Homework 1

January 23, 2022

Exercise 1. Use the following way to show that the fundamental group of a Lie group is abelian.

(1) Recall that an element $[\gamma]$ of $\pi_1(G)$ is represented by a path γ starting at the origin of G. If given such two paths γ_1, γ_2, show that there is a well-defined map from the two torus $T^2 \longrightarrow G$, such that when restriced to $S^1 \times 1$ and $1 \times S^1$ you get back γ_1 and γ_2.

(2) By part (1), the subgroup generated by $[\gamma_1]$ and $[\gamma_2]$ is contained in the image of $\pi_1(T^2)$ under the above extended map. Use this to finish the proof of $\pi_1(G)$ being abelian.

Exercise 2. Find the foundamental groups of the following Lie groups $O(n, \mathbb{R})$, $U(n)$, $SU(n)$ and $Sp(n)$. Here $Sp(n)$ is defined as the group that preserves the standard inner product on the n-dimensional quaternionic space $\mathbb{H}^n$:

$$Sp(n) := \{A \in \mathrm{M}(n, \mathbb{H}) | \langle Av, Aw \rangle = \langle v, w \rangle \ \forall v, w \in \mathbb{H}^n\}.$$

Exercise 3. Let A, B be any matrix in $\mathrm{M}(n, \mathbb{F})$ with $\mathbb{F} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Prove the following identities.

- $\exp(BAB^{-1}) = B\exp(A)B^{-1}$ if B is invertible.
- $\exp(A^*) = (\exp(A))^*$, where $*$ can either be the transpose, conjugation (on $\mathbb{C}$ and $\mathbb{H}$) or the composition of these two operations.
- $\exp : \mathrm{M}(n, \mathbb{F}) \longrightarrow \mathrm{M}(n, \mathbb{F})$ is real analytic, and the differential $d(\exp)|_0$ is nondegenerate at $T_0(\mathrm{M}(n, \mathbb{F})) \longrightarrow T_{\mathrm{Id}}(\mathrm{M}(n, \mathbb{F}))$.
- $\det(\exp(A)) = e^{\mathrm{tr}(A)}$.
- Use these properties to find the tangent space $T_{\mathrm{Id}}G$ for the following matrix groups:

$$G = GL(n, \mathbb{F}), \quad SO(n, \mathbb{R}), \quad U(n), \quad SU(n), \quad Sp(n),$$

  and compute their dimensions over $\mathbb{R}$.

Exercise 4. Show that $\mathbb{R}^3$ with the usual cross product is a Lie algebra.

Exercise 5. Recall from Exercise 2 that $Sp(1)$ consists of unit quaternions. Show that $Sp(1) \cong SU(2)$ by explicitly constructing an isomorphism.

Exercise 6. Let U be a charted open set of a manifold M, and let ξ η be two vector fields on M whose restriction on U are given by

$$\xi|_U = \sum_{i=1}^{n} a_i(x_1, \ldots, x_n)\frac{\partial}{\partial x_i}, \quad \eta|_U = \sum_{i=1}^{n} b_i(x_1, \ldots, x_n)\frac{\partial}{\partial x_i}.$$

Show that, if we define the commutator vector field $[\xi, \eta]$ locally by

$$[\xi, \eta]|_U := \sum_{i,j=1}^{n} (a_j\frac{\partial b_i}{\partial x_j} - b_j\frac{\partial a_i}{\partial x_j})\frac{\partial}{\partial x_i},$$

then $[\xi, \eta]$ is a well-defined global vector field (i.e. it is independent of choices of the chart U).

Exercise 7. Let A be a finite-dimensional algebra over $\mathbb{R}$, and let D be a derivation on A. Then

$$\exp(D) : A \longrightarrow A, \quad a \mapsto \sum_{k=0}^{\infty} \frac{D^n(a)}{n!}$$

is an algebra automorphism of A.

Exercise 8. Let A be an associative algebra.

(i) Show that there is a Lie algebra homomorphism $A^L \longrightarrow \mathrm{Der}(A)$ given by $a \mapsto [a, \text{-}]$. Such derivations on A are called *inner derivation*.

(ii) If $A = \mathbb{k}[x]$, the polynomial ring in a single variable, find all derivations on A. Are any of them inner derivations?

Exercise 9. Any (matrix) Lie group $G(\subset GL(V))$ acts on its Lie algebra $\mathfrak{g} = T_1G$ by congugation:

$$G \times \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad (g, x) \mapsto gxg^{-1}$$

(1) Show that this action preserves the Lie bracket.

(2) In the case of $SU(2)$, show that this action preserves the metric on $\mathfrak{su}(2)$ defined by $\langle A, B\rangle = \mathrm{Tr}(AB^*)$.

(3) Show that there is a well-defined group homomorphism $SU(2) \longrightarrow SO(3)$ using (2). What is the kernel of this map?