictures will give us a relation iff, whenever we cap off these pictures with any fixed cobordism from Γ to ϕ, the combination of closed foams evaluates to 0, using the algorithm we introduced in the previous subsection. We denote all relations so obtained by $R(\Gamma)$, which is a graded submodule of $\widetilde{H}(\Gamma)$.

 $\in R(\Gamma)$

By construction, we have:

$$H(\Gamma) \cong \widetilde{H}(\Gamma) / R(\Gamma).$$

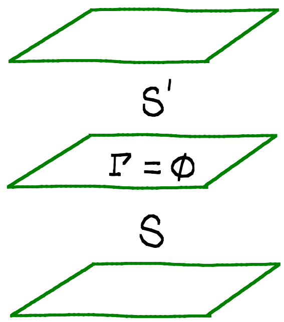
E.g. Some easy $H(\Gamma)$s.

(1). $\Gamma = \phi$.

Any closed foam S with dots gives a cobordism from ϕ to ϕ, and thus represents an element of $\tilde{H}(\Gamma)$. So

$$\tilde{H}(\phi) \cong \bigoplus \mathbb{Z}\langle S \rangle$$

But closing up S by any cobordism S' from $\phi (= \Gamma\,!)$ to ϕ is just the disjoint union of S with S'.

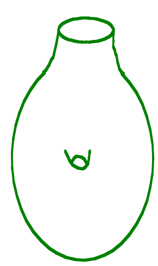


Since $\epsilon(S \sqcup S') = \epsilon(S) \cdot \epsilon(S')$, it follows that $S - \epsilon(S)\bar{\Phi}$, is in $R(\Gamma)$, where $\bar{\Phi}$ denotes the empty cobordism between ϕ and ϕ. We conclude that

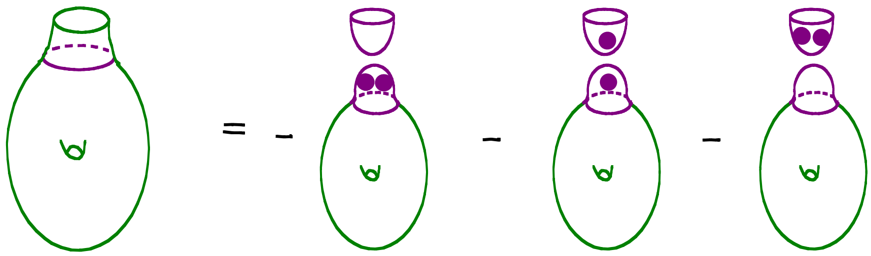
$$H(\phi) \cong \mathbb{Z}\langle \bar{\Phi} \rangle$$

(2). $\Gamma = \circlearrowright$.

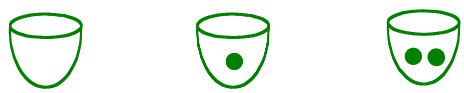
Again, generators of $H(\phi)$ are given by foams from ϕ to $\circlearrowright$:



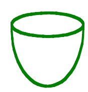
Using the surgery trick above, we can always reduce such foams to the case of a cup with no more than 2 dots on it:



The closed foams below the cups are then evaluated using ϵ. It follows that $H(\circlearrowright)$ is generated by the cups:



Moreover, there is no linear relation among these cups since they have different degrees. We thus recover our earlier assignment $H(\circlearrowright) = H^*(\mathbb{CP}^2)$, as graded abelian groups.

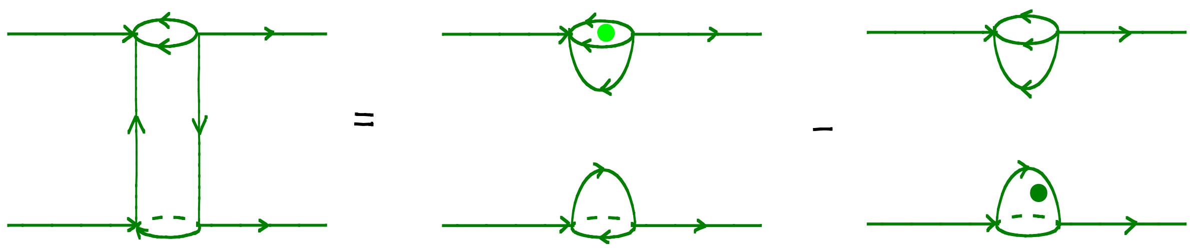
$H(\circlearrowright) \cong \mathbb{Z}\langle$  $\rangle \oplus \mathbb{Z}\langle$  $\rangle \oplus \mathbb{Z}\langle$  $\rangle$

In summary, so far we have defined the graded abelian groups $H(\Gamma)$ for any web Γ. In the following we will check that this assignment categorifies Kuperberg's spider calculus and define the $\mathfrak{sl}_3$ link homology.

Categorification of Kuperberg's spiders

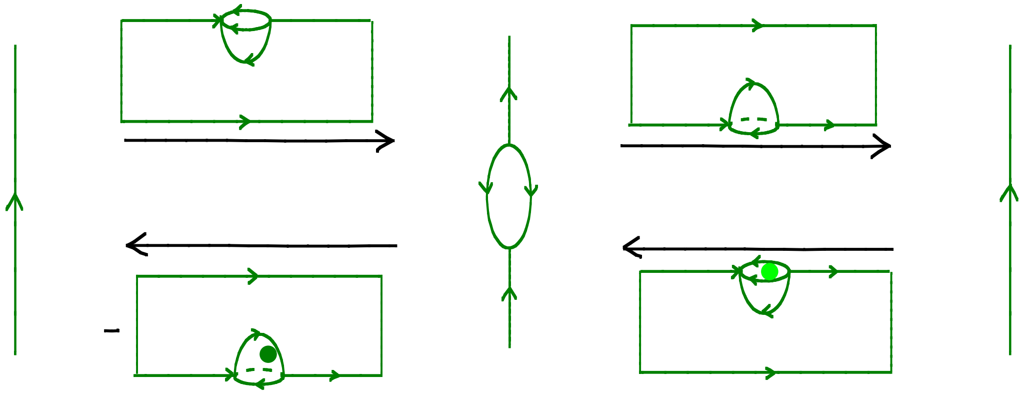
Now we check that our assignment $\Gamma \longmapsto H(\Gamma)$ lifts Kuperberg's graphical calculus. Details are left as exercises.

Lemma 2. We have the following cobordism decomposition:

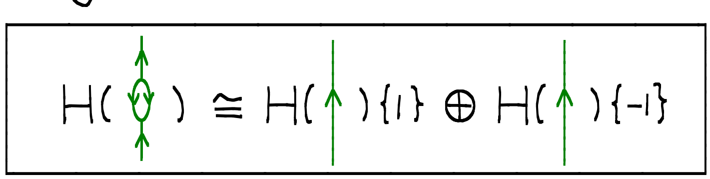


Proof omitted. This is done by "cupping" and "capping" off both sides of the equation, and doing surgeries on both sides along singular circles. See M. Khovanov, $\mathfrak{sl}(3)$ Link Homology.

Now we can set up maps between a digon and a line:

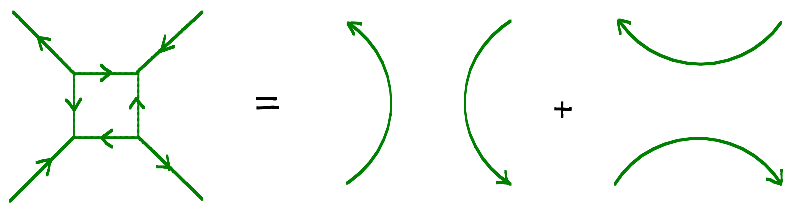


Using lemma 2 one checks that this gives an orthogonal decomposition of the identity morphism of the digon. Taking into account of the degree shifts ($\chi(\partial S)-2\chi(S)+2\cdot\#\text{dots}$), we obtain:

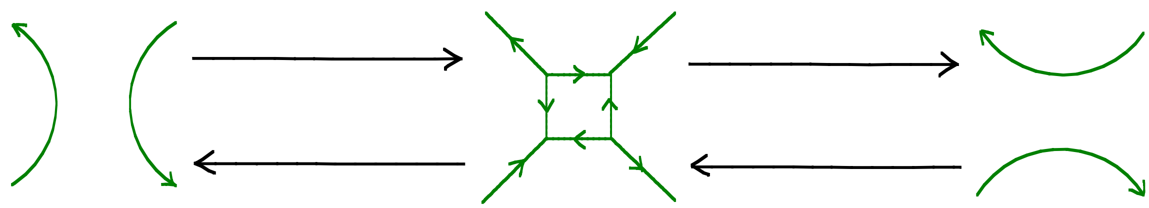


This categorifies Kuperberg's removing a digon face relation.

To check the square removal relation:

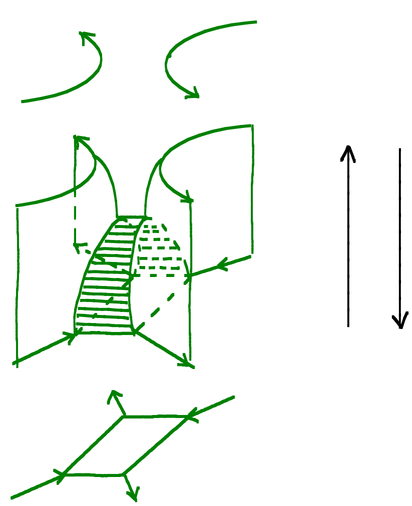


we need to set up maps as above

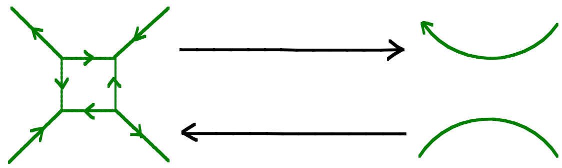


The maps are the most obvious foams that remove singular points in pairs:

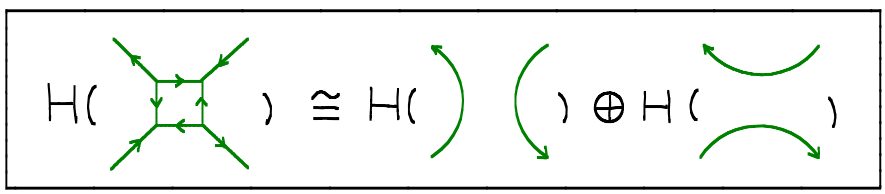
For instance:



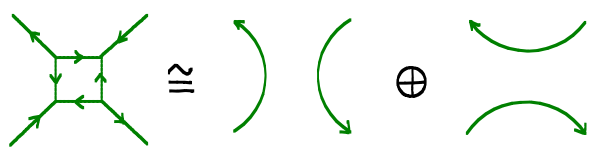
Rotating the picture by 90° gives the other cobordism between



One checks that these give an orthogonal decomposition of the identity morphism of the square web, which implies that



Rmk: A better way to think about these relations is to take the universal additive category generated by these pictures, in which we have



Again, since any web contains either circles, digons or squares, we can inductively reduce $H(\Gamma)$ to $H(\bigcirc \bigcirc \cdots \bigcirc)\{n\}$ for any web Γ.

Cor. 3. $H(\Gamma)$ is a graded free abelian group.
Pf: This follows from the circle case and inductively,

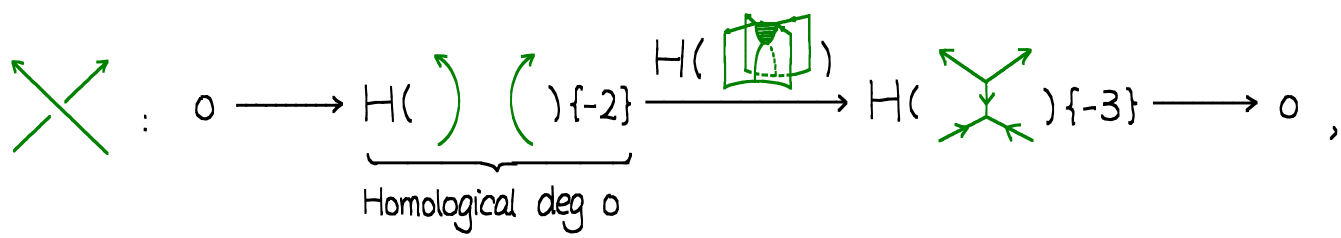
$$H(\Gamma_1 \sqcup \Gamma_2) \cong H(\Gamma_1) \otimes H(\Gamma_2) \qquad \square$$

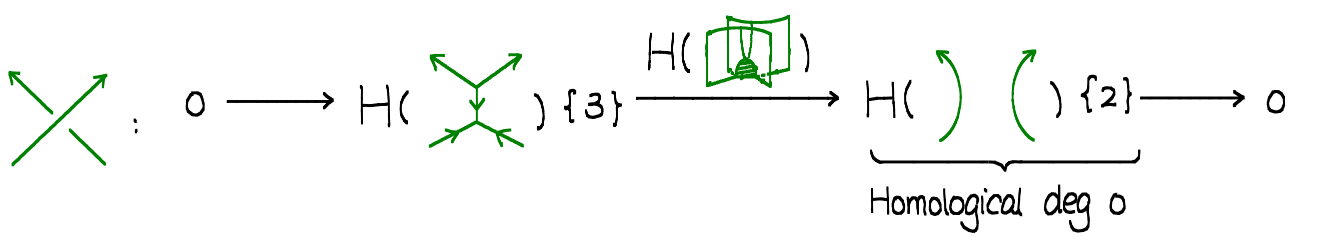
Rmk: Our construction doesn't give a preferred basis of $H(\Gamma)$.

$\mathfrak{sl}_3$ link homology

After the above effort of categorifying Kuperberg's spider calculus, we are now ready to construct the $\mathfrak{sl}_3$ link homology.

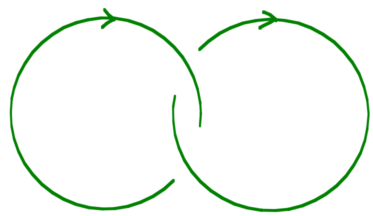
Now to the resolution of crossings, we apply:





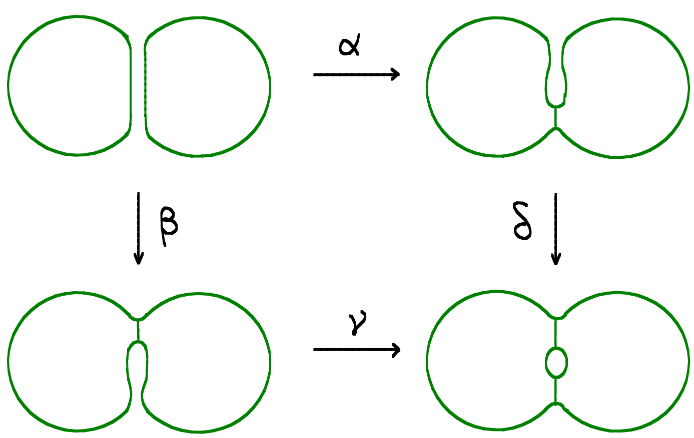
Then for any link diagram D with n crossings, we take its complete resolution to obtain 2^n webs, apply H, and obtain an n-dim'l cube of commutative diagrams of free abelian groups, just as we did for the Jones polynomial. The commutativity of the diagrams follows as before, since different paths of maps in the cube come from different orders of composing far apart foams, which are isotopic. It follows that if we adopt the sign convention of tensor products of complexes, the n-dim'l cube diagram becomes a multi-complex. We further collapse the multi-grading into a single grading. The total complex so obtained is our definition of $H(D)$.

E.g. The Hopf link diagram

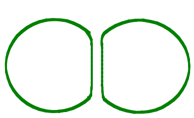
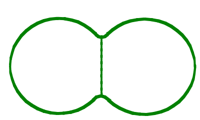
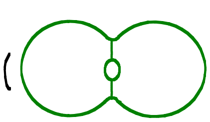


D

The complete resolution:



Thus $H(D)$ is the complex

$0 \longrightarrow H($  $) \xrightarrow{(\alpha,\,\beta)} H($  $)^{\oplus 2} \xrightarrow{\binom{-\delta}{\gamma}} H($  $) \longrightarrow 0$

Thm 4.

(1). If D_1 and D_2 are related by a Reidemeister move,

$$H(D_1) \cong H(D_2)$$

in $\mathrm{Comp}(gr.Ab)$, the homotopy category of complexes of graded abelian groups.

(2). $H(D)$ is a link invariant.

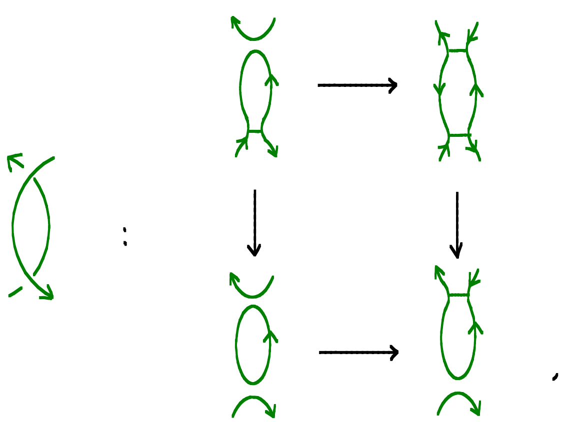
(3). $\chi(H(L)) = P(L)$.

Sketch of proof.

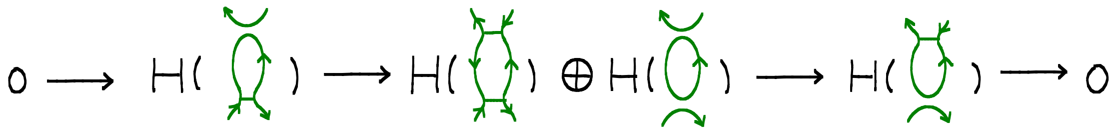
(2) follows from (1), and (3) is by construction.

The proof of (1) is similar as for the Jones polynomial case. A slight complication comes from different orientations of R-moves. For instance, we get different resolutions for RII moves with different orientations:

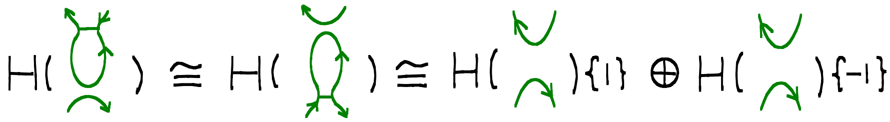
We take the last one as an illustration. The complete resolution of the l.h.s. is:

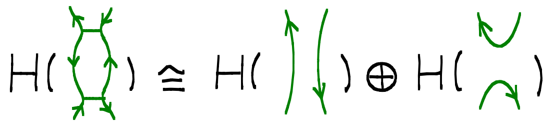


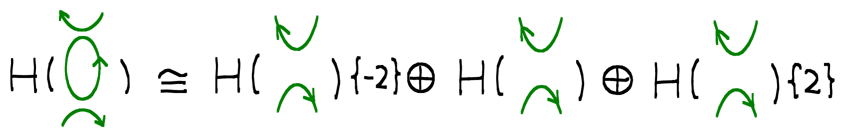
so that the total complex is:

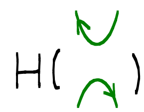


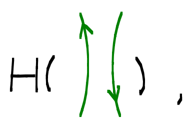
But





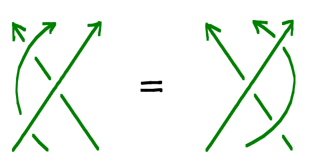


One checks that all (shifted) terms  cancel out and we are left with

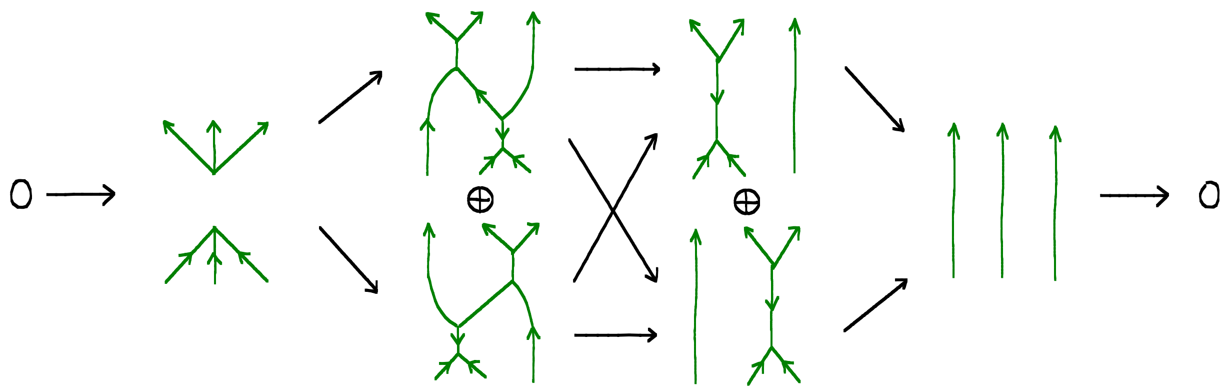


as desired.

The RIII move need only be checked for one particular orientation once the invariance under RI and RII is established, since other RIII moves are equivalent to any chosen one modulo RI and RII moves. We will sketch the following one:



In fact, both sides, after taking complete resolution, give the same total complex (with H omitted):



The invariance follows. ◻

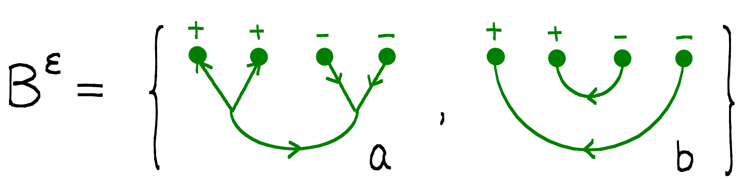
Extension to tangles

In this subsection, we sketch the extension of $\mathfrak{sl}_3$ homology theory to the 2 category of tangles. This is in analogy with the $\mathfrak{sl}_2$ case (the Jones polynomial).

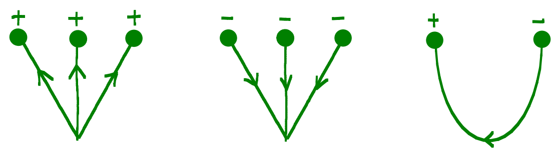
First of all we introduce the analogue of crossingless matchings. We will denote by ϵ any sequence of signs $\pm$, for instance, $\varepsilon = (++--)$. For any such an ε, we define the set B^{ε}:

$$B^{\varepsilon} \triangleq \{\text{ non-elliptic webs with boundary } \varepsilon \}$$

E.g. $\varepsilon = (++--)$



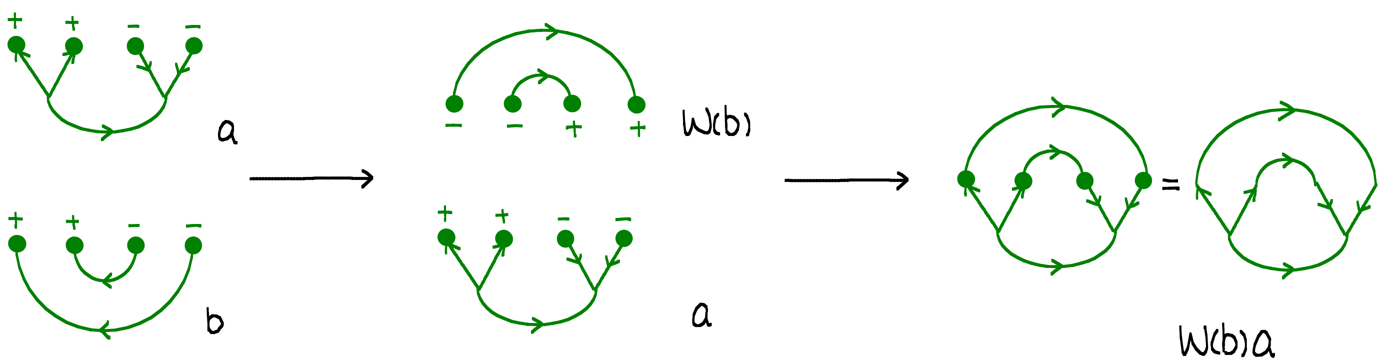
Notice that B^{ϵ} gives a basis of $\mathfrak{sl}_3$ ($U_q(\mathfrak{sl}_3)$)-invariants of the module $V^{\otimes\epsilon}$ (viewed as $\mathfrak{sl}_3$-module maps $\mathbb{C} \longrightarrow V^{\otimes\epsilon}$). $B^{\epsilon} = \phi$, unless $3 \mid \#(\overset{+}{\bullet}) - \#(\overset{-}{\bullet})$, since this holds for the basic building blocks:



Now for any such ε, define H^{ε} by

$$H^{\varepsilon} \triangleq \bigoplus_{a,b \in B^{\varepsilon}} H(W(b)\cdot a),$$

where W denotes reflecting webs about the horizontal axis and reversing orientations on arcs,

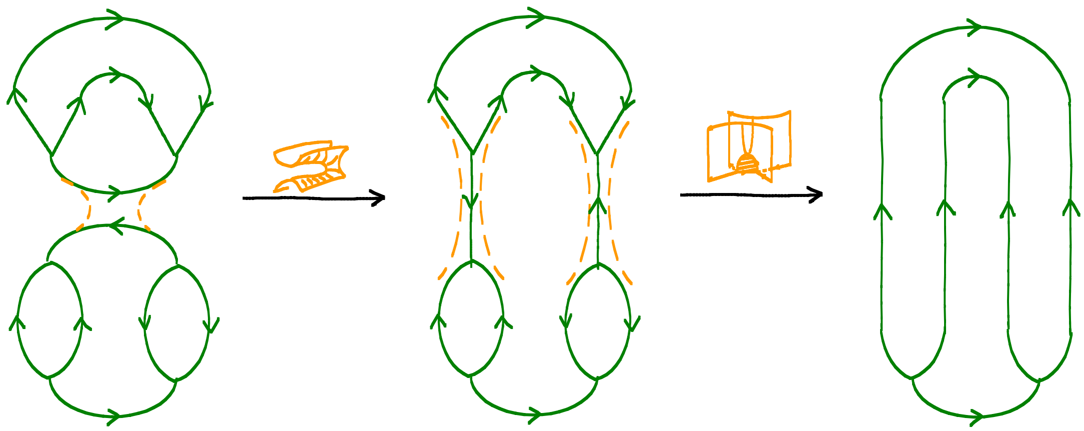


and H denotes our previous evaluation of closed webs.

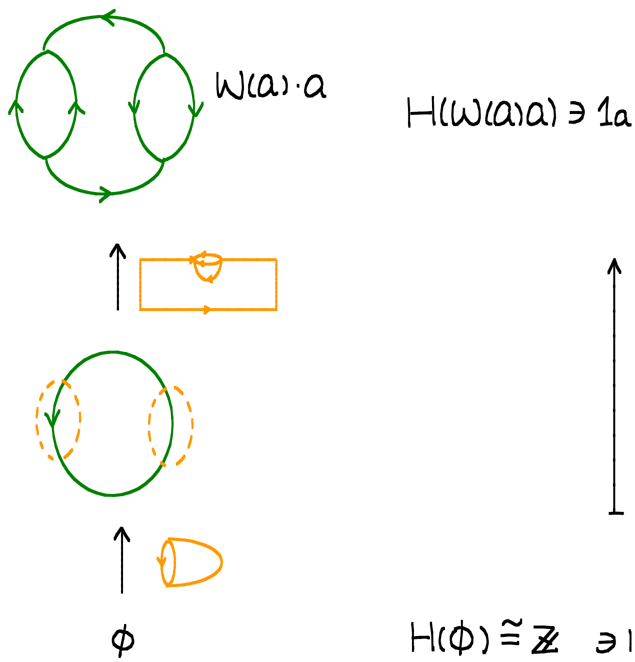
Again, we set

$$H(W(d)c) \otimes H(W(b)a) \longrightarrow \delta_{b,c} \cdot H(W(d)a)$$

as in the $\mathfrak{sl}_2$ case, and when $c=b$, the product is given by evaluating the canonical cobordism from $W(d)\cdot b \sqcup W(b)\cdot a$ to $W(d)a$:

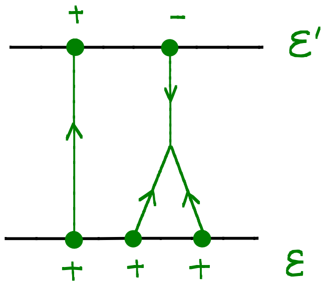


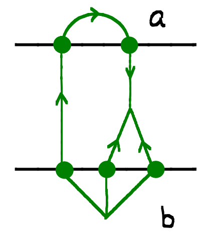
The product is associative for the same reason as in the $\mathfrak{sl}_2$ case. The unit is given by $1 = \sum_{a \in B^{\varepsilon}} 1_a$, where $1_a \in H(W(a)a)$ is given by the canonical cobordism from ϕ to $W(a)a$:



Moreover, to morphisms between ε and ε', i.e. webs $T_{\varepsilon}^{\varepsilon'}$ with boundaries $\varepsilon' \sqcup \varepsilon$, we assign the bimodule

$$H(T_{\varepsilon}^{\varepsilon'}) \triangleq \bigoplus_{b \in B^{\varepsilon}, a \in B^{\varepsilon'}} H(W(a) T_{\varepsilon}^{\varepsilon'} b)$$



$T^{\varepsilon'}_{\varepsilon}$ $\longmapsto$ $H(T^{\varepsilon'}_{\varepsilon}) = \bigoplus_{b\in B^{\varepsilon},\, a\in B^{\varepsilon'}} H\Big($  $\Big)$

It's readily seen that if $T_1{}^{\varepsilon''}_{\varepsilon'}$, $T_2{}^{\varepsilon'}_{\varepsilon}$ can be composed,

$$H(T_1 \cdot T_2) = H(T_1) \otimes_{H^{\varepsilon'}} H(T_2)$$

Next, to foams S as cobordisms between webs $T_1{}^{\varepsilon'}_{\varepsilon}$, $T_2{}^{\varepsilon'}_{\varepsilon}$, we assign the morphism of $(H^{\varepsilon'}, H^{\varepsilon})$ bimodules $H(S)$, defined by closing up S in all possible ways $W(a)\times[0,1]$, $b\times[0,1]$, $a\in B^{\varepsilon'}$, $b\in B^{\varepsilon}$:

$$H(S) \triangleq \bigoplus_{a\in B^{\varepsilon'},\, b\in B^{\varepsilon}} H(W(a)\times[0,1]\cdot S\cdot b\times[0,1]) : H(T_1) \longrightarrow H(T_2)$$

So far, this assigment gives us a 2-functor from the 2-category of webs and foams inside $\mathbb{R}^3$ to the 2-category to bimodules and bimodule homomorphisms (compare with flat tangles in the $\mathfrak{sl}_2$ case).

Finally, this 2 functor extends to the 2 category of tangle cobordisms to the 2 category of Comp(bimodules) as we did in the $\mathfrak{sl}_2$ case. The only difference being that matchings / flat tangles / F are now replaced by B^{ε} / webs / H.

Rmk: The rings H^ε are expected to be related to various parabolic categories $\mathcal{O}$ of $\mathfrak{sl}_3$, or the cohomology rings of some Springer varieties of $\mathfrak{sl}_3$.

Comparison with $\mathfrak{sl}_2$ case

In the $\mathfrak{sl}_2$ case (the Jones polynomial), we defined the projective modules for any $a \in B_n$:

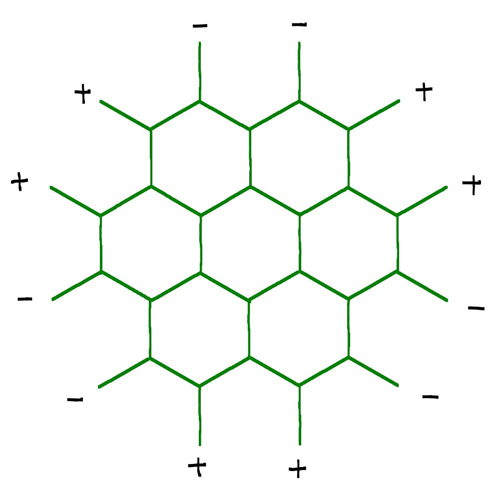
$$P_a = \bigoplus_{b \in B_n} F(W(b)a)$$

These projective modules were shown to be indecomposable, and $\mathrm{End}_{H^n}(P_a) \cong A^{\otimes N}$ for some N.

Now in our $\mathfrak{sl}_3$ theory, one only has indecomposability of P_a where

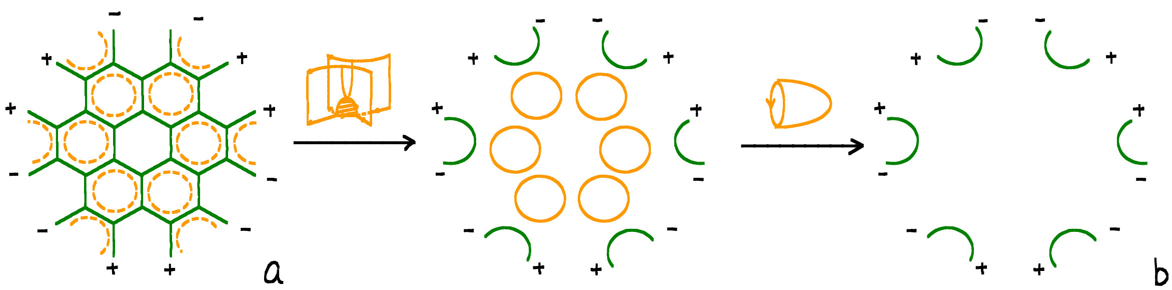
$$P_a = H^\varepsilon \cdot 1_a = \bigoplus_{b \in B^\varepsilon} H(W(b)a)$$

when ε is short enough. For instance, the first case when P_a is decomposable occurs when $\varepsilon = (++--++--++--)$



a

In fact, there is a canonical cobordism taking this web to b:



Denote the composite cobordism from a to b above by S, and S^* the same cobordism foam viewed backwards from b to a. Then one get

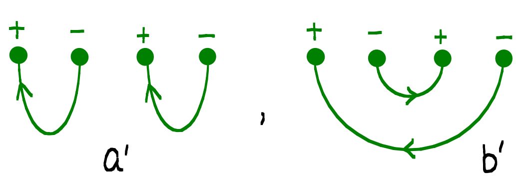
$$P_a \underset{H(S^*)}{\overset{H(S)}{\rightleftarrows}} P_b$$

One checks that $H(S) \circ H(S^*) = \pm \mathrm{Id}\, P_b$, so that P_b is a direct summand of P_a. Hence P_a is not indecomposable. However, P_b is indecomposable since

$$\mathrm{End}_{H^{\varepsilon}}(P_b) \cong (\Bbbk[X]/(X^3))^{\otimes 6}$$

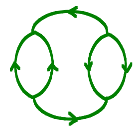
is a local ring.

Another interesting phenomenon that occurs here but not in the $\mathfrak{sl}_2$ theory is that different ordering of sequences may give rise to rings that are not Morita equivalent. For instance, consider $\varepsilon' = (+-+-)$ which is a different ordering of $\varepsilon = (++--)$. We have

$B^{\varepsilon'} = \Big\{$  $\Big\}$

In H^{ε}-mod,

$$\dim \mathrm{End}_{H^{\varepsilon}}(P_a) = \dim H(W(a)a)$$

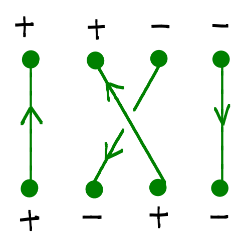
$$= \dim H($$  $$)$$

$$= [2]^2 \cdot [3]\,|_{q=1}$$

$$= 12$$

But $\dim \mathrm{End}_{H^{\varepsilon'}}(P_{a'})$ (resp. b') $= 9$. It follows that these rings are not Morita equivalent.

However, these rings are derived Morita equivalent, in the sense that the complex of bimodules ${}_{H^{\varepsilon}}H(T)_{H^{\varepsilon'}}$, where T is the tangle:

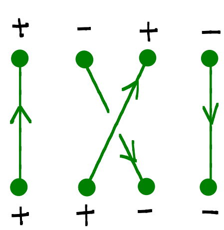


T

gives rise to an invertible map:

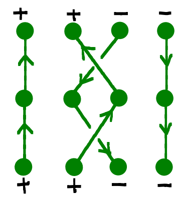
$$F(T) \otimes_{H^{+-+-}} (-) : \mathrm{Comp}(H^{+-+-}) \longrightarrow \mathrm{Comp}(H^{++--})$$

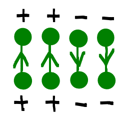
between the homotopy categories of complexes of projective modules over these rings. It's invertible since T is invertible with inverse:



T'

$$H(T)\circ H(T') = H(T\circ T')$$

$$= H\Big($$  $$\Big)$$

$$= H\Big($$  $$\Big)$$

$$= \mathrm{Id}_{\mathrm{Comp}(H^{++--})} ,$$

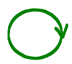
and similarly $H(T')\circ H(T) = \mathrm{Id}_{\mathrm{Comp}(H^{+-+-})}$

We will give a brief recap of (derived) Morita theory in the next section.

Open problems about $\mathfrak{sl}_3$ link homology

(1). There is no computer program to compute it at the moment.

(2). The analogue of Rasmussen's invariant.
In the works of Scott-Nieh, Vaz-Mackaay, they defined deformed versions of $H(\Gamma)$ for webs Γ using equivariant cohomology of $\mathbb{CP}^2$, Fl_3:

	H	$\widetilde{H}$
ϕ	$\mathbb{Z}$	$H^*_{U(3)}(pt, \mathbb{Z}) \cong H^*(BU(3), \mathbb{Z}) \cong \mathbb{Z}[t_1, t_2, t_3]$, $\deg t_i = 2i$
	$H^*(\mathbb{CP}^2)$	$H^*_{U(3)}(\mathbb{CP}^2, \mathbb{Z}) \cong \dfrac{\mathbb{Z}[X, t_1, t_2, t_3]}{(X^3 - t_1X^2 - t_2X - t_3)}$
	$H^*(Fl_3)$	$H^*_{U(3)}(Fl_3, \mathbb{Z}) \cong H^*_{T^3}(pt, \mathbb{Z}) \cong \mathbb{Z}[x_1, x_2, x_3]$, $t_1 = x_1+x_2+x_3$, $t_2 = x_1x_2+x_1x_3+x_2x_3$, $t_3 = x_1x_2x_3$

One defines chain complexes of link diagrams similarly as for H. For instance, this assignment satisfies, for any link diagrams K, L

$$C(K \sqcup L) = C(K) \otimes_{\widetilde{H}(\phi)} C(L).$$

The analogue of Rasmussen's invariant is defined by first setting $t_1, t_2 = 0$. Then,

$$\widetilde{H}(K) = (t\text{-torsion}) \oplus \widetilde{H}(K)^{\text{free}}$$

and

$$\widetilde{H}(K)^{\text{free}} \cong \widetilde{H}(\bigcirc)\{-2s'(K)\}$$

There are examples where $s'(K)$ is not propotional to $s(K)$. However, it works equally as well as $s(K)$ to prove Milnor's conjecture.

A problem about $s'(K)$ is to determine which diagrams are adequate for $\widetilde{H}$.